TECHNISCHE UNIVERSITÄT DRESDEN

International Master Program in Computational Logic

Fakultät Informatik

Technische Universität Dresden

Algorithms for Computing Least Common Subsumers w.r.t. General $\mathcal{FL}_0$-TBoxes

Master's Thesis

Author:
Adrian Nuradiansyah,
MNr. 4108192

Advisor:
Dipl. Inf. Benjamin Zarriess

Supervisor:
PD.Dr.-Ing.habil,
Anni-Yasmin Turhan

Abstract

Generalizations of a collection of concepts can be computed by the least common subsumer (lcs) which is a useful inference for building knowledge bases. For general $\mathcal{FL}_0$-TBoxes the lcs need not exist. In this thesis, we devise a condition to check whether a concept is the lcs of two concepts w.r.t. a general $\mathcal{FL}_0$-TBox. We also define the characterizations for the existence of the lcs. Last, we show that if the lcs exists, then we can compute the lcs and the upper bound for the role-depth of the lcs.

Declaration of Authorship

I, Adrian Nuradiansyah of Technische Universität Dresden, being a candidate of Master of Science in Computational Logic, hereby declare that this thesis is my own work. Any help that I have received in my research work has been acknowledged. I certify that I have not used any auxiliary sources and literature except those cited in the thesis.

Author	:	Adrian Nuradiansyah
Matriculation Number	:	4108192
Title	:	Algorithms for Computing Least Common Subsumers in General $\mathcal{FL}_0$-TBox
Signed	:	
Date	:	

Acknowledgments

First and foremost, I would like to tell my greatest grateful to Allah SWT. for giving me all favors and blessings such that I am able to finish my own master thesis. Shalawat is always granted to Muhammad SAW. which is being my life inspiration. I also would like to say my acknowledgments to:

- My parents and my sister for always giving me strength and motivation from a distance of 15,400 km;
- Dr. Anni-Yasmin Turhan, for her excellent supervision during the work of my thesis;
- Benjamin Zarriess, for his excellent advice and support during the work of my thesis;
- Prof. Steffen Hölldobler, as the coordinator of Master of Computational Logic in TU Dresden, for his constructive advice and encouragement for my study in Dresden within 2 years;
- Prof. Franz Baader, as the head of the chair for Automata Theory in TU Dresden, for his permission to allow me to work my thesis in his research group;
- Erasmus+ Swap and Transfer for providing me substantial funds and supports during my study in TU Dresden for two years;
- My buddies from computational logic program and Indonesian community in Dresden for their foods, jokes, and cheerfulness that are completely able to relax my mind during the work of thesis.

Contents

1 Introduction

Nowadays, ontologies have become popular resources to store knowledge bases across domain areas and to describe controlled vocabularies. Moreover, ontologies are good to support automated reasoning which is able to exploit implicit information. Since the standardization of web ontology language (OWL) ([Gro04],[Gro12]) was designed, many knowledge engineers build and maintain their applications by enriching their data vocabularies to describe more notions from their application domain in a precise way.

The formalism underlying OWL are Description Logics (DLs) [BCM+03]. Description Logics (DLs) are introduced as a decidable fragment of First Order Logic (FOL). The basic vocabularies of DL are comprised of unary predicates called concept names and binary predicates called role names. Using a point of view of programming languages, a concept name can be described as a class of objects and a role name represents a relation between two objects. For instance, we have *Composer* or *Painter* as a concept whose the objects may be people who work as a composer or a painter, respectively. On the other side, *compose* or *draw* describe relations between a person, which is a composer or a painter, with a type of art he composes. To give brief explanations about what constructors used in DLs, we use a very basic DL, namely $\mathcal{ALC}$, whose constructors are commonly used in FOL. They are conjunction ($\sqcap$), disjunction ($\sqcup$), negation ($\neg$), existential restrictions ($\exists$), and value restrictions ($\forall$). We take one example using these constructors by considering some concept and role names mentioned before, such as

$$Composer \sqcap \neg Painter \sqcap \exists .compose(Song \sqcup Poem) \sqcap \forall gender.Male,$$

where $Song, Poem$, and $Male$ are concept names and *gender* is a role name. The logical statement above describes a set of objects which are composers, but not painters, who compose a song or a poem, and all of them are male. Next, the set of all objects in a specific domain is denoted by the top concept $\top$. Conversely, the empty set is represented by the bottom concept $\bot$.

In Description Logic (DL) system [BCM+03], such a knowledge is captured by three components. First, we have a description language that defines the formal syntax and semantics. Second, we have a knowledge base that consists of *TBox* and *ABox*. *TBoxes* defines the terminologies occur in an application domain. It contains a set of implication between DL-concepts. A general TBox allows complex concepts to occur on the right- and the left-hand side of implication. An *ABox* captures facts in the form of an individual of a DL-concept and a relationship between individuals in a specific "world" of a knowledge base. Last, the system has a reasoning component that derives implicit facts from the represented knowledge.

In DLs, the classical standard inferences, like *subsumption* and *instance checking*, are already well-investigated. *Subsumption* is the reasoning task to check whether a sub-/superconcept relationship holds between the pair of concepts, whilst the *instance checking* determines for a given individual whether it belongs to a given concept with respect to a given knowledge base. The non-standard inferences that describe the generalization of a given pair of concepts or a single individual are called *least common subsumer* (lcs) and *most specific concept* (msc). Intuitively, the lcs yields a concept which captures all commonalities of pairs of input concepts such that this concept is the least common among other concepts that subsume the input concepts. This is also defined analogously for msc, but only to generalize an individual into a concept. Therefore, we are able to see that the computation for the lcs and msc are based on subsumption and instance checking, respectively.

In practice, the lcs and msc support building and maintaining the knowledge base. A knowledge engineer can gain more benefits to define a relevant concept that generalizes a pair of concept or an individual. For instance, the resulted lcs- or msc-concept can be processed and investigated by the knowledge engineer, if necessary, to be added thereafter in a knowledge base as new information. One of the applications where non-standard inferences services, such as lcs or msc, can greatly enhance the usability of DL-systems was already well-investigated in the domain area of process engineering [BT01].

Unfortunately, neither the lcs nor the msc need to exist, if compute w.r.t. general TBoxes in some description logics, for instance $\mathcal{EL}$ [Baa03]. As we know that $\mathcal{EL}$ is also one of inexpressive descripton logics that only has a few number of constructors. Here we are interested to investigate the existence of lcs in the other lightweight DLs, which is $\mathcal{FL}_0$. This specific DL only allows the occurrence of top concept, conjunction over complex concepts, and value restrictions. For DL $\mathcal{FL}_0$, the computational worst-case complexity for standard inferences, such as subsumption, reaches the class `ExpTime` or worse [BBL05]. Even though $\mathcal{FL}_0$ is a fragment of $\mathcal{ALC}$ and both of them share the same complexity class for deciding subsumption, the investigation to look for a well-behaved approach for the characterizations of the subsumption in $\mathcal{FL}_0$ is still required. So far, the characterizations for subsumption in $\mathcal{FL}_0$ itself have been approached by automata theory ([BCM$^+$03],[Pen15]), deciding inclusion between two models of input concepts [Pen15], and structural algorithmic solutions that construct models by non-deterministic construction rules ([HST99],[Pen15].

Some related works on computing the least common subsumer, which consider the presence of value restrictions, have been devised previously through various kinds of methods. For instance, [K98] devised an automata-based algorithm for computing the lcs in $\mathcal{ALN}$ that allows value restrictions with other constructors. Moreover, a description-graph approach was also employed for unfoldable TBoxes in [BTK03], where an lcs computation algorithm for the DL $\mathcal{FL}_0^+$ was devised. This DL augments $\mathcal{FL}_0$ with transitive roles.

Another related works to compute the lcs, which consider value restrictions combined with additional constructors, such as primitive negation, existential restrictions, or number restrictions, are also probed in [BK98] and [KM00]. Here they use a graph-based approach, where concept descriptions are represented in a description graph. However, all literatures that are mentioned previously did not include general $\mathcal{FL}_0$-TBoxes during the computation.

Now, let us consider the following motivating example. Let $\mathcal{T}_{ex}$ be a cyclic $\mathcal{FL}_0$-TBox and consists of the following axioms:

$$\begin{array}{rcl} A_1 & \sqsubseteq & B_1 \sqcap B_2 \\ A_2 & \sqsubseteq & B_1 \sqcap B_3 \\ B_1 \sqcap B_2 & \sqsubseteq & \forall r.B_1 \sqcap \forall r.B_2 \\ B_1 \sqcap B_3 & \sqsubseteq & \forall r.B_1 \sqcap \forall r.B_3 \end{array}$$

Then, we want to compute the lcs of concepts A_1 and A_2. With respect to $\mathcal{T}_{ex}$, the lcs of A_1 and A_2 does not exist because the cyclic definitions of A_1 and A_2 allow us to always have infinite number of common subsumers of A_1 and A_2 without having the least one. However, if we extend $\mathcal{T}_{ex}$ with an additional axiom $B_1 \sqsubseteq \forall r.B_1$, then the lcs of A_1 and A_2 w.r.t. $\mathcal{T}_{ex}$ exists, which is B_1.

In this thesis, we are about to handle three following problems whose the solutions will be described in different further sections. Now, let C, D be $\mathcal{FL}_0$-concepts and $\mathcal{T}$ be a general $\mathcal{FL}_0$-TBox.

Problems:

I. Let E be an $\mathcal{FL}_0$-concept. Is concept E the lcs of C and D w.r.t. $\mathcal{T}$?

II. Does the lcs of C and D w.r.t. $\mathcal{T}$ exist?

III. If the lcs of C and D w.r.t. $\mathcal{T}$ exists, then what is the lcs? And how big is the size of the lcs?

In order to provide solutions for the problems above, we will present characterizations for the existence of the lcs w.r.t. general $\mathcal{FL}_0$-TBoxes. We will also compute the lcs and the size of the lcs if it exists.

This thesis is organized as follows: First, we introduce basic notions in Description Logic $\mathcal{FL}_0$ and least common subsumer (lcs) in Section 2. Two primary means to characterize the existence of the lcs, which are functional models and simulation relation, will be described in Section 3 and 4, respectively. Afterwards, we show characterizations to decide whether a concept is the lcs of two input concepts w.r.t. a TBox to address Problem I above. The next contribution of this thesis, presented in Section 6, is to characterize the existence of the lcs as questioned in Problem II above. Then, we address Problem III that considers if the lcs of input concepts w.r.t. a TBox exists, then we compute the lcs and measure how big the lcs is. This computation is described in Section 7. We end this thesis with some conclusions and future works.

2 Description Logic $\mathcal{FL}_0$ and Least Common Subsumer

The main description logic in this thesis, namely $\mathcal{FL}_0$, will be discussed from its very basic notions in terms of syntactical and semantical notations in the first subsection. Next, the non-standard inference in terminological knowledge, which is least common subsumer will be explained in more detail in the second subsection. Last, we show how to normalize our general $\mathcal{FL}_0$-TBox to a specific normal form, called plane-axiom-normal-form (PANF). Eventually, we will have an assumptions how our input concepts and TBox should look like for the problem of existence of the lcs and computing the lcs, if it exists.

2.1 Description Logic $\mathcal{FL}_0$

In this thesis, we restrict all notions and definitions to the description logic $\mathcal{FL}_0$. Let N_C and N_R be a set of concept names and a set of role names, respectively. In the following, we use $A, B \in N_C$ for concept names and $r, s \in N_R$ for role names. $\mathcal{FL}_0$-*concepts* are built inductively by using the following structures:

$$C,D ::= \top \mid A \mid C \sqcap D \mid \forall r.C$$

In order to define the formal semantics of $\mathcal{FL}_0$, the notion of an interpretation is introduced. An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty domain $(\Delta^{\mathcal{I}})$ and $\cdot^{\mathcal{I}}$ as a function which interprets $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The mapping $\cdot^{\mathcal{I}}$ is extended to $\mathcal{FL}_0$-concepts which is defined in Table 1.

Name	Syntax	Semantic
Top	$\top$	$\Delta^{\mathcal{I}}$
Conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
Value Restriction	$\forall r.C$	$\{d \in \Delta^{\mathcal{I}} \mid \forall e \in \Delta^{\mathcal{I}} : (d,e) \in r^{\mathcal{I}} \text{ implies } e \in C^{\mathcal{I}}\}$

Table 1: Syntax and Semantics of Description Logic $\mathcal{FL}_0$

A general *TBox* is a finite set of General Concept Inclusions (GCIs) of the form $C \sqsubseteq D$. An interpretation $\mathcal{I}$ *satisfies* $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. $\mathcal{I}$ is called a *model* of $\mathcal{T}$ iff it satisfies all GCIs in the TBox.

Any $\mathcal{FL}_0$-concept is written in the form $\forall r_1.\forall r_2 \cdots \forall r_n A$ where A is a concept name and $r_i \in N_R$, for all $1 \leq i \leq n$. We shall abbreviate the prefix "$\forall r_1.\forall r_2 \cdots \forall r_n$" by "$\forall w$" where the word $w = r_1 r_2 \ldots r_n$ and $w \in N_R^*$. For the case $n = 0$, we write "$\forall \varepsilon.A$" to replace "A". Therefore, we consider a normal form for a given concept and TBox to simplify the structural approach used during computing generalizations. This normal form is called *concept-conjunction-normal-form* (CCNF) [Pen15]. A concept is in CCNF iff it is of the form

$$\forall w_1.A_1 \sqcap \ldots \sqcap \forall w_n.A_n,$$

where $A_i \in N_C$ and $w_i \in N_R^*$, for all $1 \leq i \leq n$.

The following rules written in Table 2 are applied exhaustively during the normalization to CCNF, both for a concept and a given TBox.

Rules	General Form		CCNF
NF1	$C \sqcap \top$	$\rightsquigarrow$	C
NF2	$\top \sqcap C$	$\rightsquigarrow$	C
NF3	$\forall w.\top$	$\rightsquigarrow$	$\top$
NF4	$\forall w.(C_1 \sqcap \ldots \sqcap C_n)$	$\rightsquigarrow$	$\forall w.C_1 \sqcap \ldots \sqcap \forall w.C_n$

Table 2: CCNF Normalization Rules for TBoxes

It is easy to see that every CCNF concept resulted by one of the normalization rules is still equivalent to the general one, i.e. they have the same extension in any interpretation. The terminological reasoning task called *subsumption* is used to check whether a concept generalizes another concept with respect to TBox $\mathcal{T}$. Formally, it is defined as follows:

Definition 2.1. (*Subsumption and Equivalence*)
Let $\mathcal{T}$ be a TBox and C, D be $\mathcal{FL}_0$-concepts. C is subsumed *by D w.r.t. $\mathcal{T}$ (denoted by $C \sqsubseteq_\mathcal{T} D$) iff $C^\mathcal{I} \subseteq D^\mathcal{I}$ for all models $\mathcal{I}$ of $\mathcal{T}$.*
Two concepts C and D are equivalent *w.r.t. $\mathcal{T}$ iff $C \sqsubseteq_\mathcal{T} D$ and $D \sqsubseteq_\mathcal{T} C$.*

2.2 Least Common Subsumer

As previously mentioned, subsumption is an important basic inference to compute the least common subsumer (lcs) w.r.t. general $\mathcal{FL}_0$-TBoxes. Now we have the definition of lcs as follows:

Definition 2.2. (*Least Common Subsumer*)
Let $\mathcal{T}$ be a general $\mathcal{FL}_0$-TBox and C, D be $\mathcal{FL}_0$-concepts. An $\mathcal{FL}_0$-concept E is the least common subsumer (lcs) of C and D w.r.t. $\mathcal{T}$ $lcs_\mathcal{T}(C, D)$ iff:

- *$C \sqsubseteq_\mathcal{T} E$ and $D \sqsubseteq_\mathcal{T} E$*
- *For each concept F such that $C \sqsubseteq_\mathcal{T} F$ and $D \sqsubseteq_\mathcal{T} F$, then $E \sqsubseteq_\mathcal{T} F$.*

Since we are dealing with general TBoxes, the occurrence of a cyclic definition should be considered. This cyclic definition can affect the common subsumer of input concepts to not be expressed as a finite concept. To avoid this problem, it is possible to limit the role-depth of the computed concept which leads us to compute the *role-depth* bounded least common subsumer (k-lcs). Suppose a concept C is of the form $\forall w_1.A_1 \sqcap \ldots \sqcap \forall w_n.A_n$. The notion of *role-depth* ($rd(C)$) of concept C is defined as follows:

$$rd(C) = max(\{|w_i| \mid \forall w_i.A_i \text{ is a conjunct in } C \text{ and } 1 \leq i \leq n\})$$

Definition 2.3. (*k-lcs*)
Let $\mathcal{T}$ be an $\mathcal{FL}_0$ TBox, $k \in \mathbb{N}$, and C, D be $\mathcal{FL}_0$-concepts. The $\mathcal{FL}_0$-concept K is the role-depth bounded least common subsumer *of C and D w.r.t. $\mathcal{T}$ (k-$lcs_{\mathcal{T}}(C, D)$) iff*

- *$rd(K) \leq k$;*
- *$C \sqsubseteq_{\mathcal{T}} K$ and $D \sqsubseteq_{\mathcal{T}} K$;*
- *For each $\mathcal{FL}_0$ concept K' with $rd(K') \leq k$ it holds that $C \sqsubseteq_{\mathcal{T}} K'$ and $D \sqsubseteq_{\mathcal{T}} K'$ implies $K \sqsubseteq_{\mathcal{T}} K'$.*

If we take the example of $\mathcal{T}_{ex}$ in Introduction, then the 0-$lcs_{\mathcal{T}_{ex}}(C, D)$ is B_1, the 1-$lcs_{\mathcal{T}_{ex}}(C, D)$ is $B_1 \sqcap \forall r.B_1$, and the 2-$lcs_{\mathcal{T}_{ex}}(C, D)$ is $B_1 \sqcap \forall r.B_1 \sqcap \forall rr.B_1$.

2.3 Normalizing $\mathcal{FL}_0$-TBoxes into PANF

In order to simplify the structural investigation for the model we use during computing the existence of lcs, we normalize our TBox to a specific structure called plane-axiom-normal-form (PANF). We will see that by considering our TBox in PANF, then we may reduce the decision problem for the existence of the lcs of concepts descriptions w.r.t. a TBox to a decision problem for the existence of the lcs of two concept names w.r.t. a TBox.

First of all, we consider to identify which basic elements that are presents in the given $\mathcal{FL}_0$-concepts or TBoxes. This set of basic elements is called as a *signature* adopted from [Pen15].

Definition 2.4. (*Signature*)
For $\mathcal{FL}_0$-concepts C and D, the set of all occurring concept names and role names is called the signature *$sig(C), sig(D)$, respectively. The definition of a signature can also be extended to a GCI or even a TBox as follows:*

- *$sig(C \sqsubseteq D) := sig(C) \cup sig(D)$*
- *$sig(\mathcal{T}) := \bigcup_{C \sqsubseteq D} sig(C \sqsubseteq D)$*

For the sake of convenience, sig can also be extended to accept multiple arguments, i.e., $sig(X_1, \ldots, X_n) := \bigcup_{i=1}^{n} sig(X_i)$, where any X_i is either an $\mathcal{FL}_0$-TBox, an $\mathcal{FL}_0$-concept, or a a GCI.

In the following for given input concepts C, D and a general $\mathcal{FL}_0$-TBox $\mathcal{T}$, we always assume that $sig(C), sig(D) \subseteq sig(\mathcal{T})$.

Now let us recall that an $\mathcal{FL}_0$-concept is in CCNF iff it is of the form

$$\forall w_1.A_1 \sqcap \ldots \sqcap \forall w_n.A_n$$

where $w_i \in N_R^*$ and $A_i \in N_C$, for all $1 \leq i \leq n$. An $\mathcal{FL}_0$-TBox $\mathcal{T}$ is in PANF iff all left- and right-hand sides of all GCIs in $\mathcal{T}$ are in CCNF and every value restriction $\forall w.A$, occurring in $\mathcal{T}$, has $|w| \leq 1$ [Pen15]. This transformation firstly requires our TBox in CCNF and introduces new concept names to abbreviate complex value restrictions. It results in PANF by applying the following rules to the GCIs in $\mathcal{T}$ exhaustively. The rules below are the additional ones from Table 2.

Rules	**CCNF**		**PANF**
NF5	$C_1 \sqcap \forall rw.A \sqcap C_2 \sqsubseteq D$	$\rightsquigarrow$	$C_1 \sqcap \forall r.B \sqcap C_2 \sqsubseteq D, \forall w.A \sqsubseteq B, B \sqsubseteq \forall w.A$
NF6	$D \sqsubseteq C_1 \forall rw.A \sqcap C_2$	$\rightsquigarrow$	$D \sqsubseteq C_1 \sqcap \forall r.B \sqcap C_2, \forall w.A \sqsubseteq B, B \sqsubseteq \forall w.A$

Table 3: PANF Normalization Rules for TBoxes

By introducing fresh concept names and new GCIs to the TBox, then equivalence between the original TBox $\mathcal{T}$ and the new TBox $\widehat{\mathcal{T}}$ w.r.t. all interpretations does not make sense here because there will be concept names $A \in sig(\widehat{\mathcal{T}}) \setminus sig(\mathcal{T})$ that are not mapped to a subset of the domain of the interpretations of $\mathcal{T}$. Therefore, it is more sensible to say that models of the original TBox can be extended to be a model of the new TBox since $sig(\mathcal{T}) \subseteq sig(\widehat{\mathcal{T}})$ such that a model of $\widehat{\mathcal{T}}$ should also be a model of $\mathcal{T}$. In order to generalize this idea, please consider the following definition about a *conservative extension* adopted from [Pen15].

Definition 2.5. (*Conservative Extension*)
Given general $\mathcal{FL}_0$ TBoxes $\mathcal{T}$ and $\widehat{\mathcal{T}}$, we say that $\widehat{\mathcal{T}}$ is a conservative extension *of $\mathcal{T}$ if*

- *$sig(\mathcal{T}) \subseteq sig(\widehat{\mathcal{T}})$,*
- *every model of $\widehat{\mathcal{T}}$ is a model of $\mathcal{T}$, and*
- *For every model $\mathcal{I}_1$ of $\mathcal{T}$ there exists a model $\mathcal{I}_2$ of $\widehat{\mathcal{T}}$ such that the extensions of the concept names and role names from $sig(\mathcal{T})$ coincide in $\mathcal{I}_1$ and $\mathcal{I}_2$, i.e.,*
  - *$A^{\mathcal{I}_1} = A^{\mathcal{I}_2}$ for all concept names $A \in sig(\mathcal{T})$, and*
  - *$r^{\mathcal{I}_1} = r^{\mathcal{I}_2}$ for all role names $r \in sig(\mathcal{T})$.*

Now let us consider the following lemma which states the list of properties for a PANF TBox taken from [Pen15].

Lemma 2.6. *Let $\mathcal{T}$ be a general $\mathcal{FL}_0$-TBox, $\widehat{\mathcal{T}}$ be the PANF TBox of $\mathcal{T}$, and C, D be $\mathcal{FL}_0$-concepts.*

1. *$\widehat{\mathcal{T}}$ is a conservative extension of $\mathcal{T}$*
2. *$C \sqsubseteq_{\mathcal{T}} D$ iff $C \sqsubseteq_{\widehat{\mathcal{T}}} D$ holds for any concept descriptions with $sig(C), sig(D) \subseteq sig(\mathcal{T})$.*
3. *If $\mathcal{T}$ is in CCNF, then $\mathcal{T}$ is transformed into $\widehat{\mathcal{T}}$ using the rules of Table 3,*

- *with a linear number of rule applications in the size of* $\mathcal{T}$, *and*
- $\widehat{\mathcal{T}}$ *is polynomial in the size of* $\mathcal{T}$.

4. *Let* $\mathcal{T}' = \mathcal{T} \cup \{A_C \sqsubseteq C, D \sqsubseteq A_D\}$ *with two fresh concept names* A_C, A_D *not occurring in* $sig(C), sig(D)$, *and* $sig(\mathcal{T})$. *It holds that*

$$C \sqsubseteq_{\mathcal{T}} D \textit{ iff } A_C \sqsubseteq_{\mathcal{T}'} A_D$$

The lemma above shows that for every general $\mathcal{FL}_0$-TBox $\mathcal{T}$, there exists a PANF TBox $\widehat{\mathcal{T}}$ obtained from $\mathcal{T}$ through the normalization rules written in Table 3. Furthermore, Claim 2 of the lemma above convinces us that the subsumption between two concepts still holds although the corresponding original TBox is already extended to a PANF TBox. Then, by introducing Claim 4 of the lemma above, it shows us that deciding subsumption between concept description w.r.t. a general TBox $\mathcal{T}$ coincides with deciding subsumption between two fresh concept names, not occurring in $\mathcal{T}$, w.r.t. the PANF TBox of $\mathcal{T}$.

Based on Lemma 2.6, we can derive more characterizations for the existence of the lcs of input concepts. First, we show that the $lcs_{\mathcal{T}}(C, D)$ is equal to the $lcs_{\widehat{\mathcal{T}}}(C, D)$, where $\widehat{\mathcal{T}}$ is the PANF of $\mathcal{T}$. Second, we show that deciding the existence of the lcs of concept descriptions w.r.t. a TBox $\mathcal{T}$ are the same as deciding the existence of the lcs of two fresh concept names w.r.t. a conservative extension of $\mathcal{T}$.

Lemma 2.7. *Let* $\mathcal{T}_1$ *be a general* $\mathcal{FL}_0$*-TBox,* $\mathcal{T}_2$ *is the PANF of* $\mathcal{T}_1$, *and* C, D *be* $\mathcal{FL}_0$*-concepts which are built only from* N_C *and* N_R. *It holds that*

1. *If* E *is the* $lcs_{\mathcal{T}_1}(C, D)$, *then* E *is the* $lcs_{\mathcal{T}_2}(C, D)$,
2. *Let* A_C *and* A_D *be fresh concept names not occurring in* N_C *and* $sig(\mathcal{T}_1)$ *such that* $\mathcal{T}_1' := \{A_C \sqsubseteq C, A_D \sqsubseteq D\} \cup \mathcal{T}_1$. *It holds that*

$$E \textit{ is the } lcs_{\mathcal{T}_1}(C, D) \textit{ iff } E \textit{ is the } lcs_{\mathcal{T}_1'}(A_C, A_D)$$

Proof:

1. Let E be the $lcs_{\mathcal{T}_1}(C, D)$ By Definition 2.2, we know that

$$C \sqsubseteq_{\mathcal{T}_1} E,\ D \sqsubseteq_{\mathcal{T}_1} E \text{ and for all } F \in cs_{\mathcal{T}_1}(C, D), \text{ we have } E \sqsubseteq_{\mathcal{T}_1} F.$$

Since $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$, by Claim 2 of Lemma 2.6, we have

$$C \sqsubseteq_{\mathcal{T}_2} E,\ D \sqsubseteq_{\mathcal{T}_2} E \text{ and for all } F \in cs_{\mathcal{T}_2}(C, D), \text{ we have } E \sqsubseteq_{\mathcal{T}_2} F.$$

which implies that E is the $lcs_{\mathcal{T}_2}(C, D)$.

2. **"⇒":** If E is the $lcs_{\mathcal{T}_1}(C, D)$, then by Definition 2.2, we know that

$$C \sqsubseteq_{\mathcal{T}_1} E,\ D \sqsubseteq_{\mathcal{T}_1} E \text{ and for all } F \in cs_{\mathcal{T}_1}(C, D), \text{ we have } E \sqsubseteq_{\mathcal{T}_1} F.$$

Since $\mathcal{T}_1'$ is a conservative extension of $\mathcal{T}_1$, by Claim 2 of Lemma 2.6, we have

$$C \sqsubseteq_{\mathcal{T}_1'} E,\ D \sqsubseteq_{\mathcal{T}_1'} E \text{ and for all } F \in cs_{\mathcal{T}_1'}(C, D), \text{ we have } E \sqsubseteq_{\mathcal{T}_1'} F.$$

Since $A_C \sqsubseteq_{\mathcal{T}_1'} C$ and $A_D \sqsubseteq_{\mathcal{T}_1'} D$, we also have

$$A_C \sqsubseteq_{\mathcal{T}_1'} E,\ A_D \sqsubseteq_{\mathcal{T}_1'} E \text{ and for all } F \in cs_{\mathcal{T}_1'}(A_C, A_D), \text{ we have } E \sqsubseteq_{\mathcal{T}_1'} F.$$

It implies that E is the $lcs_{\mathcal{T}_1'}(A_C, A_D)$.

"⇐": Now, if E is not the $lcs_{\mathcal{T}_1}(C, D)$, then one of the three properties of Definition 2.2 may not be satisfied.

- $C \not\sqsubseteq_{\mathcal{T}_1} E$.
  It means that there is a model $\mathcal{I}$ of $\mathcal{T}_1$ such that
  $$C^{\mathcal{I}} \not\subseteq E^{\mathcal{I}}. \tag{1}$$
  Since $\mathcal{T}_1'$ is a conservative extension of $\mathcal{T}_1$, there is a model $\mathcal{I}'$ of $\mathcal{T}_1'$ such that $\forall A \in N_C$, we have $A^{\mathcal{I}'} = A^{\mathcal{I}}$, $A_C^{\mathcal{I}'} = C^{\mathcal{I}'} = C^{\mathcal{I}}$, and $E^{\mathcal{I}'} = E^{\mathcal{I}}$. Together with (1) we have $A_C \not\sqsubseteq_{\mathcal{T}_1'} E$. Clearly, we have E is not the $lcs_{\mathcal{T}_1'}(A_C, A_D)$.
- $D \not\sqsubseteq_{\mathcal{T}_1} E$.
  Using the same argument as $C \not\sqsubseteq_{\mathcal{T}_1} E$.
- Assume that $C \sqsubseteq_{\mathcal{T}_1'} E$ and $D \sqsubseteq_{\mathcal{T}_1'} E$, but $\exists F \in cs_{\mathcal{T}_1}(C, D)$ such that $E \not\sqsubseteq_{\mathcal{T}_1} F$.
  Since $\mathcal{T}_1'$ is a conservative extension of $\mathcal{T}_1$, $sig(E), sig(F) \subseteq sig(\mathcal{T}_1)$, and by Claim 2 of Lemma 2.6, we have $E \not\sqsubseteq_{\mathcal{T}_1'} F$. Therefore, E is not the $lcs_{\mathcal{T}_1'}(A_C, A_D)$.

□

We have introduced another normal form for $\mathcal{FL}_0$-TBoxes, namely PANF, as well as defined additional properties to decide the existence of the lcs of input concepts. However, the normalization of TBoxes into PANF introduces new concept names. The question is whether the lcs w.r.t. PANF TBoxes can be expressed without these additional concept names.

The answer is definitely yes. It can be solved by replacing these additional concept names, occurring in the lcs-concept, with complex concepts that use these concept names on the right- and the left-side of GCIs of the PANF TBox. This is the reason why we introduce two GCIs for a new concept name B in both sides when defining PANF as presented in Table 3. Now we are ready to find the upper bound for the role-depth of the lcs of input concept names w.r.t. a PANF TBox, if the lcs exists. This also means that in the following our TBoxes are only written in PANF.

From now on, by Lemma 2.7, we can assume that the inputs for deciding the existence of the lcs and computing the lcs, if it exists, are two concept names occurring in a TBox in PANF without loss of generality. This also enables us to reduce all three research questions mentioned in Introduction to the problems with the same questions, but the TBox $\mathcal{T}$ is in PANF and the concepts C and D are concept names occurring in $\mathcal{T}$.

Before showing the characterization for the existence of least common subsumer, we need a basic means to characterize it. Those will be discussed in the next section, namely functional model of a concept w.r.t. a TBox and the graph of a functional model.

3 Functional Models and Graph Models

Here we describe two models for $\mathcal{FL}_0$, namely functional model and graph model. According to its structure, the first one has infinite number of domain elements, meanwhile the second one is the finite type of functional model. It will be shown that even though they have different structures, they are actually semantically equivalent.

3.1 Functional Model of a Concept w.r.t. a TBox

In the previous section, we have introduced the description logic $\mathcal{FL}_0$ and least common subsumer as non-standard inferences we can apply for input concepts w.r.t. a given TBox. As mentioned previously that computing subsumption is a basic inference for the existence of least common subsumer. However, to check subsumption relationship, firstly we need to have an appropriate tree-model structure that represents a given concept w.r.t. general $\mathcal{FL}_0$-TBox. Beforehand, we consider a set of value restrictions which concept C entails through general $\mathcal{FL}_0$-TBox. This will be represented in two different sets which are $L_\mathcal{T}(C)$ and $L_\mathcal{T}(C, A)$, where $A \in N_C$.

Definition 3.1. ($L_\mathcal{T}(C)$ *and* $L_\mathcal{T}(C, A)$) [Pen15]
Let $\mathcal{T}$ *be a TBox,* C *be an* $\mathcal{FL}_0$*-concept, and* $A \in N_C$*. The* set of value restrictions entailed by an $\mathcal{FL}_0$-concept C *is represented into two different sets as follows:*

1. $L_\mathcal{T}(C) := \{(w, A) \mid C \sqsubseteq_\mathcal{T} \forall w.A\}$.
2. $L_\mathcal{T}(C, A) := \{w \mid (w, A) \in L_\mathcal{T}(C)\}$.

The definition above states that $L_\mathcal{T}(C)$ is the set of pairs (w, A), where $w \in N_R^*$ and $A \in N_C$, such that $C \sqsubseteq_\mathcal{T} \forall w.A$. Meanwhile, $L_\mathcal{T}(C, A)$ merely extracts the set of words w from $L_\mathcal{T}(C)$.

In general, $L_\mathcal{T}(C)$ can be directly seen as a complete n-ary tree, where $n = |N_R|$, whose nodes are words over N_R^* and each node or element is labeled by a set of concept names. This leads us to an idea to make a tree-like representation of $L_\mathcal{T}(C)$, so called functional model, where every node $w \in N_R^*$, of the tree, has exactly one successor for every role name in N_R.

Definition 3.2. *(Functional Model of a Concept w.r.t. a TBox)* [Pen15]
An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ *is a* functional model of C w.r.t. $\mathcal{T}$ *iff it satisfies the following properties:*

1. $\Delta^\mathcal{I} = N_R^*$ *and* $\forall r \in N_R$*, then* $(u,v) \in r^\mathcal{I}$ *iff* $v = ur$*.*
   This property represents the tree-structure of this model, such that each element has exactly n-successors where $n = \mid N_R \mid$*. For each* $r \in N_R$*, each element is only mapped to one r-successor element. If* $\mathcal{I}$ *only satisfies this first property, then it is called a* functional interpretation*.*
2. *Let* $\mathcal{I}$ *be a functional interpretation.* $\forall C \sqsubseteq D \in \mathcal{T}$*, it holds that* $C^\mathcal{I} \subseteq D^\mathcal{I}$*.*
   It explicitly means that if $\mathcal{I}$ *should satisfy all GCIs in* $\mathcal{T}$*, then it is called a* functional model of TBox*.*

3. *Let $\mathcal{I}$ be a functional model of $\mathcal{T}$ and $\varepsilon \in C^{\mathcal{I}}$.*
   The tree-like model contains an initial element denoted by the empty word ε meaning that $\mathcal{I}$ satisfies C at the root of the tree. It also implies that $\mathcal{I}$ is a functional model of C w.r.t. $\mathcal{T}$.

It is easy to see that two functional interpretations $\mathcal{I}_1, \mathcal{I}_2$ over the same set of role names always have the same domain and are structurally identical w.r.t. the interpretation of role names. Due to that, we can easily apply intersection and subset operator to each interpretation $\mathcal{I}_1, \mathcal{I}_2$ that are described in the following definition.

Definition 3.3. (*Intersection, Union, and Subset over Functional Interpretations*)
For functional interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ over the same domain N_R^*, we have

1. *An intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ of functional interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ over the domain N_R^*, such that*
   (a) *For all $A \in N_C$, we have $A^{\mathcal{I}_1 \cap \mathcal{I}_2} = A^{\mathcal{I}_1} \cap A^{\mathcal{I}_2}$* [Pen15]*;*
   (b) *For any $\mathcal{FL}_0$-concept C,we have $C^{\mathcal{I}_1 \cap \mathcal{I}_2} = C^{\mathcal{I}_1} \cap C^{\mathcal{I}_2}$* [Pen15]*;*
2. *A union $\mathcal{I}_1 \cup \mathcal{I}_2$ of functional interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ over the domain N_R^*, such that*
   (a) *For all $A \in N_C$, we have $A^{\mathcal{I}_1 \cup \mathcal{I}_2} = A^{\mathcal{I}_1} \cup A^{\mathcal{I}_2}$;*
   (b) *For any $\mathcal{FL}_0$-concept C,we have $C^{\mathcal{I}_1 \cup \mathcal{I}_2} = C^{\mathcal{I}_1} \cup C^{\mathcal{I}_2}$*
3. *$\mathcal{I}_1 \subseteq \mathcal{I}_2$ iff $\forall A \in N_C$, it holds that $A^{\mathcal{I}_1} \subseteq A^{\mathcal{I}_2}$* [Pen15].

Now, given a functional interpretation $\mathcal{I}$, it is more convenient to only have an interpretation $\mathcal{I}_{C,\mathcal{T}}$, so called *least functional model*, that contains exactly the minimal information a functional interpretation $\mathcal{I}$ must contain for all value restrictions in order to be a model of TBox and $\varepsilon \in C^{\mathcal{I}}$. This type of interpretation is obtained in the following way.

Definition 3.4. (*Least Functional Model of Concept w.r.t. TBox*) [Pen15]
Let $\mathbb{I}_{C,\mathcal{T}}$ be the set of all functional models of C w.r.t. $\mathcal{T}$. The functional model

$$\mathcal{I}_{C,\mathcal{T}} = \bigcap_{\mathcal{J} \in \mathbb{I}_{C,\mathcal{T}}} \mathcal{J}$$

is the least functional model (LFM) of C w.r.t. $\mathcal{T}$ *s.t. $\mathcal{I}_{C,\mathcal{T}} \subseteq \mathcal{J}$ for all $\mathcal{J} \in \mathbb{I}_{C,\mathcal{T}}$.*

Since the lcs of two given concepts is the most specific subsumer, the LFMs of input concepts are intuitively more suitable to be a basic representation during computing the existence of lcs. It is due to the fact that the LFMs of input concepts also contain the most specific information from the input concepts w.r.t. a given TBox.

The next lemma, taken from [Pen15], consists of claims which state important properties of a functional model of TBox $\mathcal{T}$ as well as shows the correlation between the set $L_{\mathcal{T}}(C)$ and the least functional model $\mathcal{I}_{C,\mathcal{T}}$. Last, this lemma also says that for deciding subsumption between two concepts C and D, it is enough to decide the inclusion between the LFMs of C and D.

Lemma 3.5.

1. *Let $\mathcal{I}$ be a functional model of TBox $\mathcal{T}$, it holds for any $u \in N_R^*, r \in N_R$, and $A \in N_C$ that*

$$u \in (\forall r.A)^{\mathcal{I}} \text{ iff } ur \in (\forall \varepsilon.A)^{\mathcal{I}}$$

2. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be functional models of TBox $\mathcal{T}$ over the same domain N_R^*. The intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ is again a functional model of $\mathcal{T}$.*

3. *For any $\mathcal{FL}_0$-concept C and any concept name $A \in N_C$, it holds that*

$$L_{\mathcal{T}}(C, A) = A^{\mathcal{I}_{C,\mathcal{T}}}.$$

4. *Let C and D be $\mathcal{FL}_0$-concepts. It holds that*

$$C \sqsubseteq_{\mathcal{T}} D \text{ iff } \mathcal{I}_{D,\mathcal{T}} \subseteq \mathcal{I}_{C,\mathcal{T}}$$

Now we are interested to take a subtree, from a given functional interpretation $\mathcal{I}$, which is rooted in an element of $\mathcal{I}$.

Definition 3.6. (*subtree of a functional interpretation*) *Let $\mathcal{I}$ be a functional interpretation. The* subtree $(\mathcal{I}, u)$ *of $\mathcal{I}$ rooted in $u \in \Delta^{\mathcal{I}}$ is defined as follows:*

- $\Delta^{(\mathcal{I},u)} := \{w \in N_R^* \mid uw \in \Delta^{\mathcal{I}}\}$;
- $r^{(\mathcal{I},u)} := r^{\mathcal{I}}$, *for all* $r \in N_R$.
- $A^{(\mathcal{I},u)} := \{w \in N_R^* \mid uw \in A^{\mathcal{I}}\}$, *for all* $A \in N_C$;

We show that a subtree of a functional model of a TBox is also a model of the TBox.

Lemma 3.7. *Let $\mathcal{I}$ be a functional model of TBox $\mathcal{T}$. The* subtree $(\mathcal{I}, u)$ *of $\mathcal{I}$ is a model of $\mathcal{T}$.*

Proof: Let $E \sqsubseteq F \in \mathcal{T}$ be a GCI and $v \in \Delta^{(\mathcal{I},u)}$. Assume that $v \in E^{(\mathcal{I},u)}$. We have to show that $v \in F^{(\mathcal{I},u)}$. W.l.o.g. E and F are defined in the following PANF form:

$$\begin{aligned} E &= A_1 \sqcap \ldots \sqcap \forall A_k \sqcap \forall r_1.B_1 \sqcap \ldots \sqcap \forall r_\ell.B_\ell \\ F &= A'_1 \sqcap \ldots \sqcap A'_m \sqcap \forall r'_1.B'_1 \sqcap \ldots \sqcap \forall r'_n.B'_n \end{aligned}$$

where $A_i, A'_{i'}, B_j$, and $B'_{j'} \in N_C$ and $r_j, r'_{j'} \in N_R$, for all $1 \leq i \leq k$, $1 \leq i' \leq m$, $1 \leq j \leq \ell$, and $1 \leq j' \leq n$.

If $v \in E^{(\mathcal{I},u)}$, then $v \in A_i^{(\mathcal{I},u)}$ and $uv \in (\forall r_j.B_j)^{(\mathcal{I},u)}$, for all $1 \leq i \leq k$ and all $1 \leq j \leq \ell$. By Definition 3.6, we know that $uv \in A_i^{\mathcal{I}}$ and regarding the tree-structure of a functional interpretation, we have $vr_j \in B_j^{(\mathcal{I},u)}$ which implies that $uvr_j \in B_j^{\mathcal{I}}$ and $uv \in (\forall r_j.B_j)^{\mathcal{I}}$, for all $1 \le i \le k$ and all $1 \le j \le \ell$. It means that $uv \in E^{\mathcal{I}}$.

Now, since $\mathcal{I}$ is a functional model of $\mathcal{T}$, we have $uv \in F^{\mathcal{I}}$ such that $uv \in A_{i'}^{\prime\mathcal{I}}$ and $uv \in (\forall r'_{j'}.B_{j'}^{\prime\mathcal{I}})$ which implies that $uvr'j' \in B_{j'}^{\prime\mathcal{I}}$, for all $1 \le i' \le m$ and $1 \le j' \le n$. Because v and $vr'_{j'}$ are domain elements of $\Delta^{(\mathcal{I},u)}$, we have $v \in A_{i'}^{(\mathcal{I},u)}$ and $vr'_{j'} \in B_{j'}^{(\mathcal{I},u)}$ which also follows that $v \in (\forall r'_{j'}.B_{j'})^{(\mathcal{I},u)}$ by the structure of a functional interpretation, for all $1 \le i' \le m$ and $1 \le j' \le n$. Therefore, it holds that $v \in F^{(\mathcal{I},u)}$.

□

For any subtree, we are interested to only label elements up to a certain depth $\ell \in \mathbb{N}$. We call this special subtree as the ℓ-subtree $(\mathcal{I}^\ell, u)$ of $(\mathcal{I}, u)$. In general, the ℓ-subtree $(\mathcal{I}^\ell, u)$ of $(\mathcal{I}, u)$ only labels elements w of $(\mathcal{I}, u)$ where $|w| \le \ell$.

Definition 3.8. (ℓ*-subtree of* $(\mathcal{I}, u)$)
Let $(\mathcal{I}, u)$ *be a subtree of a functional interpretation* $\mathcal{I}$ *and* $\ell \in \mathbb{N}$. $(\mathcal{I}^\ell, u)$ *is the* ℓ-subtree *of* $(\mathcal{I}, u)$ *which is defined as follows:*

- $\Delta^{(\mathcal{I}^\ell,u)} := \Delta^{(\mathcal{I},u)}$;
- $r^{(\mathcal{I}^\ell,u)} := r^{(\mathcal{I},u)}$, *for all* $r \in N_R$;
- $A^{(\mathcal{I}^\ell,u)} := \{w \in A^{(\mathcal{I},u)} \mid (|w| \le \ell\}$, *for all* $A \in N_C$.

Next, the ℓ-subtree $(\mathcal{I}^\ell, u)$ of $(\mathcal{I}, u)$ can be translated into a complex concept called ℓ-*characteristic concept.*

Definition 3.9. ($\mathcal{FL}_0$*-characteristic concept*)
Let $(\mathcal{I}, u)$ *be a subtree of functional model of TBox* $\mathcal{T}$ *rooted in* $u \in \Delta^{\mathcal{I}}$. *The* k-characteristic concept $X^k(\mathcal{I}, u)$, *where* $k \in \mathbb{N}$, *is defined as follows:*

$$X^k(\mathcal{I}, u) := \bigsqcap\{\forall w.A \mid w \in \Delta^{(\mathcal{I},u)}, w \in N_R^*, |w| \le k, w \in A^{(\mathcal{I},u)}\}$$

The definition above also explicitly results in the k-characteristic concept in CCNF. In order to make this definition well-defined, one should observe that by only using finite number of concept and role names occurring in a TBox and for each $k \ge 0$, there are only, up to equivalence, finitely many k-characteristic concepts of an interpretation rooted in a domain element. Now, we have to show another property that reveals a relationship between $(\mathcal{I}^k, u)$ and the LFM of a concept $K = X^k(\mathcal{I}, u)$ w.r.t. the empty TBox.

Lemma 3.10. *Let* $(\mathcal{I}^k, u)$ *be the* k*-subtree of* $(\mathcal{I}, u)$ *and* $K = X^k(\mathcal{I}, u)$. *It holds that*

$$(\mathcal{I}^k, u) = \mathcal{I}_{K,\emptyset}.$$

Proof:
We only need to prove that for all $w \in N_R^*$ and all $A \in N_C$,

$$w \in A^{(\mathcal{I}^k,u)} \text{ iff } w \in A^{\mathcal{I}_{K,\emptyset}}$$

"⇒": Let $w \in A^{(\mathcal{I}^k,u)}$. It follows that $|uw| \leq k$ and $w \in A^{(\mathcal{I},u)}$. Let us compute $K = X^k(\mathcal{I}, u)$. By definition of k-characteristic concept and since $|w| \leq k$, we also have $\forall w.A$ as a subconcept that occurs in K, such that $w \in A^{\mathcal{I}_{K,\emptyset}}$ by definition of interpretation of an $\mathcal{FL}_0$-concept.

"⇐": Let $w \in A^{\mathcal{I}_{K,\emptyset}}$. By definition of interpretation of an $\mathcal{FL}_0$-concept, we know that there is $\forall w.A$ as a subconcept in K. Since $K = X^k(\mathcal{I}, u)$ and $rd(K) \leq k$, we know that $|w| \leq k$ which implies that $uw \in A^{(\mathcal{I}^k,u)}$.

□

Previously, we defined the LFM of a concept C w.r.t. a TBox $\mathcal{T}$ by looking for the least model w.r.t. subset relationship (Definition 3.4). Next, we show an alternative way to compute the LFM in stepwise with the inputs are only $\mathcal{T}$ and set of concept names $N_{C,\mathcal{T}}$ occurring in $\mathcal{T}$. This computation results in a functional interpretation whose all elements are also assigned by a set of concept names. We will show that the resulting functional interpretation is equal to the LFM.

Definition 3.11. *Let $\mathcal{T}$ be a TBox in PANF and $C \in N_{C,\mathcal{T}}$. We define an infinite sequence of functional interpretations*

$$\mathcal{Y}^0_{C,\mathcal{T}}, \mathcal{Y}^1_{C,\mathcal{T}}, \mathcal{Y}^2_{C,\mathcal{T}}, \ldots$$

inductively as follows

$$A^{\mathcal{Y}^0_{C,\mathcal{T}}} := \{\varepsilon \mid C \sqsubseteq_\mathcal{T} A\} \text{ for all } A \in N_C$$

and for all $n > 0$ we define

$$A^{\mathcal{Y}^n_{C,\mathcal{T}}} := \{wr \in N_R* \mid |w| = n-1, r \in N_R(\bigsqcap_{w \in B^{\mathcal{Y}^{n-1}_{C,\mathcal{T}}}, B \in N_C} B) \sqsubseteq_\mathcal{T} \forall r.A\} \text{ for all } A \in N_C$$

Please note that the $\mathcal{Y}^0_{C,\mathcal{T}}, \mathcal{Y}^1_{C,\mathcal{T}}, \mathcal{Y}^2_{C,\mathcal{T}}, \ldots$ are functional interpretation and thus the domain elements and interpretation of role names are fixed. Furthermore, we define the infinite union as a functional interpretation as follows:

$$\mathcal{Y}_{C,\mathcal{T}} := \bigcup_{n=0}^{\infty} \mathcal{Y}^n_{C,\mathcal{T}}$$

Please note that every functional interpretation $\mathcal{Y}^i_{C,\mathcal{T}}$ assigns elements w, where $|w| = i$, to each concept name. This gives us the following property of $\mathcal{Y}_{C,\mathcal{T}}$ that for each $w \in N_R$, with $|w| = n$, and $A \in N_C$, we have

$$w \in A^{\mathcal{Y}_{C,\mathcal{T}}} \text{ iff } w \in A^{\mathcal{Y}^n_{C,\mathcal{T}}}$$

Now, we show that $\mathcal{Y}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$.

Lemma 3.12. *$\mathcal{Y}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$.*

Proof:
It has to be shown that $\mathcal{Y}_{C,\mathcal{T}}$ satisfies all the GCIs in $\mathcal{T}$. Let $L \sqsubseteq R \in \mathcal{T}$ be a GCI in $\mathcal{T}$. Since $\mathcal{T}$ is in PANF, we assume that L and R have the following form:

$$L = P_1 \sqcap \dots \sqcap P_n \sqcap \forall r_1.A_1 \sqcap \dots \sqcap \forall r_m.A_m \text{ and} \tag{2}$$

$$R = P'_1 \sqcap \dots \sqcap P'_{n'} \sqcap \forall r'_1.A'_1 \sqcap \dots \sqcap \forall r'_{m'}.A'_{m'} \tag{3}$$

where

$$P_1, \dots, P_n, P'_1, \dots, P'_{n'} \text{ and } A_1, \dots, A_m, A'_1, \dots, A'_{m'}$$

are concept names and $r_0, \dots, r_m$, $r'_0, \dots, r'_{m'}$ are role names for some $n, n', m, m' \geq 0$.

Let $w \in N_R^*$. We have to show that $w \in L^{\mathcal{Y}_{C,\mathcal{T}}}$ implies $w \in R^{\mathcal{Y}_{C,\mathcal{T}}}$. Let $n = |w|$ and

$$M := \bigsqcap_{w \in B^{\mathcal{Y}^n_{C,\mathcal{T}}}, B \in N_C} B \tag{4}$$

By induction on n we show that

$$M \sqsubseteq_{\mathcal{T}} A \text{ for some } A \in N_C \text{ implies that } A \textit{ is a conjunct in } M.$$

$n = 0:$ It follows that $w = \varepsilon$. By definition we have $\varepsilon \in B^{\mathcal{Y}^0_{C,\mathcal{T}}}$ iff $C \sqsubseteq_{\mathcal{T}} B$. Therefore, (4) yields $C \sqsubseteq_{\mathcal{T}} M$. Consequently, $M \sqsubseteq_{\mathcal{T}} A$ for a concept name $A \in N_C$ implies also $C \sqsubseteq_{\mathcal{T}} A$. It follows that $\varepsilon \in A^{\mathcal{Y}^0_{C,\mathcal{T}}}$ and A is a conjunct in M.

$n \to n+1:$ Assume $w = w'r$ and with $|w'| = n$. Let

$$Q' := \bigsqcap_{w' \in B^{\mathcal{Y}^n_{C,\mathcal{T}}}, B \in N_C} B \text{ and } Q := \bigsqcap_{w \in B^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}, B \in N_C} B$$

By definition of $\mathcal{Y}^{n+1}_{C,\mathcal{T}}$ and $w = w'r$ with $|w'| = n$ it holds that $Q' \sqsubseteq_{\mathcal{T}} \forall r.Q$. Obviously, $Q' \sqsubseteq_{\mathcal{T}} \forall r.Q$ and $Q \sqsubseteq_{\mathcal{T}} A$ implies $Q' \sqsubseteq_{\mathcal{T}} \forall r.A$. By definition of $\mathcal{Y}^{n+1}_{C,\mathcal{T}}$ it follows that $w = w'r \in A^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}$ and therefore A is a conjunct in Q.

Suppose $L \sqsubseteq R \in \mathcal{T}$ and $w \in L^{\mathcal{Y}_{C,\mathcal{T}}}$. By assumption on the form of L (2) we have

$$w \in (P_1 \sqcap \dots \sqcap P_n)^{\mathcal{Y}_{C,\mathcal{T}}} \text{ and } w \in (\forall r_1.A_1 \sqcap \dots \sqcap \forall r_m.A_m)^{\mathcal{Y}_{C,\mathcal{T}}}.$$

Since $|w| = n$ the definition of $\mathcal{Y}_{C,\mathcal{T}}$ yields

$$w \in (P_1 \sqcap \cdots \sqcap P_n)^{\mathcal{Y}^n_{C,\mathcal{T}}} \text{ and } wr_1 \in (A_1)^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}, \ldots, wr_m \in (A_m)^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}.$$

It follows that the concept names $P_1, \ldots, P_n$ are conjuncts in M (see (4)). Obviously, it holds that

$$M \sqsubseteq_{\mathcal{T}} P_1 \sqcap \cdots \sqcap P_n.$$

Since $|w| = n$ and $wr_1 \in (A_1)^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}, \ldots, wr_m \in (A_m)^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}$, the definition of $\mathcal{Y}^{n+1}_{C,\mathcal{T}}$ and (4) implies

$$M \sqsubseteq_{\mathcal{T}} \forall r_1.A_1, \ldots M \sqsubseteq_{\mathcal{T}} \forall r_m.A_m.$$

Thus, we have $M \sqsubseteq_{\mathcal{T}} L$. Since $L \sqsubseteq R \in \mathcal{T}$ we have also

$$M \sqsubseteq_{\mathcal{T}} \forall r'_1.A'_1, \ldots, M \sqsubseteq_{\mathcal{T}} \forall r'_{m'}.A'_{m'}$$

for all value restrictions in R (see (3)). Consequently, by definition of $\mathcal{Y}^{n+1}_{C,\mathcal{T}}$ it follows that

$$wr'_1 \in (A'_1)^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}, \ldots, wr'_{m'} \in (A'_{m'})^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}. \tag{5}$$

And likewise we have that $M \sqsubseteq_{\mathcal{T}} L$ and $L \sqsubseteq R \in \mathcal{T}$ implies

$$M \sqsubseteq_{\mathcal{T}} P'_1, \ldots, M \sqsubseteq_{\mathcal{T}} P'_{n'}$$

for all concept names on top level of R. Consequently, the names $P'_1, \ldots, P'_{n'}$ are conjuncts in M as shown above. Thus, we have

$$w \in (P'_1 \sqcap \ldots \sqcap P'_{n'})^{\mathcal{Y}^n_{C,\mathcal{T}}}.$$

By construction of $\mathcal{Y}_{C,\mathcal{T}}$ and (5) we obtain that

$$w \in (P'_1 \sqcap \ldots \sqcap P'_{n'})^{\mathcal{Y}_{C,\mathcal{T}}} \text{ and } wr'_1 \in (A'_1)^{\mathcal{Y}_{C,\mathcal{T}}}, \ldots, wr'_{m'} \in (A_{m'})^{\mathcal{Y}_{C,\mathcal{T}}}.$$

It is implied that $w \in R^{\mathcal{Y}_{C,\mathcal{T}}}$.

□

Finally, we can show that $\mathcal{Y}_{C,\mathcal{T}}$ is actually the LFM $\mathcal{I}_{C,\mathcal{T}}$.

Lemma 3.13. $\mathcal{Y}_{C,\mathcal{T}} = \mathcal{I}_{C,\mathcal{T}}$

Proof:
Lemma 3.12 implies that $\mathcal{Y}_{C,\mathcal{T}}$ is a functional model of C w.r.t. $\mathcal{T}$. It is implied that $\mathcal{I}_{C,\mathcal{T}} \subseteq \mathcal{Y}_{C,\mathcal{T}}$. It remains to be shown that

$$w \in A^{\mathcal{Y}_{C,\mathcal{T}}} \text{ implies } w \in A^{\mathcal{I}_{C,\mathcal{T}}}$$

for all $w \in N_R^*$ and all $A \in N_C$. The proof is by induction on $n = |w|$.

$n = 0$: In this case $w = \varepsilon$. By definition of $\mathcal{Y}_{C,\mathcal{T}}$ we have that $\varepsilon \in A^{\mathcal{Y}_{C,\mathcal{T}}}$ implies $\varepsilon \in A^{\mathcal{Y}^0_{C,\mathcal{T}}}$ which implies $C \sqsubseteq_\mathcal{T} A$. Since $\mathcal{I}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$ and $\varepsilon \in C^{\mathcal{I}_{C,\mathcal{T}}}$ it follows that $\varepsilon \in A^{\mathcal{I}_{C,\mathcal{T}}}$.

$n \to n+1$ Assume $w = w'r$ with $|w'| = n$ and $w'r \in A^{\mathcal{Y}_{C,\mathcal{T}}}$. The definition of $\mathcal{Y}_{C,\mathcal{T}}$ yields $w'r \in A^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}$. Let $Q := \bigsqcap_{w' \in B^{\mathcal{Y}^n_{C,\mathcal{T}}}, B \in N_C}$. Obviously, $w' \in Q^{\mathcal{Y}_{C,\mathcal{T}}}$. The definition of $\mathcal{Y}^{n+1}_{C,\mathcal{T}}$, $|w'r| = n+1$ and $w'r \in A^{\mathcal{Y}^{n+1}_{C,\mathcal{T}}}$ imply that $Q \sqsubseteq_\mathcal{T} \forall r.A$. The induction hypothesis applied to w' and $w' \in Q^{\mathcal{Y}_{C,\mathcal{T}}}$ yields $w'Q^{\mathcal{I}_{C,\mathcal{T}}}$. Since $\mathcal{I}_{C,\mathcal{T}}$ is a model of $\mathcal{T}$ we get $w'r \in A^{\mathcal{I}_{C,\mathcal{T}}}$, which finishes the proof.

□

As a consequence of the lemma above, we can show the following property of $\mathcal{I}_{C,\mathcal{T}}$.

Lemma 3.14. *Let $\mathcal{I}_{C,\mathcal{T}}$ be the LFM of a concept C w.r.t. a TBox $\mathcal{T}$, $w \in N_R^*$, $r \in N_R$, and $A \in N_C$. It holds that*

$$wr \in A^{\mathcal{I}_{C,\mathcal{T}}} \text{ iff } (\bigsqcap_{w \in B^{\mathcal{I}_{C,\mathcal{T}}}, B \in N_C} B) \sqsubseteq_\mathcal{T} \forall r.A$$

The lemma above shows us that for each r-successor wr of $w \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$, where $w \in N_R^*$, the conjunction of all concept names labeling w entails $\forall r.A$, where $A \in N_C$ and $wr \in A^{\mathcal{I}_{C,\mathcal{T}}}$. Next, we see the finite type of LFMs described in the next subsection. Lemma 3.14 also states that by assuming our input is a TBox $\mathcal{T}$ in PANF and a concept name C occurring in $\mathcal{T}$, the stepwise construction of $\mathcal{Y}_{C,\mathcal{T}}$ implicitly only gives a local information for each element w of $\mathcal{Y}_{C,\mathcal{T}}$, to the r-successor of it, for all $r \in N_R$. In the other words, by using the entailment property described in Lemma 3.14, all elements w of $\mathcal{Y}_{C,\mathcal{T}}$ only know a concept name that labels the r-successor of w, for all $r \in N_R$. It also means that all elements w can not access the information about the label of all elements which are the descendants of the r-successor of w, for all $r \in N_R$.

3.2 Graph of Functional Model

Now, given the LFM $\mathcal{I}_{C,\mathcal{T}}$ of a concept C w.r.t. a TBox $\mathcal{T}$, then we can view $\mathcal{I}$ as a function that assigns a set of concept names $N_{C,\mathcal{T}}$ occurring in $\mathcal{T}$ to each element $w \in \Delta^\mathcal{I}$. Formally, for all $w \in \Delta^\mathcal{I} = N_R^*$, where $\mathcal{I}$ is the LFM of a concept C w.r.t. a TBox $\mathcal{T}$, we have a function

$$\mathcal{I}_{C,\mathcal{T}}(w) = \{A \in N_{C,\mathcal{T}} \mid w \in A^{\mathcal{I}_{C,\mathcal{T}}}\}.$$

However, the LFMs still has infinite number of domain elements. Therefore, it brings us to an idea to have the LFMs that only has a finite number of domain elements and we change the form of this infinite model in a cyclic fashion, so-called *graph model*.

To start running this idea, we should consider that an LFM may have infinitely many domain elements with the same non-empty label. Hence, we need to recognize those recurring labels and block the corresponding nodes to enable us to not create multiple same subtrees afterwards. Firstly we need to define an equivalence relation $\sim_{\mathcal{I}_{C,\mathcal{T}}}$ on N_R^* based on $\mathcal{I}_{C,\mathcal{T}}$.

Definition 3.15. *Given the LFM $\mathcal{I}_{C,\mathcal{T}}$. An* equivalence relation $\sim_{\mathcal{I}_{C,\mathcal{T}}}$ *on* $\Delta^{\mathcal{I}_{C,\mathcal{T}}} = N_R^*$ *is defined as follows:*

$$\text{For all } u, v \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}\text{: } u \sim_{\mathcal{I}_{C,\mathcal{T}}} v \text{ iff } \mathcal{I}_{C,\mathcal{T}}(u) = \mathcal{I}_{C,\mathcal{T}}(v)$$

Note that for any $w \in N_R^*$, it holds that $\mathcal{I}_{C,\mathcal{T}}(w) \subseteq N_{C,\mathcal{T}}$. Using Lemma 3.14 it can be shown that the equivalence relation $\sim_{\mathcal{I}_{C,\mathcal{T}}}$ is preserved when we go one step further down in the tree.

Lemma 3.16. *For all $r \in N_R$, it holds that*

$$\text{If } u \sim_{\mathcal{I}_{C,\mathcal{T}}} v\text{, then } ur \sim_{\mathcal{I}_{C,\mathcal{T}}} vr$$

Proof:
We have $ur \in A^{\mathcal{I}_{C,\mathcal{T}}}$

iff $(\bigsqcap_{u \in B^{\mathcal{I}_{C,\mathcal{T}}}, B \in N_C} B) \sqsubseteq_{\mathcal{T}} \forall r.A$ (by Lemma 3.14)

iff $(\bigsqcap_{v \in B^{\mathcal{I}_{C,\mathcal{T}}}, B \in N_C} B) \sqsubseteq_{\mathcal{T}} \forall r.A$ (with $u \sim_{\mathcal{I}_{C,\mathcal{T}}} v$)

iff $vr \in A^{\mathcal{I}_{C,\mathcal{T}}}$ (by Lemma 3.14)

It follows that $ur \sim_{\mathcal{I}_{C,\mathcal{T}}} vr$.

□

Thus, we can construct an *equivalence class of a word*.

Definition 3.17. (*Equivalence Class of Words*) *Let $\mathcal{I}_{C,\mathcal{T}}$ be the LFM of a concept w.r.t. a TBox and $u \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$. The* equivalence class of words *u is defined as follows:*

$$[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} := \{v \in \Delta^{\mathcal{I}_{C,\mathcal{T}}} \mid u \sim_{\mathcal{I}_{C,\mathcal{T}}} v\}$$

It is easy to see that for each LFM $\mathcal{I}_{C,\mathcal{T}}$, the equivalence relation $\sim_{\mathcal{I}_{C,\mathcal{T}}}$ leads to a partition of the domain of $\mathcal{I}_{C,\mathcal{T}}$ into finitely many equivalence classes, i.e., $\sim_{\mathcal{I}_{C,\mathcal{T}}}$ has finite index. It is also due to that there are only finitely many sets of value restrictions which are assigned to each domain element of $\mathcal{I}_{C,\mathcal{T}}$.

Now we are ready to build the graph model of the LFM of a concept w.r.t. a TBox. Let us take the LFM $\mathcal{I}_{C,\mathcal{T}}$ of concept C w.r.t. TBox $\mathcal{T}$ as an input, we construct the *graph model* $\mathcal{J}_{C,\mathcal{T}} = (\Delta^{\mathcal{J}_{C,\mathcal{T}}}, \cdot_{\mathcal{J}})$ of $\mathcal{T}$ whose the domain elements are the equivalence classes of words.

Definition 3.18. (*Graph Model of a Concept w.r.t. a TBox*)
Let $\mathcal{I}_{C,\mathcal{T}}$ be the LFM of a concept C w.r.t. a TBox $\mathcal{T}$ and $\sim_{\mathcal{I}_{C,\mathcal{T}}}$ be an equivalence relation on $\Delta^{\mathcal{I}_{C,\mathcal{T}}}$. The corresponding graph model $\mathcal{J}_{C,\mathcal{T}}$ *of C w.r.t. $\mathcal{T}$ is defined as follows:*

- $\Delta^{\mathcal{J}_{C,\mathcal{T}}} := \{[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} u \in N_R^*\}$;
- $r^{\mathcal{J}_{C,\mathcal{T}}} := \{([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \mid \exists u' \in [u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$, *and* $\exists v' \in [v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ *with* $(u', v') \in r^{\mathcal{I}_{C,\mathcal{T}}}\}$, *for all* $r \in N_R$;
- $A^{\mathcal{J}_{C,\mathcal{T}}} := \{[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \mid u \in A^{\mathcal{I}_{C,\mathcal{T}}}\}$ *for all* $A \in N_C$.

Please note that for the context of graph models, $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ is already viewed as an element of model, no longer as a class of words. Obviously, $\mathcal{J}_{C,\mathcal{T}}$ is finite since there are only finitely many equivalence class. We show some properties of $\mathcal{J}_{C,\mathcal{T}}$ that directly follow from the definition.

Lemma 3.19. *Let* $\mathcal{I}_{C,\mathcal{T}}$ *and* $\mathcal{J}_{C,\mathcal{T}}$ *be defined as above.*

1. *For each* $w \in N_R^*$ *and* $r \in N_R$ *it holds that* $([w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [wr]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in r^{\mathcal{J}_{C,\mathcal{T}}}$.
2. *For each* $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$ *and each* $r \in N_R$ *there exists exactly one element* $[v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$ *such that* $([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in r^{\mathcal{J}_{C,\mathcal{T}}}$.
3. *Let* $w \in N_R^*$
   (a) $w' \in A^{\mathcal{I}_{C,\mathcal{T}}}$ *iff* $[w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in A^{\mathcal{J}_{C,\mathcal{T}}}$ *for all* $A \in N_C$ *and all* $w' \in [w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$
   (b) $w' \in (\forall r.A)^{\mathcal{I}_{C,\mathcal{T}}}$ *iff* $[w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in (\forall r.A)^{\mathcal{J}_{C,\mathcal{T}}}$ *for all* $A \in N_C$, $r \in N_R$ *and all* $w' \in [w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$.

Proof:

1. Let $w \in N_R^*$ and $r \in N_R$. Since $\mathcal{I}_{C,\mathcal{T}}$ is a functional interpretation we have $(w, wr) \in r^{\mathcal{I}_{C,\mathcal{T}}}$. The definition of $\mathcal{J}_{C,\mathcal{T}}$ implies that $([w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [wr]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in r^{\mathcal{J}_{C,\mathcal{T}}}$.
2. Let $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$. Assume to the contrary that there are two different elements $[v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$, $[v']^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$ such that
$$[v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \neq [v']^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \tag{6}$$
and $([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in r^{\mathcal{J}_{C,\mathcal{T}}}$ and $([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [v']^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in r^{\mathcal{J}_{C,\mathcal{T}}}$. By definition of $\mathcal{J}_{C,\mathcal{T}}$ there are $x, y \in [u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ such that $xr \in [v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ and $yr \in [v']^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$. Thus, we have $x \sim_{\mathcal{I}_{C,\mathcal{T}}} y$ which implies $xr \sim_{\mathcal{I}_{C,\mathcal{T}}} yr$ using Lemma 3.16. Obviously, $xr \sim_{\mathcal{I}_{C,\mathcal{T}}} yr$ is a contradiction to (6).
3. It follows directly using the definition of $\mathcal{J}_{C,\mathcal{T}}$.

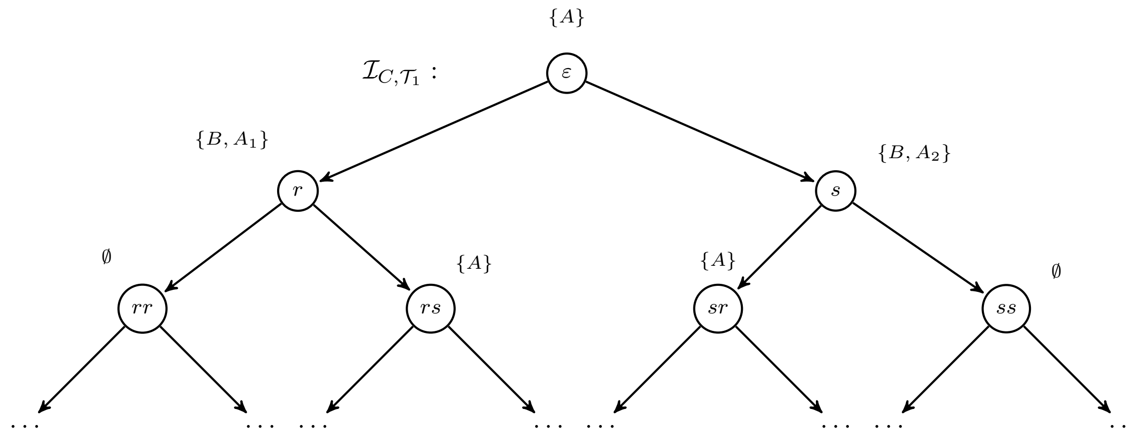
□

The lemma above states that for all $w \in N_R^*$, all $r \in N_R$, and the r-successor wr of w, their equivalence classes $[w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ and $[wr]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ are also a pair of elements connected via role r in $\mathcal{J}_{C,\mathcal{T}}$. In addition, every element $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ of $\mathcal{J}_{C,\mathcal{T}}$ only has exactly one r-successor, which is $[v]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$, for all $r \in N_R$. It indicates that the graph model is also functional. Now, let us consider the following example of computing a graph model.

Example 3.20.
$\mathcal{T}_1$:

$$\begin{array}{rcl} \{A & \sqsubseteq & \forall r.B \sqcap \forall r.A_1 \\ A & \sqsubseteq & \forall s.B \sqcap \forall s.A_2 \\ A_1 & \sqsubseteq & \forall s.A \\ A_2 & \sqsubseteq & \forall r.A\} \end{array}$$

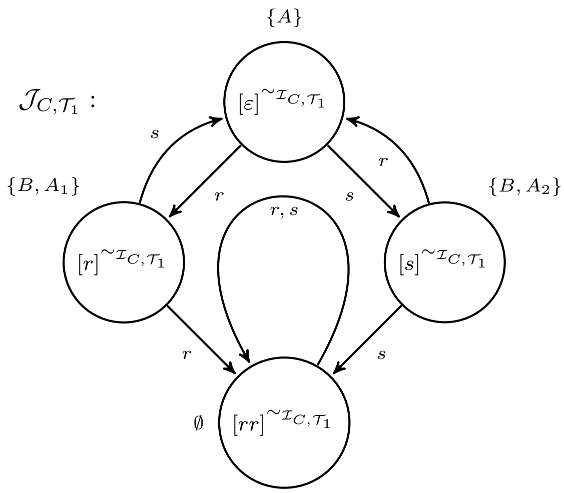
Let $C = A$. The LFM $\mathcal{I}_{C,\mathcal{T}_1}$ is computed as follows:



From the structure of $\mathcal{I}_{C,\mathcal{T}_1}$ described above, we obtain equivalence classes of words as follows:

- $[\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}_1}}} = \{\varepsilon, rs, sr, \ldots\}$
  where for all $w \in [\varepsilon]$, we have $\mathcal{I}_{C,\mathcal{T}_1}(w) = \{A\}$
- $[r]^{\sim_{\mathcal{I}_{C,\mathcal{T}_1}}} = \{r, rsr, srr, \ldots\}$
  where for all $w \in [r]$, we have $\mathcal{I}_{C,\mathcal{T}_1}(w) = \{B, A_1\}$
- $[s]^{\sim_{\mathcal{I}_{C,\mathcal{T}_1}}} = \{s, rss, srs, \ldots\}$
  where for all $w \in [s]$, we have $\mathcal{I}_{C,\mathcal{T}_1}(w) = \{B, A_2\}$
- $[rr]^{\sim_{\mathcal{I}_{C,\mathcal{T}_1}}} = \{rr, ss, rrr, rrs, ssr, sss, \ldots\}$
  where for all $w \in [rr]$, we have $\mathcal{I}_{C,\mathcal{T}_1}(w) = \emptyset$

The corresponding graph model $\mathcal{J}_{C,\mathcal{T}_1}$ of C w.r.t. $\mathcal{T}_1$ is computed as follows:



Definition 3.18 still provides an input for computing the graph model in the form of the LFM of a concept w.r.t. a TBox, which is in infinite type. In order to ensure that any algorithm computing a graph model terminates and only deals with finite inputs, now we have to consider that the inputs are only a TBox $\mathcal{T}$ in PANF and a concept name $C \in N_{C,\mathcal{T}}$. Recall that for building equivalence classes, we have to enumerate all possible sets of concept names from $N_{C,\mathcal{T}}$ for each element of $w \in \Delta^{\mathcal{I}}$, where $\mathcal{I}_{C,\mathcal{T}}$ and the number of possible sets of concept name to enumerate is finite and bounded by $2^{N_{C,\mathcal{T}}}$. We call every $X \in 2^{N_{C,\mathcal{T}}}$ as a *label set* because it is a candidate of set of concept names that will label an element in $\mathcal{I}_{C,\mathcal{T}}$. Now, in order to have a computable algorithm for a graph model, let us define another finite interpretation based on subsets of $N_{C,\mathcal{T}}$.

Definition 3.21. *Given C and $\mathcal{T}$ as above we define an interpretation $\widehat{\mathcal{J}}_{C,\mathcal{T}}$ as follows:*

- $\Delta^{\widehat{\mathcal{J}}_{C,\mathcal{T}}} := 2^{N_{C,\mathcal{T}}}$;
- $A^{\widehat{\mathcal{J}}_{C,\mathcal{T}}} := \{X \mid A \in X\}$ *for all* $A \in N_C$;
- $r^{\widehat{\mathcal{J}}_{C,\mathcal{T}}} := \{(X, Y) \mid Y = \{B \in N_{C,\mathcal{T}} \mid \left(\bigsqcap X\right) \sqsubseteq_{\mathcal{T}} \forall r.B\}\}$ *for all* $r \in N_R$.

Note that for a concept name $B \in N_C$ and $X \in 2^{N_{C,\mathcal{T}}}$ we have that $\left(\bigsqcap X\right) \sqsubseteq_{\mathcal{T}} \forall r.B$ implies $B \in N_{C,\mathcal{T}}$. We may call for each $(X, Y) \in r^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$, Y is *reachable* from X iff $Y = \{B \in N_{C,\mathcal{T}} \mid \left(\bigsqcap X\right) \sqsubseteq_{\mathcal{T}} \forall r.B\}$. The interpretation $\widehat{\mathcal{J}}_{C,\mathcal{T}}$ is computable because for a given $X \in 2^{N_{C,\mathcal{T}}}$ the set $\{B \in N_{C,\mathcal{T}} \mid \left(\bigsqcap X\right) \sqsubseteq_{\mathcal{T}} \forall r.B\}$ is computable using a decision procedure for subsumption in $\mathcal{FL}_0$.

Next, we define a total function $\mu_{C,\mathcal{T}} : \Delta^{\mathcal{J}_{C,\mathcal{T}}} \to \Delta^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$ as follows: For each $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$, we define

$$\mu_{C,\mathcal{T}}([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) := \{B \in N_C \mid [u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in B^{\mathcal{J}_{C,\mathcal{T}}}\}.$$

It follows that

$$\mu_{C,\mathcal{T}}([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in 2^{N_{C,\mathcal{T}}} \text{ and } \mu_{C,\mathcal{T}}([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) = \mathcal{I}_{C,\mathcal{T}}(u).$$

The *image set* of $\mu_{C,\mathcal{T}}$, denoted by $\mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})$, is given by

$$\{X \in \Delta^{\widehat{\mathcal{J}}_{C,\mathcal{T}}} \mid \mu_{C,\mathcal{T}}(\sigma) = X \text{ for some } \sigma \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}\}.$$

Note that in general $\mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}}) \subsetneq \Delta^{\widehat{\mathcal{J}}_{C,\mathcal{T}}} = 2^{N_{C,\mathcal{T}}}$. It might be the case that the image set of $\mu_{C,\mathcal{T}}$ and $2^{N_{C,\mathcal{T}}}$ are not equal.

Example 3.22. *Consider the TBox $\mathcal{T} = \{A \sqsubseteq \forall r.B, B \sqsubseteq A\}$ and let $C = A$ and $N_R = \{r\}$. There are only two equivalence classes of $\sim_{\mathcal{I}_{C,\mathcal{T}}}$, namely*

$$[\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} = \{\varepsilon\} \text{ and } [r]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} = \{r, rr, rrr, rrrr, \ldots\}.$$

And we have

$$\mu_{C,\mathcal{T}}([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) = \{A\} \text{ and } \mu_{C,\mathcal{T}}([r]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) = \{A, B\}.$$

Furthermore,

$$\mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}}) = \{\{A\}, \{A, B\}\} \text{ and } 2^{N_{C,\mathcal{T}}} = \{\emptyset, \{A\}, \{B\}, \{A, B\}\}.$$

Using Lemma 3.19, we will see the following properties relating $\mathcal{J}_{C,\mathcal{T}}$, $\widehat{\mathcal{J}}_{C,\mathcal{T}}$, and $\mu_{C,\mathcal{T}}$.

Lemma 3.23. *Let $\mathcal{J}_{C,\mathcal{T}}$, $\widehat{\mathcal{J}}_{C,\mathcal{T}}$ and $\mu_{C,\mathcal{T}}$ be as above.*

1. *If $(\sigma_1, \sigma_2) \in r^{\mathcal{J}_{C,\mathcal{T}}}$, then $(\mu_{C,\mathcal{T}}(\sigma_1), \mu_{C,\mathcal{T}}(\sigma_2)) \in r^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$ for all $r \in N_R$.*
2. *If $(X, Y) \in r^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$ and $\mu_{C,\mathcal{T}}([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) = X$ for some $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$, then*

$$\mu_{C,\mathcal{T}}([ur]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) = Y \text{ for all } r \in N_R.$$

Proof:

1. Let $(\sigma_1, \sigma_2) \in r^{\mathcal{J}_{C,\mathcal{T}}}$ for some $r \in N_R$. According to Point 1 of Lemma 3.19 and Point 2 of Lemma 3.19 there exists $w \in N_R^*$ such that

$$\sigma_1 = [w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$$
$$\sigma_2 = [wr]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}.$$

It follows that $(w, wr) \in r^{\mathcal{I}_{C,\mathcal{T}}}$ for the LFM $\mathcal{I}_{C,\mathcal{T}}$ and

$$\mu_{C,\mathcal{T}}(\sigma_1) = \{B \in N_C \mid w \in B^{\mathcal{I}_{C,\mathcal{T}}}\}$$
$$\mu_{C,\mathcal{T}}(\sigma_2) = \{B \in N_C \mid wr \in B^{\mathcal{I}_{C,\mathcal{T}}}\}.$$

It now follows from Lemma 3.14, that

$$\mu_{C,\mathcal{T}}(\sigma_2) = \{B \in N_{C,\mathcal{T}} \mid \left(\bigsqcap \mu_{C,\mathcal{T}}(\sigma_1)\right) \sqsubseteq_{\mathcal{T}} \forall r.B\},$$

which yields $(\mu_{C,\mathcal{T}}(\sigma_1), \mu_{C,\mathcal{T}}(\sigma_2)) \in r^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$.

2. Let $(X, Y) \in r^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$ and $\mu_{C,\mathcal{T}}([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) = X$ for some $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$. It follows that

$$X = \{B \in N_C \mid u \in B^{\mathcal{I}_{C,\mathcal{T}}}\}, \tag{7}$$

because $u \in B^{\mathcal{I}_{C,\mathcal{T}}}$ iff $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in B^{\mathcal{J}_{C,\mathcal{T}}}$ for all concept names $B \in N_C$. Since $(X, Y) \in r^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$ we have

$$Y = \{B \in N_{C,\mathcal{T}} \mid \left(\bigsqcap X\right) \sqsubseteq_{\mathcal{T}} \forall r.B\}. \tag{8}$$

Point 1 of Lemma 3.19 implies that $([u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [ur]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in r^{\mathcal{J}_{C,\mathcal{T}}}$. We have $B \in \mu_{C,\mathcal{T}}([ur]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}})$ for some $B \in N_C$

iff $[ur]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in B^{\mathcal{J}_{C,\mathcal{T}}}$

iff $ur \in B^{\mathcal{I}_{C,\mathcal{T}}}$ (by definition of $\mathcal{J}_{C,\mathcal{T}}$)

iff $\left(\bigsqcap X\right) \sqsubseteq_{\mathcal{T}} \forall r.B$ (by Lemma 3.14 and (7))

iff $B \in N_{C,\mathcal{T}}$ and $B \in Y$ (by (8)).

Thus, $\mu_{C,\mathcal{T}}([ur]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) = Y$ as required.

□

The lemma above shows us that for all pairs of elements (σ_1, σ_2) connected via a role $r \in N_R$ in $\mathcal{J}_{C,\mathcal{T}}$, a total function $\mu_{C,\mathcal{T}}$ preserves the resulting image sets X and Y of σ_1 and σ_2, respectively, as a pair of element connected via r in $\widehat{\mathcal{J}}_{C,\mathcal{T}}$. We can now give characterization of the image set $\mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})$ that allows us to effectively compute it.

Lemma 3.24. *Let $\mathcal{I}_{C,\mathcal{T}}$, $\mathcal{J}_{C,\mathcal{T}}$, $\widehat{\mathcal{J}}_{C,\mathcal{T}}$ and $\mu_{C,\mathcal{T}}$ be as above and let*

$$X_\varepsilon = \{B \in N_{C,\mathcal{T}} \mid C \sqsubseteq_{\mathcal{T}} B\}.$$

1. *It holds that $\mu_{C,\mathcal{T}}([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) = X_\varepsilon$.*

2. *Let $Y \in 2^{N_{C,\mathcal{T}}}$. It holds that $Y \in \mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})$ iff there exist $n \geq 0$, sets $X_0, \ldots X_n \in 2^{N_{C,\mathcal{T}}}$ and role names $r_1, \ldots, r_n$ such that the following holds*

   - *$X_0 = X_\varepsilon$ and $X_n = Y$ and*

- $(X_i, X_{i+1}) \in r_{i+1}^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$ *for all* $i = 0, \dots, n-1$.

Proof:

1. Let $B \in N_C$. We have $B \in \mu_{C,\mathcal{T}}([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}})$

   iff $[\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in B^{\mathcal{J}_{C,\mathcal{T}}}$

   iff $\varepsilon \in B^{\mathcal{I}_{C,\mathcal{T}}}$

   iff $C \sqsubseteq_{\mathcal{T}} \forall\varepsilon.B$ (property of the LFM)

   iff $B \in N_{C,\mathcal{T}}$ and $B \in X_\varepsilon$ (as defined above).

2. "⇒:" Let $Y \in 2^{N_{C,\mathcal{T}}}$ and assume $Y \in \mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})$. The definition of $\mu_{C,\mathcal{T}}$ ensures that there is an element $w \in N_R^*$ such that

   $$Y = \mu_{C,\mathcal{T}}([w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}).$$

   Obviously, since $w \in N_R^*$ there exists an $n \geq 0$ and role names $r_1, \dots, r_n$ such that

   $$w = r_1 \cdots r_n$$

   In case $n = 0$ we assume $w = \varepsilon$. The definition of $\mathcal{I}_{C,\mathcal{T}}$ implies that

   $$(\varepsilon, r_1) \in {r_1}^{\mathcal{I}_{C,\mathcal{T}}}, \dots, (r_1 \cdots r_{n-1}, w) \in {r_n}^{\mathcal{I}_{C,\mathcal{T}}}.$$

   Point 1 of Lemma 3.19 yields

   $$([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [r_1]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in {r_1}^{\mathcal{J}_{C,\mathcal{T}}}, \dots, ([r_1 \cdots r_{n-1}]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \in {r_n}^{\mathcal{J}_{C,\mathcal{T}}}.$$

   Let

   $$\begin{aligned} X_0 &:= \mu_{C,\mathcal{T}}([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \\ X_1 &:= \mu_{C,\mathcal{T}}([r_1]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \\ &\vdots \\ X_{n-1} &:= \mu_{C,\mathcal{T}}([r_1 \cdots r_{n-1}]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \\ X_n &:= \mu_{C,\mathcal{T}}([w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}). \end{aligned}$$

   The first claim implies that $X_0 = X_\varepsilon$ and we have $X_n = Y$. Using the first claim of Lemma 3.23 it follows that $(X_i, X_{i+1}) \in {r_{i+1}}^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$ for all $i = 0, \dots, n-1$. Thus, the sequence $X_0, \dots, X_n$ is as required.

"⇐:" Let $Y \in 2^{N_{C,\mathcal{T}}}$, $n \geq 0$ such that there are sequences $X_0, \ldots, X_n \in 2^{N_{C,\mathcal{T}}}$ and $r_1, \ldots, r_n \in N_R$ such that

- $X_0 = X_\varepsilon$ and $X_n = Y$ and
- $(X_i, X_{i+1}) \in r_{i+1}^{\widehat{\mathcal{J}}_{C,\mathcal{T}}}$ for all $i = 0, \ldots, n-1$.

We have to show that $Y \in \mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})$. According to the first claim we have

$$X_0 = \mu_{C,\mathcal{T}}([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}).$$

The second claim of Lemma 3.23 implies that

$$\begin{aligned}
X_0 &= \mu_{C,\mathcal{T}}([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \\
X_1 &= \mu_{C,\mathcal{T}}([r_1]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \\
&\vdots \\
X_{n-1} &= \mu_{C,\mathcal{T}}([r_1 \cdots r_{n-1}]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}) \\
X_n &= \mu_{C,\mathcal{T}}([w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}).
\end{aligned}$$

Consequently, $X_n = Y \in \mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})$.

□

The Lemma above shows us a computable way to have a graph model. First, we determine which set $X \in 2^{N_{C,\mathcal{T}}}$ that labels the element ε of $\mathcal{J}_{C,\mathcal{T}}$. The label set of the element is denoted by X_ε. Starting from this label set, we look for another label sets that label all elements $[u]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \mathcal{J}_{C,\mathcal{T}}$, such that all these label sets are reachable from X_ε. The reachability property can be seen obviously from the Point 2 of Lemma 3.24. Each label set is reachable from another label set via a role $r_i \in N_R$. As a consequence from Lemma 3.24, we get the following lemma.

Lemma 3.25. *Given C and $\mathcal{T}$ the image set $\mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})$ is effectively computable.*

Proof. According to Lemma 3.24 the set X_ε is computable. Furthermore, the finite interpretation $\widehat{\mathcal{J}}_{C,\mathcal{T}}$ is also computable. Lemma 3.24 implies that the image set consists of all sets in $2^{N_{C,\mathcal{T}}}$ that are reachable from X_ε in $\widehat{\mathcal{J}}_{C,\mathcal{T}}$. Thus, the image set is computable using reachability checks from X_ε in $\widehat{\mathcal{J}}_{C,\mathcal{T}}$. □

Finally, the graph model $\mathcal{J}_{C,\mathcal{T}}$ can be obtained from $\widehat{\mathcal{J}}_{C,\mathcal{T}}$. We define an interpretation $\mathcal{Z}_{C,\mathcal{T}}$ as follows:

$$\Delta^{\mathcal{Z}_{C,\mathcal{T}}} := \Delta^{\widehat{\mathcal{J}}_{C,\mathcal{T}}} \cap \mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}});$$
$$A^{\mathcal{Z}_{C,\mathcal{T}}} := A^{\widehat{\mathcal{J}}_{C,\mathcal{T}}} \cap \mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}}) \text{ for all } A \in N_C;$$
$$r^{\mathcal{Z}_{C,\mathcal{T}}} := r^{\widehat{\mathcal{J}}_{C,\mathcal{T}}} \cap (\mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}}) \times \mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})) \text{ for all } r \in N_R.$$

Since $\widehat{\mathcal{J}}_{C,\mathcal{T}}$ and $\mu_{C,\mathcal{T}}(\Delta^{\mathcal{J}_{C,\mathcal{T}}})$ are computable it follows that $\mathcal{Z}_{C,\mathcal{T}}$ is also computable. From Lemma 3.23 and Lemma 3.24 it follows that $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{Z}_{C,\mathcal{T}}$ are isomorphic, where the function $\mu_{C,\mathcal{T}} : \Delta^{\mathcal{J}_{C,\mathcal{T}}} \to \widehat{\mathcal{J}}_{C,\mathcal{T}}$ is an isomorphism. Thus, the interpretations $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{Z}_{C,\mathcal{T}}$ are identical up to renaming of domain elements.

In the following, if the equivalence class of words is clear from the context, then we shall omit the superscript index $\mathcal{I}_{C,\mathcal{T}}$ to simplify the notation. Next, we want to unravel $\mathcal{J}_{C,\mathcal{T}}$ to re-obtain the LFM $\mathcal{I}_{C,\mathcal{T}}$. In order to show it, we need to define a functional interpretation in the form of the tree-unraveling of $\mathcal{J}_{C,\mathcal{T}}$. Therefore, firstly, we introduce the notion of a finite path occurring in a graph model. Let $\mathcal{J}_{C,\mathcal{T}}$ be a graph model. A finite path $\pi = [u_0]r_1[u_1]r_2[u_2]r_3 \ldots r_n[u_n]$ occurring in $\mathcal{J}_{C,\mathcal{T}}$ consists of domain elements $[u_{i-1}]$ and $[u_i]$ that are connected via r_i, where $r_i \in N_R$, for all $1 \leq i \leq n$. Next, let $w = r_1 \ldots r_n$, then we can simply abbreviate the definition of a finite path before as follows $\pi = [\varepsilon]w[u_n]$. To simplify the writing of elements in $\mathcal{J}_{C,\mathcal{T}}$, except the root element $[\varepsilon]$, we alternatively use a notation σ.

Definition 3.26. (*Tree-unraveling of* $\mathcal{J}_{C,\mathcal{T}}$) *Let* $\mathcal{J}_{C,\mathcal{T}}$ *be the graph model of the least functional model* $\mathcal{I}$ *of a concept w.r.t. a TBox* $\mathcal{T}$*. The tree-unraveling* $\widetilde{\mathcal{I}}_{C,\mathcal{T}}$ *of* $\mathcal{J}_{C,\mathcal{T}}$ *is defined as follows:*

- $\Delta^{\widetilde{\mathcal{I}}_{C,\mathcal{T}}} := \{w \in N_R^* \mid \exists \sigma \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$ *such that* $\exists \pi = [\varepsilon]w\sigma$ *in* $\mathcal{J}_{C,\mathcal{T}}\}$;
- $r^{\widetilde{\mathcal{I}}_{C,\mathcal{T}}} := \{(w, wr) \mid \exists (\sigma, \sigma') \in r^{\mathcal{J}_{C,\mathcal{T}}}$ *such that* $\exists \pi_1 = [\varepsilon]w\sigma \wedge \exists \pi_2 = [\varepsilon]w\sigma'$ *in* $\mathcal{J}_{C,\mathcal{T}}\}$*;*
- $A^{\widetilde{\mathcal{I}}_{C,\mathcal{T}}} := \{w \mid \exists \sigma \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$ *with* $\sigma \in A^{\mathcal{J}_{C,\mathcal{T}}}$ *such that* $\exists \pi = [\varepsilon]w\sigma$ *in* $\mathcal{J}_{C,\mathcal{T}}\}$;

Next, we show that actually $\widetilde{\mathcal{I}}_{C,\mathcal{T}}$ is equal to the least functional model $\mathcal{I}_{C,\mathcal{T}}$.

Lemma 3.27. *Let* $\widetilde{\mathcal{I}}_{C,\mathcal{T}}$ *is the tree-unraveling of* $\mathcal{J}_{C,\mathcal{T}}$ *and* $\mathcal{J}_{C,\mathcal{T}}$ *is the graph-model of the LFM* $\mathcal{I}_{C,\mathcal{T}}$ *of a concept w.r.t a TBox* $\mathcal{T}$*. It holds that* $\widetilde{\mathcal{I}}_{C,\mathcal{T}} = \mathcal{I}_{C,\mathcal{T}}$

Proof:
It remains to prove the following claim that for all $w \in N_R^*$ and $A \in N_C$, we have

$$w \in A^{\widetilde{\mathcal{I}}_{C,\mathcal{T}}} \text{ iff } w \in A^{\mathcal{I}_{C,\mathcal{T}}}$$

"⇒": If $w \in A^{\widetilde{\mathcal{I}}_{C,\mathcal{T}}}$, then $\exists \sigma \in \Delta^{\mathcal{J}_{C,\mathcal{T}}}$ with $\sigma \in A^{\mathcal{J}_{C,\mathcal{T}}}$ such that $\exists \pi = [\varepsilon]w\sigma$ in $\mathcal{J}_{C,\mathcal{T}}$. By Point 3 of Lemma 3.19, we know that $\sigma = [w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}$ and it implies that $w \in A^{\mathcal{I}_{C,\mathcal{T}}}$.

"$\Leftarrow$": If $w \in A^{\mathcal{I}_{C,\mathcal{T}}}$, then $[w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in A^{\mathcal{J}_{C,\mathcal{T}}}$, by Point 3 of Lemma 3.19. It also means that there exists a path $\pi = [\varepsilon]w[w]$ in $\mathcal{J}_{\mathcal{C},\mathcal{T}}$ and by definition of tree-unraveling, we have $w \in A^{\widetilde{\mathcal{I}}_{C,\mathcal{T}}}$.

□

Now, given the intersection model $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ of the LFMs of C and D w.r.t. TBox $\mathcal{T}$, we are also interested to make the corresponding graph model of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. The idea to make this type of model for $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is firstly by making the graph models $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$. Next, we compute the product $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}$ of $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$. Last, we take the subgraph of $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}$ that only contains elements reachable from the pair of the root elements $([\varepsilon_C], [\varepsilon_D])$ of two input graph models.

Definition 3.28. (*Product of Graph Model*)
Let $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$ be the graph models of C and D w.r.t. $\mathcal{T}$, respectively. The product $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}$ of graph models $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$ is computed as follows:

- $\Delta^{\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}} := \{([u],[v]) \mid [u] \in \Delta^{\mathcal{J}_{C,\mathcal{T}}} \wedge [v] \in \Delta^{\mathcal{J}_{D,\mathcal{T}}}\}$;
- $r^{\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}} := \{(([u_1],[v_1]),([u_2,v_2])) \mid ([u_1],[u_2]) \in r^{\mathcal{J}_{C,\mathcal{T}}} \wedge ([v_1],[v_2]) \in r^{\mathcal{J}_{D,\mathcal{T}}}\}$, *for all* $r \in N_R$;
- $A^{\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}} := \{([u],[v]) \mid [u] \in A^{\mathcal{J}_{C,\mathcal{T}}} \wedge [v] \in A^{\mathcal{J}_{D,\mathcal{T}}}\}$, *for all* $A \in N_C$.

Now we want to take the subgraph of $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}$ whose all elements are only reachable from $([\varepsilon_C], [\varepsilon_D])$. It is defined as follows.

Definition 3.29. (*Subgraph of Product of Graph Models*)
Let $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}$ be the product of $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$. The subgraph $\mathcal{G}$ of the product of graph models is defined in the following:

- $\Delta^{\mathcal{G}} := \{([u],[v]) \in \Delta^{\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}} \mid \exists w \in N_R^*$ *such that* $\exists \pi = ([\varepsilon_C],[\varepsilon_D])w([u][v])$ *in* $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}\}$;
- $r^{\mathcal{G}} := \Delta^{\mathcal{G}} \cap r^{\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}}$, *for all* $r \in N_R$;
- $A^{\mathcal{G}} := \Delta^{\mathcal{G}} \cap A^{\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}}$, *for all* $A \in N_C$.

In the following we only consider the notation $\mathcal{G}$ for the subgraph of $\mathcal{J}_{C,\mathcal{T}} \times \mathcal{J}_{D,\mathcal{T}}$ that are only reachable from $([\varepsilon_C], [\varepsilon_D])$. To unravel $\mathcal{G}$, we also do the same construction for unraveling the graph model $\mathcal{J}_{C,\mathcal{T}}$ of the LFM $\mathcal{I}_{C,\mathcal{T}}$ as written in Lemma 3.27. It also results in the tree-unraveling of $\mathcal{G}$ that is equal to $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$.

Next, we will show that for any graph models $\mathcal{J}_{C,\mathcal{T}}$ of the LFM $\mathcal{I}_{C,\mathcal{T}}$ of a concept w.r.t. a TBox $\mathcal{T}$, it holds that they are semantically equivalent by means of a special type of relation that will be explained in the next section. This relation is also able to characterize more properties of all the $\mathcal{FL}_0$-models described above.

4 Simulation between Functional Interpretations

Now we want to devise a procedure to compute generalizations in $\mathcal{FL}_0$-TBox by using the same setting as defined in description logic $\mathcal{EL}$ [ZT13]. One of the notions adopted from the characterization of lcs in $\mathcal{EL}$ is a simulation relation that can be used later to characterize properties of functional models or terminological reasonings in description logic $\mathcal{FL}_0$. The following definitions about simulation is introduced as a binary relation between two interpretations in general.

Definition 4.1. (*Simulation between interpretations*)
Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations, $d \in \Delta^{\mathcal{I}_1}$, and $e \in \Delta^{\mathcal{I}_2}$. $\mathcal{S} \subseteq \Delta^{(\mathcal{I}_1,d)} \times \Delta^{(\mathcal{I}_2,e)}$ is called a simulation from $(\mathcal{I}_1, d)$ to $(\mathcal{I}_2, e)$ iff the following properties are satisfied:

1. *$(d, e) \in \mathcal{S}$;*
2. *For all $(d_1, e_1) \in \mathcal{S}$ and all $A \in N_C$, it holds that $d_1 \in A^{(\mathcal{I}_1,d)}$ implies $e_1 \in A^{(\mathcal{I}_2,e)}$;*
3. *For all role names $r \in N_R$ and all $(d_1, e_1) \in \mathcal{S}$ and $d_2 \in \Delta^{(\mathcal{I}_1,d)}$ with $(d_1, d_2) \in r^{(\mathcal{I}_1,d)}$, there exists $e_2 \in \Delta^{(\mathcal{I}_2,e)}$ such that $(e_1, e_2) \in r^{(\mathcal{I}_2,e)}$ and $(d_2, e_2) \in \mathcal{S}$.*

$(\mathcal{I}_1, d)$ is *simulated* by $(\mathcal{I}_2, e)$ (denoted by $(\mathcal{I}_1, d) \lesssim (\mathcal{I}_2, e)$) iff there exists a simulation $\mathcal{S} \subseteq \Delta^{(\mathcal{I}_1,d)} \times \Delta^{(\mathcal{I}_2,e)}$. Note that the symbol "$\lesssim$" is reflexive and transitive. If $(\mathcal{I}_1, d) \lesssim (\mathcal{I}_2, e)$ and $(\mathcal{I}_2, e) \lesssim (\mathcal{I}_1, d)$, then $(\mathcal{I}_1, d)$ and $(\mathcal{I}_2, e))$ are *simulation-equivalent* (denoted by $(\mathcal{I}_1, d) \simeq (\mathcal{I}_2, e)$). Since every functional interpretation and graph model are also interpretations, the definitions above are also applied analogously to them.

Now we can characterize some properties of functional interpretations using a simulation. Mostly, the following proof procedures, which are to employ a simulation for the characterization of subsumption and other reasoning tasks, are adopted from [LW10] and [LPW10]. First, we show that simulation is able to characterize whether an element belongs to a concept in a given model.

Lemma 4.2. *Let $\mathcal{I}$ be a functional model of $\mathcal{T}$. An element $u \in C^{\mathcal{I}}$ iff $(\mathcal{I}_{C,\mathcal{T}}, \varepsilon_C) \lesssim (\mathcal{I}, u)$.*

Proof:

"⇐": As defined in Definition 3.2 it holds that $C \in \mathcal{I}_{C,\mathcal{T}}(\varepsilon_C)$. Now, w.l.o.g., we have

$$C = A_1 \sqcap \ldots \sqcap A_n \sqcap \forall w_1.B_1 \sqcap \ldots \sqcap \forall w_m.B_m$$

where $A_i \in N_C, w_j \in N_R^+, B_j \in N_C$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Since $(\mathcal{I}_{C,\mathcal{T}}), \varepsilon_C) \lesssim (\mathcal{I}, u)$ and $\varepsilon_C \in A_i^{\mathcal{I}_{C,\mathcal{T}}}$, we know that $u \in A_i^{\mathcal{I}}$, for all $1 \leq i \leq n$. Now, it remains to show that $u \in (\forall w_j.B_j)^{\mathcal{I}}$, for all $1 \leq j \leq m$. Let $\varepsilon_C \in (\forall w_j.B_j)^{\mathcal{I}_{C,\mathcal{T}}}$. By Lemma 3.5, we have $w_j \in B_j^{\mathcal{I}_{C,\mathcal{T}}}$. It follows that there exists $v \in \Delta^{\mathcal{I}}$ such that $v \in B_j^{\mathcal{I}}$ by Property 2 of Definition 4.1. Since $(\mathcal{I}_{C,\mathcal{T}})$ and $(\mathcal{I}, u)$ are over the same set of role names N_R, we know that $|w_j| = |v| - |u|$ or $v = uw_j$ and it implies that $u \in (\forall w_j.B_j)^{\mathcal{I}}$. Finally it holds that $u \in C^{\mathcal{I}}$.

"⇒" Let $u \in C^{\mathcal{I}}$. We build a relation $\mathcal{S} \subseteq \Delta^{\mathcal{I}_{C,\mathcal{T}}} \times \Delta^{(\mathcal{I},u)}$ by setting for all $v \in \Delta^{\mathcal{I}_{C,\mathcal{T}}}$, we have $(v, v) \in \mathcal{S}$. Assume $v \in A^{\mathcal{I}_{C,\mathcal{T}}}$. By Claim 3 of Lemma 3.5, we have $v \in L_{\mathcal{T}}(C, A)$ and thus $C \sqsubseteq_{\mathcal{T}} \forall v.A$. Since $\mathcal{I}$ is a functional model of $\mathcal{T}$, it follows $u \in (\forall v.A)^{\mathcal{I}}$ from $u \in C^{\mathcal{I}}$. By Claim 1 of Lemma 3.5 and Definition 3.6, it implies $v \in A^{(\mathcal{I},u)}$. Obviously, the relation $\mathcal{S}$ just satisfied Property 2 of Definition 4.1.

Let $(v, vr) \in r^{\mathcal{I}_{C,\mathcal{T}}}$. Assume $vr \in B^{\mathcal{I}_{C,\mathcal{T}}}$. Using the same proof derivation as the case $v \in A^{\mathcal{I}_{C,\mathcal{T}}}$, we obtain $vr \in B^{(\mathcal{I},u)}$. Obviously, $(v, vr) \in r^{(\mathcal{I},u)}$ and $(vr, vr) \in \mathcal{S}$. It follows that $\mathcal{S}$ is a simulation and regarding the setting of $\mathcal{S}$, we have $(\varepsilon, \varepsilon) \in \mathcal{S}$.

□

Next, the subsumption relationship can also be characterized with the help of a simulation.

Lemma 4.3. *$C \sqsubseteq_{\mathcal{T}} D$ iff $(\mathcal{I}_{D,\mathcal{T}}, \varepsilon_D) \lesssim (\mathcal{I}_{C,\mathcal{T}}, \varepsilon_C)$.*

Proof:

"⇒": Let $C \sqsubseteq_{\mathcal{T}} D$. It follows $\varepsilon_C \in D^{\mathcal{I}_{C,\mathcal{T}}}$ from $\varepsilon_C \in C^{\mathcal{I}_{C,\mathcal{T}}}$. As an immediate consequence of Lemma 4.2, we have $(\mathcal{I}_{D,\mathcal{T}}, \varepsilon_D) \lesssim (\mathcal{I}_{C,\mathcal{T}}, \varepsilon_C)$.

"⇐": Let $u \in C^{\mathcal{I}}$. It implies that $(\mathcal{I}_{C,\mathcal{T}}, \varepsilon_C) \lesssim (\mathcal{I}, u)$. Together with $(\mathcal{I}_{D,\mathcal{T}}, \varepsilon_D) \lesssim (\mathcal{I}_{C,\mathcal{T}}, \varepsilon_C)$ and transitivity of "$\lesssim$", we get $(\mathcal{I}_{D,\mathcal{T}}, \varepsilon_D) \lesssim (\mathcal{I}, u)$ and again by Lemma 4.2, we obtain $u \in D^{\mathcal{I}}$

□

Last, as mentioned in the previous chapter that we want to show that for any graph models $\mathcal{J}$ of the LFM $\mathcal{I}$ of a concept w.r.t. a TBox, it holds that $\mathcal{J}$ and $\mathcal{I}$ are semantically-equivalent. This is described in the following lemma.

Lemma 4.4. *Let $\mathcal{I}$ be the LFM of a concept w.r.t. a TBox $\mathcal{T}$ and $\mathcal{J}$ be the graph model of $\mathcal{I}$. It holds that $\mathcal{J} \simeq \mathcal{I}$.*

Proof:

"⇒": Let us build a relation $\mathcal{S}_1 \subseteq \Delta^{\mathcal{J}} \times \Delta^{\mathcal{I}}$ by setting $\{(\sigma, w)\} \in \mathcal{S}_1$ iff there exists a path $\pi = [\varepsilon]w\sigma$ in $\mathcal{J}$. Now assume that $\sigma \in A^{\mathcal{J}}$. We have to show that $w \in A^{\mathcal{I}}$. Since there exists a path $\pi = [\varepsilon]w\sigma$ in $\mathcal{J}$, we know that $[\varepsilon] \in (\forall w.A)^{\mathcal{J}}$. As a direct consequence we know that $\varepsilon \in (\forall w.A)^{\mathcal{I}}$ and $w \in A^{\mathcal{I}}$.

Now assume that $(\sigma, \sigma') \in r^{\mathcal{J}}$ and $\sigma' \in B^{\mathcal{J}}$. By using the same argument as the case $\sigma \in A^{\mathcal{J}}$, we obtain $wr \in B^{\mathcal{I}}$, wr is the r-successor of w, where $r \in N_R$, and $(\sigma', wr) \in \mathcal{S}_1$. Moreover, it is also easy to see that $([\varepsilon], \varepsilon) \in \mathcal{S}_1$ and thus $\mathcal{S}_1$ is a simulation.

"⇐": Let us build a relation $\mathcal{S}_2 \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ by setting $\{(w, \sigma)\} \in \mathcal{S}_2$ iff there exists a path $\pi = [\varepsilon]w\sigma$ in $\mathcal{J}$. We have to show that $\sigma \in A^{\mathcal{J}}$ and assume that $w \in A^{\mathcal{I}}$. It implies that $\varepsilon \in (\forall w.A)^{\mathcal{I}}$. Since $\varepsilon \in [\varepsilon]$, where $[\varepsilon]$ is an element of $\mathcal{J}$, it implies that $[\varepsilon] \in (\forall w.A)^{\mathcal{J}}$. According to the path $\pi = [\varepsilon]w\sigma$ in $\mathcal{J}$, we have $\sigma \in A^{\mathcal{J}}$.

□

The lemma above is also applied analogously with the same proof construction to check whether $\mathcal{G}$ is simulation-equivalent to $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. As a conclusion for this section, we can note that a simulation is able to identify many properties of models for $\mathcal{FL}_0$ and even to compute the subsumption relationship between $\mathcal{FL}_0$-concept descriptions. Next, we start providing a solution for Problem I mentioned in Introduction that whether a concept is the least common subsumer of two input concepts.

5 Conditions Whether a Concept is the Least Common Subsumer

Let us recall that Problem I is as follows:

I. Let C, D, and E be $\mathcal{FL}_0$-concepts and $\mathcal{T}$ be a TBox. Is concept E the $lcs_{\mathcal{T}}(C, D)$?

As written in the end of Section 2, we may assume that C and D are concept names occurring in $\mathcal{T}$ and $\mathcal{T}$ is a PANF TBox. To address this question, we need to investigate that the set of lcs-candidates for C and D w.r.t. $\mathcal{T}$ consists of all $k - lcs_{\mathcal{T}}(C, D)$, for all $k > 0$. First of all, we show that the k-characteristic concept of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is the k-$lcs_{\mathcal{T}}(C, D)$.

Lemma 5.1. *Let $k \in \mathbb{N}$*

1. $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \in cs_{\mathcal{T}}(C, D)$.
2. *Let E be a concept with the role-depth $rd(E) \leq k$ and $C \sqsubseteq_{\mathcal{T}} E$ and $D \sqsubseteq_{\mathcal{T}} E$.*
   It holds that $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \sqsubseteq_{\mathcal{T}} E$.

Proof:

1. We prove it by induction on k.

   - For $k = 0$:
     By definition of k-characteristic concept, we know that

     $$X^0(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) := \bigsqcap \{\forall \varepsilon.A \mid \varepsilon \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}\}$$

     Assume $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) = \forall \varepsilon.A_1 \sqcap \ldots \sqcap \forall \varepsilon.A_n$. For all A_i, where $1 \leq i \leq n$, we have $\varepsilon \in A^{\mathcal{J}_{C,\mathcal{T}} \cap \mathcal{J}_{D,\mathcal{T}}}$. By Claim 3 of Definition 3.3, we have $\varepsilon \in A_i^{\mathcal{I}_{C,\mathcal{T}}}$ and $\varepsilon \in A_i^{\mathcal{J}_{D,\mathcal{T}}}$, for all $1 \leq i \leq n$. By Lemma 4.3, we get $C \sqsubseteq_{\mathcal{T}} A_i$ and $D \sqsubseteq_{\mathcal{T}} A_i$, for all $1 \leq i \leq n$. Therefore,

     $$C \sqsubseteq_{\mathcal{T}} X^0(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$$
     $$D \sqsubseteq_{\mathcal{T}} X^0(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$$

   - For $k > 0$:
     By using the definition of k-characteristic concept,

     $$\begin{aligned} X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) := \quad & X^{k-1}(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \sqcap \bigsqcap \\ & \{\forall w.A \mid w \in N_R^+, A \in N_C, |w| = k, w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}\} \end{aligned}$$

     By induction hypothesis, we know that

     $$X^{k-1}(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \in cs_{\mathcal{T}}(C, D) \tag{9}$$

     It remains to show that $\{\forall w.A \mid w \in N_R^*, A \in N_C, |w| = k, w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}\}$ is also a common subsumer of C and D w.r.t. $\mathcal{T}$. We know that $w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$ which implies that $\varepsilon \in (\forall w.A)^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$. As a direct consequence,

$$\varepsilon \in (\forall w.A)^{\mathcal{I}_{C,\mathcal{T}}} \text{ and } \varepsilon \in (\forall w.A)^{\mathcal{I}_{D,\mathcal{T}}}$$

By Lemma 4.2 and 4.3, it is implied that

$$C \sqsubseteq_{\mathcal{T}} \forall w.A \text{ and } D \sqsubseteq_{\mathcal{T}} \forall w.A$$

Together with 9, we have $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \in cs_{\mathcal{T}}(C, D)$

2. Again, we prove it by induction on k as the role-depth of concept E

- $rd(E) = 0$.
  Let E be a conjunction over concept names $\bigsqcap_{i>0} A_i$. Since $E \in cs_{\mathcal{T}}(C, D)$, then by Lemma 4.3, we have $\varepsilon_C \in E^{\mathcal{I}_{C,\mathcal{T}}}$ and $\varepsilon_D \in E^{\mathcal{I}_{D,\mathcal{T}}}$. It implies $\varepsilon_C \in A_i^{\mathcal{I}_{C,\mathcal{T}}}$, $\varepsilon_D \in A_i^{\mathcal{I}_{D,\mathcal{T}}}$, and thus $\varepsilon \in A_i^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$, for all i. By Definitions 3.3 and 3.9, we get $X^0(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \sqsubseteq_{\mathcal{T}} E$.
- $rd(E) = k > 0$.
  Let E be in the following CCNF form:

  $$E = \bigsqcap\{\forall v.A \mid v \in N_R^*, |v| < k, A \in N_C\} \sqcap \bigsqcap\{\forall w.B \mid w \in N_R^+, |w| = k, B \in N_C\}$$

  By induction hypothesis, we know that

  $$X^{k-1}(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \sqsubseteq_{\mathcal{T}} \bigsqcap\{\forall v.A \mid v \in N_R^*, |v| < k, A \in N_C\}. \tag{10}$$

  Since $E \in cs_{\mathcal{T}}(C, D)$, we have

  $$\bigsqcap\{\forall w.B \mid w \in N_R^+, |w| = k, B \in N_C\} \in cs_{\mathcal{T}}(C, D)$$

  Let $\forall w.B \in \{\forall w.B \mid w \in N_R^+, |w| = k, B \in N_C\}$. Then, it implies that

  $$C \sqsubseteq_{\mathcal{T}} \forall w.B \text{ and } D \sqsubseteq_{\mathcal{T}} \forall w.B$$

  Since $|w| = k$ and by definition of characteristic concepts, we know that $\forall w.B$ is a conjunct in $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$. If we add $\{\forall w.B \mid w \in N_R^+, |w| = k, B \in N_C\}$ in 10 as a conjunct on the left- and right-hand sides of $\sqsubseteq_{\mathcal{T}}$, then we obtain $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$ $\sqsubseteq_{\mathcal{T}}$ E.

□

Lemma 5.1 implies that the set of k-characteristic concepts of the intersection of LFMs $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T}, \varepsilon$ is the possible candidate for the $lcs_{\mathcal{T}}(C, D)$. It is stated formally in the following corollary.

Corollary 5.2. *The lcs exists if and only if there is a $k \in \mathbb{N}$ such that for all $l \in \mathbb{N}$, k-$lcs_{\mathcal{T}}(C, D) \sqsubseteq_{\mathcal{T}} l$-$lcs_{\mathcal{T}}(C, D)$.*

Next, we show that the LFM of the $lcs_{\mathcal{T}}(C, D)$ and the intersection model $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T}$ are simulation-equivalent.

Lemma 5.3. *Let E be a concept.*
E is the $lcs_{\mathcal{T}}(C, D)$ iff $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \simeq (\mathcal{I}_{E,\mathcal{T}}, \varepsilon)$.

Proof:
The structure of LFMs is a special type of the structure of canonical models in $\mathcal{EL}$. It is indicated by the fact that for all $r \in N_R$, each element of LFMs only has 1 r-successor. The modest structure for the functional model also leads us to easily prove this lemma by adopting the similar proof procedure for Lemma 12 in [ZT13] which states that the canonical model of lcs in $\mathcal{EL}$ is simulation-equivalent to the product model of two input concepts.

□

Since the LFM of the lcs of input concepts and the intersection model of input concepts are over the same domain elements N_R^*, we can say that these two models are exactly the same. It is shown in the following lemma which also provides a condition whether a concept is the lcs of two concept names w.r.t. a TBox or not.

Lemma 5.4. *Let E be a concept. E is the $lcs_{\mathcal{T}}(C, D)$ iff $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{E,\mathcal{T}}$.*

Proof:
By Corollary 5.2, we know that the $lcs_{\mathcal{T}}(C, D)$ exists iff there is a k, such that $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$ is the $lcs_{\mathcal{T}}(C, D)$. Let $K = X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$. Therefore, it brings us to the following claim.

$$K \equiv lcs_{\mathcal{T}}(C, D) \text{ iff } \mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{K,\mathcal{T}}$$

"⇐": Let F be a common subsumer of C and D w.r.t. $\mathcal{T}$. We want to show that $F \sqsubseteq_{\mathcal{T}} K$. By Claim 5 of Lemma 3.5, we know that $\mathcal{I}_{F,\mathcal{T}} \subseteq \mathcal{I}_{C,\mathcal{T}}$ and $\mathcal{I}_{F,\mathcal{T}} \subseteq \mathcal{I}_{D,\mathcal{T}}$. It implies that $\mathcal{I}_{F,\mathcal{T}} \subseteq \mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. By our assumption we know that $\mathcal{I}_{F,\mathcal{T}} \subseteq \mathcal{I}_{K,\mathcal{T}}$ which implies that $F \sqsubseteq_{\mathcal{T}} K$.

"⇒": Let $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \equiv lcs_{\mathcal{T}}(C, D)$. By Definition 3.8, we can define that the intersection model $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is the union of all ℓ-subtree of the intersection model, where $\ell \in \mathbb{N}$. Now let $X^\ell(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$ and we know that $((\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}})^\ell, \varepsilon) = \mathcal{I}_{X^\ell(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \emptyset}$.

$$\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \bigcup_{\ell=0}^{\infty} \mathcal{I}_{X^\ell(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \emptyset} \tag{11}$$

For all $\ell = 0, 1, 2, \ldots$, by Corollary 5.2, we have $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) \sqsubseteq_{\mathcal{T}} X^\ell(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$. It implies that $\varepsilon \in (X^\ell(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon))^{\mathcal{I}_{X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \mathcal{T}}}$. It is followed that $\mathcal{I}_{X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \mathcal{T}}$ is a functional model of $X^\ell(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$ w.r.t. empty TBox. Formally, we can write it as follows

$$\mathcal{I}_{X^\ell(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \emptyset} \subseteq \mathcal{I}_{X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \mathcal{T}}$$

By (11), we have $\bigcup_{\ell=0}^{\infty} \mathcal{I}_{X^\ell(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \emptyset} \subseteq \mathcal{I}_{X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \mathcal{T}}$. By the transitivity of "=", we obtain

$$\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} \subseteq \mathcal{I}_{X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \mathcal{T}} \tag{12}$$

Moreover, it is known that $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$ is the k-$lcs_{\mathcal{T}}(C, D)$. Therefore, we have $\mathcal{I}_{X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \mathcal{T}} \subseteq \mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. Together with 12, we obtain $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon), \mathcal{T}}$.

□

The lemma above convinces us that if the lcs exists, then the intersection model $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is actually the LFM of the lcs-concept of input concepts C and D w.r.t. $\mathcal{T}$. Next, let us consider the following example to compute whether a concept is the lcs based on Lemma 5.4, where the TBox taken from Introduction.

Example 5.5.

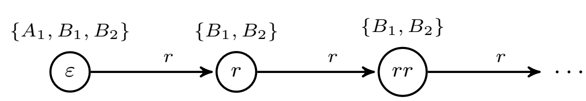
1. *Let $\mathcal{T}_{ex}$ be a TBox consisting the following GCIs.*

$$\begin{array}{rcl} \{A_1 & \sqsubseteq & B_1 \sqcap B_2, \\ A_2 & \sqsubseteq & B_1 \sqcap B_3, \\ B_1 \sqcap B_2 & \sqsubseteq & \forall r.B_1 \sqcap \forall r.B_2, \\ B_1 \sqcap B_3 & \sqsubseteq & \forall r.B_1 \sqcap \forall r.B_3\} \end{array}$$

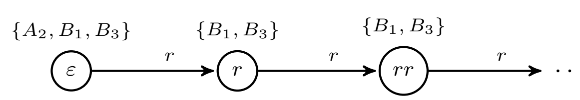
   Let $C = A_1$ and $D = A_2$. Now, we compute the $lcs_{\mathcal{T}_{ex}}(C, D)$.

   - *Build the LFMs $\mathcal{I}_{C,\mathcal{T}_{ex}}$ and $\mathcal{I}_{D,\mathcal{T}_{ex}}$*

$\mathcal{I}_{C,\mathcal{T}_{ex}}$:

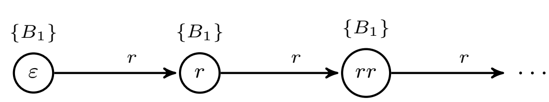


$\mathcal{I}_{D,\mathcal{T}_{ex}}$:

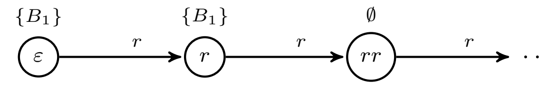


   - *Compute the intersection model $\mathcal{I}_{C,\mathcal{T}_{ex}} \cap \mathcal{I}_{D,\mathcal{T}_{ex}}$ of $\mathcal{I}_{C,\mathcal{T}_{ex}}$ and $\mathcal{I}_{D,\mathcal{T}_{ex}}$ and construct the LFM of $B_1 \sqcap \forall r.B_1 \in cs_{\mathcal{T}_{ex}}(C, D)$.*

$\mathcal{I}_{C,\mathcal{T}_{ex}} \cap \mathcal{I}_{D,\mathcal{T}_{ex}}$:



$\mathcal{I}_{A \sqcap \forall r.A, \mathcal{T}_{ex}}$:



Please note that the intersection model is not equal to the LFM of $B_1 \sqcap \forall r.B_1$ w.r.t. $\mathcal{T}_{ex}$. Therefore, $B_1 \sqcap \forall r.B_1$ is not the $lcs_{\mathcal{T}_{ex}}(C, D)$.

2. *Let $\mathcal{T}_{ex2}$ be an extended TBox of $\mathcal{T}_{ex}$ consisting the following GCIs:*

$$\begin{aligned} \{A_1 &\sqsubseteq B_1 \sqcap B_2, \\ A_2 &\sqsubseteq B_1 \sqcap B_3, \\ B_1 \sqcap B_2 &\sqsubseteq \forall r.B_1 \sqcap \forall r.B_2, \\ B_1 \sqcap B_3 &\sqsubseteq \forall r.B_1 \sqcap \forall r.B_3, \\ B_1 &\sqsubseteq \forall r.B_1\} \end{aligned}$$

Let $C = A_1$ and $D = A_2$. Now, we compute the $lcs_{\mathcal{T}_{ex2}}(C, D)$.

- *Build the LFMs $\mathcal{I}_{C,\mathcal{T}_{ex2}}$ and $\mathcal{I}_{D,\mathcal{T}_{ex2}}$ that are structurally equivalent to $\mathcal{I}_{C,\mathcal{T}_{ex2}}$ and $\mathcal{I}_{D,\mathcal{T}_{ex2}}$, respectively.*
- *Compute the intersection $\mathcal{I}_{C,\mathcal{T}_{ex2}} \cap \mathcal{I}_{D,\mathcal{T}_{ex2}}$ of $\mathcal{I}_{C,\mathcal{T}_{ex2}}$ and $\mathcal{I}_{D,\mathcal{T}_{ex2}}$ that is structurally equivalent to $\mathcal{I}_{C,\mathcal{T}_{ex}} \cap \mathcal{I}_{D,\mathcal{T}_{ex}}$. Unlike the previous example, we have B_1 as the $lcs_{\mathcal{T}_{ex2}}(C, D)$ since $\mathcal{I}_{C,\mathcal{T}_{ex2}} \cap \mathcal{I}_{D,\mathcal{T}_{ex2}} = \mathcal{I}_{B_1,\mathcal{T}_{ex2}}, \varepsilon$.*

So far, the condition to check whether a concept is the lcs only considers a functional model with infinite domain elements. Now we show another solution for addressing Problem I by means of graph models such that the problem in Problem I is decidable. First, we have to find a relation between the fact we have in Lemma 5.4 and a condition that $\mathcal{G}$ is simulation-equivalent to the graph model of the LFM of the lcs.

Lemma 5.6. *Let C,D, and E be $\mathcal{FL}_0$-concepts and $\mathcal{T}$ be a TBox.*
$(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) = (\mathcal{I}_{E,\mathcal{T}}, \varepsilon)$ *iff* $(\mathcal{G}, ([\varepsilon])) \simeq (\mathcal{J}_{E,\mathcal{T}}, [\varepsilon_E])$ *over the same set of role names.*

Proof:

"⇒": Let $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon) = (\mathcal{I}_{E,\mathcal{T}}, \varepsilon)$. For all $w \in \Delta^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}} = \Delta^{\mathcal{I}_{E,\mathcal{T}}}$, we have

$$w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}} \text{ iff } w \in A^{\mathcal{I}_{E,\mathcal{T}}}, \text{ for all } A \in N_C$$

Let us build a relation $\mathcal{S}_1 \subseteq \Delta^{\mathcal{G},[\varepsilon])} \times \Delta^{(\mathcal{J}_{E,\mathcal{T}},[\varepsilon])}$ by setting $(\sigma, [u]) \in \mathcal{S}_1$ iff there exists $w \in N_R^*$ such that there are paths $\pi_1 = [\varepsilon]w\sigma$ in $\mathcal{G}$ and $\pi_2 = [\varepsilon_E]w[u]$ in $\mathcal{J}_{E,\mathcal{T}}$. Assume that $\sigma \in A^{\mathcal{G}}$, we want to show that $[u] \in A^{\mathcal{J}_{E,\mathcal{T}}}$. If $\sigma \in A^{\mathcal{G}}$, then it implies that $\sigma \in A^{\mathcal{J}_{C,\mathcal{T}} \cap \mathcal{J}_{D,\mathcal{T}}}$. Let $\sigma = ([v_1], [v_2])$. It means that

$$[v_1] \in A^{\mathcal{J}_{C,\mathcal{T}}} \text{ and } [v_2] \in A^{\mathcal{J}_{D,\mathcal{T}}}$$

Since there is a path $\pi_1 = [\varepsilon]w\sigma$ in $\mathcal{G}$ and $\mathcal{J}_{C,\mathcal{T}} \cap \mathcal{J}_{D,\mathcal{T}}$, it also means that there are path $\pi_3 = [\varepsilon_C]w[v_1]$ in $\mathcal{J}_{C,\mathcal{T}}$ and $\pi_4 = [\varepsilon_D]w[v_2]$ and $\mathcal{J}_{D,\mathcal{T}}$. Because $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$ are models of $\mathcal{T}$, we know that

$$[\varepsilon_C] \in (\forall w.A)^{\mathcal{J}_{C,\mathcal{T}}} \text{ and } [\varepsilon_D] \in (\forall w.A)^{\mathcal{J}_{D,\mathcal{T}}}$$

It is also followed that

$$\varepsilon_C \in (\forall w.A)^{\mathcal{I}_{C,\mathcal{T}}} \text{ and } \varepsilon_D \in (\forall w.A)^{\mathcal{I}_{D,\mathcal{T}}}$$

By the definition of intersection of functional models, then we also have

$$\varepsilon \in (\forall w.A)^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$$

By our assumption, we have $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{E,\mathcal{T}}$. It implies directly that

$$\varepsilon \in (\forall w.A)^{\mathcal{I}_{E,\mathcal{T}}}$$

By Definition 3.18, we know that $[\varepsilon_E] \in (\forall w.A)^{\mathcal{J}_{E,\mathcal{T}}}$ and since $\mathcal{J}_{E,\mathcal{T}}$ is a model of $\mathcal{T}$ and there is a path $\pi_2 = [\varepsilon_E]w[u]$ in $\mathcal{J}_{E,\mathcal{T}}$, finally we have $[u] \in A^{\mathcal{J}_{E,\mathcal{T}}}$. We just showed that $\mathcal{S}_1$ satisfying Property 2 of Definition 4.1.
Now assume that we have $(\sigma, \sigma') \in r^{\mathcal{G}}$ and $\sigma' \in B^{\mathcal{G}}$. By using the same argument as the case $\sigma \in A^{\mathcal{G}}$, then obtain $[u'] \in \Delta^{\mathcal{J}_{E,\mathcal{T}}}$ with $[u'] \in A^{\mathcal{J}_{E,\mathcal{T}}}$ and $([u],[u']) \in r^{\mathcal{J}_{E,\mathcal{T}}}$. Therefore, $(\sigma',[u']) \in \mathcal{S}_1$. It is easy to see that $([\varepsilon, \varepsilon_E]) \in \mathcal{S}_1$ and thus $\mathcal{S}_1$ is a simulation.

Now, we build a relation $\mathcal{S}_2 \subseteq \Delta^{(\mathcal{J}_{E,\mathcal{T}},[\varepsilon_E])} \times \Delta^{(\mathcal{G},[\varepsilon])}$. By using the same setting as $\mathcal{S}_1$, we can prove that $\mathcal{S}_2$ is a simulation. Together with $\mathcal{S}_1$, we obtain $(\mathcal{G},[\varepsilon]) \simeq (\mathcal{J}_{E,\mathcal{T}},[\varepsilon_E])$.

"⇐": Let $(\mathcal{G},[\varepsilon]) \simeq (\mathcal{J}_{E,\mathcal{T}})$ and $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T}$ and $\mathcal{I}_{E,\mathcal{T}}$ be the tree-unraveling of $\mathcal{G}$ and $\mathcal{J}_{E,\mathcal{T}}$, respectively. Since each functional is simulation equivalent with their graph model, we also have $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T} \simeq \mathcal{I}_{E,\mathcal{T}}$. It implies that E is the $lcs_{\mathcal{T}}(C,D)$ by Lemma 5.3 and thus $\mathcal{I}_{C,T} \cap \mathcal{I}_{D,T} = \mathcal{I}_{E,\mathcal{T}}$ by Lemma 5.4. □

As a consequence, we have the following lemma to show that the problem in Problem I is decidable.

Lemma 5.7. *Let E be a concept. E is the $lcs_{\mathcal{T}}(C,D)$ iff $\mathcal{G} \simeq \mathcal{J}_{E,\mathcal{T}}$.*

Proof:
A consequence of Lemma 5.4 and 5.6.

□

Now, we are able to see that there is a relationship between the relational symbol "$\simeq$" and "$=$" when they are employed to show a relationship between $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and $\mathcal{I}_{E,\mathcal{T}}$ as well as to their corresponding graph models. Even though "$\simeq$" is weaker than "$=$", but a simulation equivalence "$\simeq$" is more helpful and suitable to decide whether a concept is the lcs of two concept names w.r.t. a TBox when we work on the graph models since they only have finitely many number of domain elements. Finally, by using Corollary 5.2, we can derive characterizations for the existence of the lcs w.r.t. $\mathcal{FL}_0$-TBoxes from Lemma 5.4 and Lemma 5.6 as stated in the following corollary.

Corollary 5.8.

1. *The* $lcs_{\mathcal{T}}(C, D)$ *exists iff there is a* $k \in \mathbb{N}$ *such that* $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$ *is equal to the LFM of* $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$ *w.r.t.* $\mathcal{T}$*.*

2. *The* $lcs_{\mathcal{T}}(C, D)$ *exists iff there is a* $k \in \mathbb{N}$ *such that* $\mathcal{G}$ *is simulation-equivalent to the graph model of the LFM of* $X^k(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, \varepsilon)$ *w.r.t.* $\mathcal{T}$*.*

However, this corollary does not show any decision procedure to compute the $lcs_{\mathcal{T}}(C, D)$ since there are infinitely many k to check whether the lcs exists in finite time. The following section will describe how to make the claims above decidable as well as to provide a solution for Problem II.

6 Characterizations for the Existence of the Least Common Subsumer

Problem II is written as follows:

II. Let C and D be $\mathcal{FL}_0$-concepts and $\mathcal{T}$ be a TBox. Does the $lcs_{\mathcal{T}}(C, D)$ exist?

As mentioned previously that we may assume that C, D are concept names occurring in a PANF TBox $\mathcal{T}$. We start answering this question by continuing the solution provided in Lemma 5.8 to check whether the $lcs_{\mathcal{T}}(C, D)$ exists. First, to simplify the notation, in the following we write X^k to abbreviate the k-characteristic concept of the intersection model $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. Let us recall that Point 1 of Lemma 5.8 may also be written in the following lemma.

Lemma 6.1. *The $lcs_{\mathcal{T}}(C, D)$ exists iff there is a $k \in \mathbb{N}$ such that*

$$\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^k,\mathcal{T}}.$$

Next, we show a property of the LFM of X^k.

Lemma 6.2. *Let $\mathcal{I}_{X^k,\emptyset}$ be the LFM of X^k w.r.t. $\emptyset$ TBox for some $k \in \mathbb{N}$. It holds that*

$$w \in A^{\mathcal{I}_{X^k,\mathcal{T}}} \text{ iff } w \in A^{\mathcal{I}_{X^k,\emptyset}}, \text{ for all } w \in N_R^* \text{ with } |w| \leq k \text{ and all } A \in N_C.$$

Proof:
By Definition 3.2, $\mathcal{I}_{X^k,\emptyset}$ is a subinterpretation of $\mathcal{I}_{X^k,\mathcal{T}}$, which means that $\mathcal{I}_{X^k,\emptyset} \subseteq \mathcal{I}_{X^k,\mathcal{T}}$. It also implies that for all $w \in N_R^*$ with $|w| \leq k$ and all $A \in N_C$ we have $w \in A^{\mathcal{I}_{X^k,\emptyset}}$ implies $w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$. Now it remains to show that

For all $w \in N_R^*$ with $|w| \leq k$ and all $A \in N_C$ we have $w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$ implies $w \in A^{\mathcal{I}_{X^k,\emptyset}}$

We can show the claim above by induction on $|w| \leq k$.

- Let $|w| = 0$
  It means that $w = \varepsilon$. If $\varepsilon \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$, then $X^k \sqsubseteq_{\mathcal{T}} A$. Since $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is a model of $\mathcal{T}$ and $\varepsilon \in (X^k)^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$, $\varepsilon \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$ follows. Therefore, by definition of k-characteristic concept, A is on the top level of X^k as a conjunct, which means that $\varepsilon \in A^{\mathcal{I}_{X^k,\emptyset}}$.

- Let $0 < |w| \leq k$
  Let $w = w'r$ for $r \in N_R$ and $w' \in N_R^*$, and $w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$. Since $|w'| < |w|$, we know that $w' \in (\forall r.A)^{\mathcal{I}_{X^k,\mathcal{T}}}$. By induction hypothesis, we know that

  For all $B_i \in N_C$, where $1 \leq i \leq n$, we have $w' \in B_i^{\mathcal{I}_{X^k,\mathcal{T}}}$ implies $w' \in B_i^{\mathcal{I}_{X^k,\emptyset}}$.

  Now assume
  $$w \notin A^{\mathcal{I}_{X^k,\emptyset}} \tag{13}$$
  then $w' \notin \forall r.A^{\mathcal{I}_{X^k,\emptyset}}$. It means that $w' \in (\forall r.A)^{\mathcal{I}_{X^k,\mathcal{T}}}$ because the presence of $\mathcal{T}$. Therefore, $B_1 \sqcap \ldots \sqcap B_n \sqsubseteq_{\mathcal{T}} \forall r.A$. By Lemma 3.10, we also have

$$w' \in B_i^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}, \text{ for all } 1 \leq i \leq n.$$

But then, $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is also a model of $\mathcal{T}$, which implies that $w' \in (\forall r.A)^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$ and $w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$. By Lemma 3.10 and definition of k-characteristic concept, it turns out that we have $w \in A^{\mathcal{I}_{X^k,\emptyset}}$, which is a contradiction to our assumption (13).

□

As what we did to construct LFMs and graph models in stepwise, here we also define a new functional interpretation which is an extended version of a given $\mathcal{I}_{X^k,\emptyset}$. This is called *extended k-subtree*. The following definition can make us understand on how to build $\mathcal{I}_{X^k,\mathcal{T}}$ in a graded way with the input is $\mathcal{I}_{X^k,\emptyset}$.

Definition 6.3. (*Extended k-subtree*)
Let $\mathcal{I}_{X^k,\emptyset}$ be the LFM of X^k w.r.t. the empty TBox for some $k \in \mathbb{N}$. We define an infinite sequence of functional interpretations.

$$\widehat{\mathcal{I}}_0, \widehat{\mathcal{I}}_1, \widehat{\mathcal{I}}_2, \ldots$$

inductively as follows

$$\widehat{\mathcal{I}}_0 := \mathcal{I}_{X^k,\emptyset} \tag{14}$$

and for all $n \geq 0$ we define

$$A^{\widehat{\mathcal{I}}_n} := A^{\widehat{\mathcal{I}}_{n-1}} \cup \{wr \in N_R^* \mid n-1 = |w| - k, (\bigsqcap_{w \in B^{\widehat{\mathcal{I}}_{n-1}}, B \in N_C} B) \sqsubseteq_{\mathcal{T}} \forall r.A\}, \text{for all } A \in N_C \tag{15}$$

Finally, we define the extended k-subtree

$$\widehat{\mathcal{I}}_{X^k,\emptyset} := \bigcup_{\ell=0}^{\infty} \widehat{\mathcal{I}}_\ell \tag{16}$$

From the definition above, we have a property for the extended k-subtree described in the following lemma.

Lemma 6.4. *Let $w \in N_R^*$ and $A \in N_C$. If $|w| \leq k$, then it holds that*

$$w \in A^{\widehat{\mathcal{I}}_{X^k,\emptyset}} \text{ iff } w \in A^{\mathcal{I}_{X^k,\emptyset}}.$$

And if $|w| > k$, then it holds that

$$w \in A^{\widehat{\mathcal{I}}_{X^k,\emptyset}} \text{ iff } w \in A^{\widehat{\mathcal{I}}_n},$$

where $n = |w| - k$.

Proof: This is explicitly written in the definition of extended k-subtree (Definition 6.3).

□

Next, we need to show that $\widehat{\mathcal{I}}_{X^k,\emptyset}$ is a model of $\mathcal{T}$.

Lemma 6.5. *$\widehat{\mathcal{I}}_{X^k,\emptyset}$ is a model of $\mathcal{T}$.*

Proof:
It has to be shown that $\widehat{\mathcal{I}}_{X^k,\emptyset}$ satisfies all the GCIs in $\mathcal{T}$. Let $L \sqsubseteq R \in \mathcal{T}$ be a GCI in $\mathcal{T}$. Since $\mathcal{T}$ is in PANF, we assume that L and R have the following form:

$$L = P_1 \sqcap \ldots \sqcap P_n \sqcap \forall r_1.A_1 \sqcap \ldots \sqcap \forall r_m.A_m \tag{17}$$

$$R = P'_1 \sqcap \ldots \sqcap P'_{n'} \sqcap \forall r'_1.A'_1 \sqcap \ldots \sqcap \forall r'_{m'}.A'_{m'} \tag{18}$$

where $P_i, P'_{i'}, A_j, A'_{j'}$ are concept names and $r_j, r_{j'}$ are role names for all $1 \leq i \leq n$, $1 \leq j \leq m$, $1 \leq i' \leq n'$, and $1 \leq j' \leq m'$.

Let $w \in N_R^*$. We have to show that $w \in L^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$ implies $w \in R^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$. We distinguishes the two cases $|w| < k$ and $|w| \geq k$.

- For $|w| < k$
  Assume $w \in L^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$. Consequently with (17),

  $$w \in P_i^{\widehat{\mathcal{I}}_{X^k,\emptyset}} \text{ and } w \in (\forall r_j.A_j)^{\widehat{\mathcal{I}}_{X^k,\emptyset}}, \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.$$

  The assumption $|w| < k$ implies $|wr_j| \leq k$ for all $1 \leq j \leq m$. With Lemma 6.4 it follows that

  $$w \in P_i^{\widehat{\mathcal{I}}_{X^k,\emptyset}} \text{ and } wr_j \in A_j^{\widehat{\mathcal{I}}_{X^k,\emptyset}}, \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.$$

  Lemma 6.2 now yields

  $$w \in P_i^{\mathcal{I}_{X^k,\mathcal{T}}} \text{ and } wr_j \in A_j^{\mathcal{I}_{X^k,\mathcal{T}}}, \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.$$

  Thus, $w \in L^{\mathcal{I}_{X^k,\mathcal{T}}}$ and since $\mathcal{I}_{X^k,\mathcal{T}}$ is a model of $\mathcal{T}$ it follows $w \in R^{\mathcal{I}_{X^k,\mathcal{T}}}$. By assumption on the form of R (18) we get

  $$w \in P_{i'}^{\mathcal{I}_{X^k,\mathcal{T}}} \text{ and } wr_{j'} \in A_{j'}^{\mathcal{I}_{X^k,\mathcal{T}}}, \text{ for all } 1 \leq i' \leq n' \text{ and } 1 \leq j' \leq m'.$$

  Again using Lemma 6.2 and Lemma 6.4 we obtain

  $$w \in P_{i'}^{\mathcal{I}_{X^k,\emptyset}} \text{ and } wr_{j'} \in A_{j'}^{\mathcal{I}_{X^k,\emptyset}}, \text{ for all } 1 \leq i' \leq n' \text{ and } 1 \leq j' \leq m'.$$

  $$w \in P_{i'}^{\widehat{\mathcal{I}}_{X^k,\emptyset}} \text{ and } wr_{j'} \in A_{j'}^{\widehat{\mathcal{I}}_{X^k,\emptyset}}, \text{ for all } 1 \leq i' \leq n' \text{ and } 1 \leq j' \leq m'.$$

  and thus $w \in R^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$.

- For $|w| \geq k$.
  Let $n = |w| - k$ and
  $$M := \bigsqcap_{w \in B^{\widehat{\mathcal{I}}_n}, B \in N_C} \tag{19}$$
  By induction on n we show that

  $M \sqsubseteq_{\mathcal{T}} A$ for some $A \in N_C$ implies that A *is a conjunct in* M.

  - For $n = 0$:
    It follows that $|w| = k$ and $\widehat{\mathcal{I}}_0 = \mathcal{I}_{X^k,\emptyset}$. With Lemma 6.2, $w \in M^{\mathcal{I}_{X^k,\emptyset}}$ implies $w \in M^{\mathcal{I}_{X^k,\mathcal{T}}}$ and because $M \sqsubseteq_{\mathcal{T}} A$ and $\mathcal{I}_{X^k,\mathcal{T}}$ it follows that $w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$. Lemma 6.2 implies $w \in A^{\mathcal{I}_{X^k,\emptyset}}$ and therefore $w \in A^{\widehat{\mathcal{I}}_0}$, which implies that A is a conjunct in M by definition of M (see (19)).
  - $n \to n + 1$:
    Assume w = w'r and with $n = |w'| - k$ and $n + 1 = |w| - k$. Let
    $$Q' := \bigsqcap_{w' \in B^{\widehat{\mathcal{I}}_n}, B \in N_C} \text{ and } Q := \bigsqcap_{w \in B^{\widehat{\mathcal{I}}_{n+1}}, B \in N_C}$$
    By definition of $\widehat{\mathcal{I}}_{n+1}$ and $w = w'r$ with $n + 1 = |w| - k$ it holds that $Q' \sqsubseteq_{\mathcal{T}} \forall r.Q$. Obviously, $Q' \sqsubseteq_{\mathcal{T}} \forall r.Q$ and $\forall Q \sqsubseteq_{\mathcal{T}} A$ implies $Q' \sqsubseteq_{\mathcal{T}} \forall r.A$. By definition of $\widehat{\mathcal{I}}_{n+1}$ it follows that $w = w'r \in A^{\widehat{\mathcal{I}}_{n+1}}$ and therefore A is a conjunct in Q.
    Suppose $L \sqsubseteq R \in \mathcal{T}$ and $w \in L^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$. By assumption on the form of L (17) we have
    $$w \in P_i^{\widehat{\mathcal{I}}_{X^k,\emptyset}} \text{ and } wr_j \in A_j^{\widehat{\mathcal{I}}_{X^k,\emptyset}}, \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.$$
    Lemma 6.4 and $n = |w| - k$ with $|w| \geq k$ yields
    $$w \in P_i^{\widehat{\mathcal{I}}_n} \text{ and } wr_j \in A_j^{\widehat{\mathcal{I}}_{n+1}}, \text{ for all } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.$$
    It follows that the concept names P_i, for all $1 \leq i \leq n$ are conjuncts in M (see (19)). Obviously, it holds that
    $$M \sqsubseteq_{\mathcal{T}} P_i \text{ for all } 1 \leq i \leq n.$$
    Since $n = |w| - k$ and $wr_j \in A_j^{\widehat{\mathcal{I}}_{n+1}}$, for all $1 \leq j \leq m$, the definition of $\widehat{\mathcal{I}}_{n+1}$ and (19) implies
    $$M \sqsubseteq_{\mathcal{T}} \forall r_j.A_j, \text{ for all } 1 \leq j \leq m.$$
    Thus we have $M \sqsubseteq_{\mathcal{T}} L$. Since $L \sqsubseteq R \in \mathcal{T}$ we have also
    $$M \sqsubseteq_{\mathcal{T}} \forall r'_{j'}.A'_{j'}$$
    for all value restrictions in R (see (18)). Consequently by definition of $\widehat{\mathcal{I}}_{n+1}$ it follows that
    $$wr'_{j'} \in A'^{\widehat{\mathcal{I}}_{n+1}}_{j'}, \text{for all } 1 \leq j' \leq m' \tag{20}$$

And likewise we have that $M \sqsubseteq_{\mathcal{T}} L$ and $L \sqsubseteq R \in \mathcal{T}$ implies

$$M \sqsubseteq_{\mathcal{T}} P'_{i'}, \text{ for all } 1 \leq i' \leq n'$$

for all concept names on top level of R. Consequently the names $P'_{i'}$, for all $1 \leq i' \leq n'$, are conjuncts in M as shown above. Thus we have $w \in P_{i'}'^{\widehat{\mathcal{I}}_n}$. Together with (20) and Lemma 6.4 it is implied that $w \in R^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$.

□

Last, we show that actually $\widehat{\mathcal{I}}_{X^k,\emptyset}$ is equal to $\mathcal{I}_{X^k,\mathcal{T}}$.

Lemma 6.6. $\widehat{\mathcal{I}}_{X^k,\emptyset} = \mathcal{I}_{X^k,\mathcal{T}}$.

Proof:
Lemma 6.5 implies that $\widehat{\mathcal{I}}_{X^k,\emptyset}$ is a functional model of X^k and $\mathcal{T}$. Since $\widehat{\mathcal{I}}_{X^k,\mathcal{T}}$ is the LFM of X^k and $\mathcal{T}$, by Definition 3.4 $\mathcal{I}_{X^k,\mathcal{T}} \subseteq \widehat{\mathcal{I}}_{X^k,\emptyset}$. It remains to be shown that $\widehat{\mathcal{I}}_{X^k,\emptyset} \subseteq \mathcal{I}_{X^k,\mathcal{T}}$. Let $A \in N_C$ and $w \in N_R^*$. We have to show that

$$w \in A^{\widehat{\mathcal{I}}_{X^k,\emptyset}} \text{ implies } w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$$

Due to Lemma 6.2 and 6.4 this holds if $|w| \leq k$. We have to prove this also for the case $|w| \leq k$. The proof is by induction on n with $n = |w| - k$.

- $n = 1$: Assume w = w'r with $|w'| = k$. Due to Lemma 6.4 and $|w'| - k = 1$ we have that $w'r \in A^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$ implies $w'r \in A^{\widehat{\mathcal{I}}_1}$. By definition of $\widehat{\mathcal{I}}_1$ and $w = w'r$ it follows that

$$Q \sqsubseteq_{\mathcal{T}} \forall r.A \text{ with } Q = \bigsqcap_{w' \in B^{\widehat{\mathcal{I}}_0}, B \in N_C} B.$$

  Since $w' \in Q^{\widehat{\mathcal{I}}_0}$ and $\widehat{\mathcal{I}}_0 = \widehat{\mathcal{I}}_{X^k,\emptyset}$, Lemma 6.2 and $|w'| = k$ imply that $w' \in Q^{\mathcal{I}_{X^k,\mathcal{T}}}$. Since $\mathcal{I}_{X^k,\mathcal{T}}$ is a model of $\mathcal{T}$, $Q \sqsubseteq_{\mathcal{T}} \forall r.A$ and $w' \in Q^{\mathcal{I}_{X^k,\mathcal{T}}}$ yield $w'r \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$.

- $n \to n{+}1$ Assume $w = w'r$ for some $r \in N_R$ and $w' \in N_R^*$ with $n := |w'| - k$ and $n+1 = |w| - k$. Due to Lemma 6.4 and $n + 1 = |w| - k$ we have that $w \in A^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$ implies $w \in A^{\widehat{\mathcal{I}}_{n+1}}$. By definition of $\widehat{\mathcal{I}}_{n+1}$ and $w = w'r$ it follows that

$$Q \sqsubseteq_{\mathcal{T}} \forall r.A \text{ with } Q = \bigsqcap_{w' \in B^{\widehat{\mathcal{I}}_n}, B \in N_C} B.$$

  Since $w' \in Q^{\widehat{\mathcal{I}}_n}$ and $n = |w'| - k$, Lemma 6.4 implies that $w' \in Q^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$. Note that Q is a conjunction of concept names. Therefore the induction hypothesis for w' implies $w' \in Q^{\mathcal{I}_{X^k,\mathcal{T}}}$. Since $\mathcal{I}_{X^k,\mathcal{T}}$ is a model of $\mathcal{T}$, $Q \sqsubseteq_{\mathcal{T}} \forall r.A$ and $w' \in Q^{\mathcal{I}_{X^k,\mathcal{T}}}$ imply that $w'r \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$.

□

We have defined the notion of extended k-subtree which turns out, has the same structure as the LFM of X^k w.r.t. $\mathcal{T}$. We continue our investigation to decide the existence of lcs by considering

Example 5.5. According to that example, the $lcs_{\mathcal{T}_{ex2}}(C, D)$ is captured through the equality between the intersection model $\mathcal{I}_{C,\mathcal{T}_{ex2}} \cap \mathcal{I}_{D,\mathcal{T}_{ex2}}$ and the LFM of a concept A w.r.t. $\mathcal{T}_{ex2}$.

Still on the same example, in particular for the TBox $\mathcal{T}_{ex}$, the $lcs_{\mathcal{T}_{ex}}(C, D)$ is not captured through the equality between the intersection model $\mathcal{I}_{C,\mathcal{T}_{ex}} \cap \mathcal{I}_{D,\mathcal{T}_{ex}}$ and the LFM of $A \sqcap \forall r.A$ w.r.t. $\mathcal{T}_{ex}$. The equality for the label of w, both in $\mathcal{I}_{C,\mathcal{T}_{ex2}} \cap \mathcal{I}_{D,\mathcal{T}_{ex2}}$ and $\mathcal{I}_{E,\mathcal{T}_{ex2}}$, where $E = lcs_{\mathcal{T}_{ex2}}(C, D)$, for all $w \in N_R^*$ leads us to formalize this consideration in a general way through the following definition.

Definition 6.7. *Let $\mathcal{I}$ be a functional model of a TBox and $w \in \Delta^{\mathcal{I}} = N_R^*$ and $Q = \bigsqcap_{\substack{w \in B^{\mathcal{I}},\\ B \in N_C}} B$. w is label synchronous in $\mathcal{I}$ iff*

$$(\mathcal{I}, w) = \mathcal{I}_{Q,\mathcal{T}}$$

The use of the word "synchronous" in the definition above means that there is a sameness or synchronization between the label of an element w of $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}})$ and the label of the root element ε_Q in $\mathcal{I}_{Q,\mathcal{T}}$. We simply call an element that does not satisfy this definition is called as an *label-asynchronous element.* This definition also states that if an element w is label-synchronous, then $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, w)$ represents the LFM of a concept w.r.t $\mathcal{T}$. It is obvious to see that the concept is the concept $Q = \bigsqcap_{\substack{w \in B^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}},\\ B \in N_C}} B$.

The next lemmas states the properties of label-synchronous elements. First, we show that the r-successor of a label-synchronous element is also label-synchronous for all $r \in N_R$.

Lemma 6.8. *Let $\mathcal{I}$ be a functional model of a TBox and $w \in \Delta^{\mathcal{I}} = N_R^*$. If w is label-synchronous in $\mathcal{I}$, then all its successors are label-synchronous.*

Proof:
If w is label-synchronous, then there is a concept $Q = \bigsqcap_{\substack{w \in B^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}},\\ B \in N_C}} B$ such that

$$(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, w) = (\mathcal{I}_{Q,\mathcal{T}}), \tag{21}$$

which implies that $wr \in A^{(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, w)}$ iff $r \in A^{\mathcal{I}_{Q,\mathcal{T}}}$, for all $r \in N_R$. Therefore, for all successors of w, we have

$$Q' = \bigsqcap_{\substack{wr \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}},\\ A \in N_C}} A \quad = \bigsqcap_{\substack{r \in A^{\mathcal{I}_{Q,\mathcal{T}}},\\ A \in N_C}} A \tag{22}$$

By definition of 3.1, we have $Q \sqsubseteq_{\mathcal{T}} \forall r.Q'$. We have to show that wr is label-synchronous, which means that it is enough to show that $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr) = \mathcal{I}_{Q',\mathcal{T}}$.

It is clear that $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr)$ is a functional model of concept Q' w.r.t. $\mathcal{T}$. Since $\mathcal{I}_{Q',\mathcal{T}}$ is the LFM of concept Q' w.r.t. $\mathcal{T}$, we have $\mathcal{I}_{Q',\mathcal{T}} \subseteq (\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr)$. It remains to show that $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr) \subseteq \mathcal{I}_{Q',\mathcal{T}}$ or for all $x \in \Delta^{(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr)}$, where $x \in N_R^*$, and all $A \in N_C$, we have

$$x \in A^{(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr)} \text{ implies } x \in A^{\mathcal{I}_{Q',\mathcal{T}}}.$$

We show the claim above by induction on $|x|$.

- $|x| = 0$
  By (22), we know that for all $A \in N_C$, $\varepsilon \in A^{(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr)}$ implies $\varepsilon \in A^{\mathcal{I}_{Q',\mathcal{T}}}$.

- $|x| > 0$
  Now let $x = x'r$, $x \in A^{(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr)}$, and $x' \in P_i^{(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, wr)}$, where $P_i \in N_C$, for all $1 \leq i \leq n$. By induction hypothesis, we know that $x' \in P_i^{\mathcal{I}_{Q',\mathcal{T}}}$. Now assume that $x \notin A^{\mathcal{I}_{Q',\mathcal{T}}}$ which implies $Q' \not\sqsubseteq_{\mathcal{T}} \forall x.A$ by definition 3.1. Moreover,

  $$\text{Since } Q \sqsubseteq_{\mathcal{T}} \forall r.Q', \text{ it is also followed that } Q \not\sqsubseteq_{\mathcal{T}} \forall rx.A. \tag{23}$$

  By (21), we know that for all $A \in N_C$, $x \in A^{(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, w)}$ implies $x \in A^{\mathcal{I}_{Q,\mathcal{T}}}$ and by definition 3.1, we have $Q \sqsubseteq_{\mathcal{T}} \forall rx.A$, which is a contradiction to our assumption (23). Therefore $x \in A^{\mathcal{I}_{Q',\mathcal{T}}}$.

□

Then, let us show that all elements with the depth $\geq k$ in $\mathcal{I}_{X^k,\mathcal{T}}$ are label-synchronous in $\mathcal{I}_{X^k,\mathcal{T}}$.

Lemma 6.9. *Let $k \in \mathbb{N}$. For all $w \in N_R^*$ with $|w| \geq k$ it holds that w is label-synchronous in $\mathcal{I}_{X^k,\mathcal{T}}$.*

Proof:
Let $w \in N_R^*$ with $|w| \geq k$. We show that w is label-synchronous in $\widehat{\mathcal{I}}_{X^k,\emptyset}$. Let

$$Q := \bigsqcap_{w \in B^{\widehat{\mathcal{I}}_{X^k,\emptyset}}, B \in N_C} B \tag{24}$$

We have to show that $(\widehat{\mathcal{I}}_{X^k,\emptyset}, w) = \mathcal{I}_{Q,\mathcal{T}}$. It can be shown that $\varepsilon \in Q^{(\widehat{\mathcal{I}}_{X^k,\emptyset}, w)}$. By Lemma 6.5 $\widehat{\mathcal{I}}_{X^k,\emptyset}$ is a model of $\mathcal{T}$ and by Lemma 3.7 $(\widehat{\mathcal{I}}_{X^k,\emptyset}, w)$ is also a model of $\mathcal{T}$. It follows that $(\widehat{\mathcal{I}}_{X^k,\emptyset}, w)$ is a functional model of Q w.r.t. $\mathcal{T}$. Therefore, $\mathcal{I}_{Q,\mathcal{T}} \subseteq (\widehat{\mathcal{I}}_{X^k,\emptyset}, w)$. It remains to be shown that $(\widehat{\mathcal{I}}_{X^k,\emptyset}, w) \subseteq \mathcal{I}_{Q,\mathcal{T}}$. Let $A \in N_C$ and $u \in N_R^*$. We show by induction on $|u|$ that $u \in A^{(\widehat{\mathcal{I}}_{X^k,\emptyset}, w)}$ implies $u \in A^{\mathcal{I}_{Q,\mathcal{T}}}$.

- $|u| = 0$
  Let $n = |w| - k$. Lemma 6.4 implies that Q (see (24)) satisfies

  $$Q := \bigsqcap_{w \in B^{\widehat{\mathcal{I}}_n}, B \in N_C} B$$

  $u = \varepsilon \in A^{(\widehat{\mathcal{I}}_{X^k,\emptyset}, w)}$ implies $w \in A^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$, which implies $w \in A^{\widehat{\mathcal{I}}_n}$ and A is a conjunct in Q. Since $u = \varepsilon \in Q^{\mathcal{I}_{Q,\mathcal{T}}}$ it follows that $u = \varepsilon \in A^{\mathcal{I}_{Q,\mathcal{T}}}$.

- $|u| > 0$
  Let $u = u'r$ for some $r \in N_R$ and $u' \in N_R^*$ and let $n = |wu'| - k$. We have to show that $u'r \in A^{(\widehat{\mathcal{I}}_n,w)}$ implies $u'r \in A^{\mathcal{I}_{Q,\mathcal{T}}}$.
  $u'r \in A^{(\widehat{\mathcal{I}}_{X^k,\emptyset},w)}$ implies $wu'r \in A^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$. Lemma 6.4 implies that $wu'r \in A^{\widehat{\mathcal{I}}_{n+1}}$ because $n = |wu'| - k$ and therefore $n + 1 = |wu'| - k$. By definition $\widehat{\mathcal{I}}_{n+1}$ and since $wu'r \in A^{\widehat{\mathcal{I}}_{n+1}}$ we have
  $$M \sqsubseteq_{\mathcal{T}} \forall r.A \text{ with } M = \bigsqcap_{wu' \in B^{\widehat{\mathcal{I}}_n}, B \in N_C} B.$$
  It follows that $wu' \in M^{\widehat{\mathcal{I}}_n}$ implies $wu' \in M^{\widehat{\mathcal{I}}_{X^k,\emptyset}}$ by Lemma 6.4 and $n = |wu'| - k$. Consequently, by definition of $\widehat{\mathcal{I}}_{X^k,\emptyset}$, we get $u' \in M^{(\widehat{\mathcal{I}}_{X^k,\emptyset},w)}$. Since $|u'| \leq |u|$ and M is a conjunction of concept names the induction hypothesis yields $u' \in M^{\mathcal{I}_{Q,\mathcal{T}}}$. Because $\mathcal{I}_{Q,\mathcal{T}}$ is a model of $\mathcal{T}$ and $M \sqsubseteq_{\mathcal{T}} \forall r.A$ holds, it follows that $u'r \in A^{\mathcal{I}_{Q,\mathcal{T}}}$. Since by assumption we have $u = u'r$ this finishes the proof of the induction step.

□

Furthermore, the equality between $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and $\mathcal{I}_{X^k,\mathcal{T}}$ is also influenced by the fact that if all elements $w \in N_R^*$ with $|w| \geq k$ are label-synchronous in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$.

Lemma 6.10. *If there exists a $k \in \mathbb{N}$ such that for all $w \in N_R^*$ with $|w| \geq k$ the element w is label-synchronous in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$, then it holds that $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^k,\mathcal{T}}$.*

Proof:
Since X^k is the k-$lcs_{\mathcal{T}}(C, D)$, it implies that $\mathcal{I}_{X^k,\mathcal{T}} \subseteq \mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. Now, it remains to show that for all $w \in N_R^*$ and all $A \in N_C$, we have

$$w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}} \text{ implies } w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}.$$

By Lemma 3.10 and 6.2, we directly have

$$\text{For all } w \in N_R^* \text{ with } |w| < k \text{ and all } A \in N_C, w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}} \text{ implies } w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}.$$

Now it remains to show that for all $w \in N_R^*$ with $|w| \geq k$ and all $A \in N_C$, $w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$ implies $w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$. Here we also assume that w is label-synchronous and we proof by induction on n, where $n = |w| - k$.

- $n = 0$
  It means that $|w| = k$ and by Lemma 6.9 w is label-synchronous. Then, by Lemma 3.10 and 6.2, we know that $w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$ implies $w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$, for all $A \in N_C$.

- $n \to n + 1$
  Let $w = w'r$, where $r \in N_R$, $w' \in N_R^*$, and $w \in A^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$. Let $w' \in B_i^{\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}}$, where $B_i \in N_C$, for all $1 \leq i \leq n$ and by our assumption w' is label-synchronous. By Definition 6.7, we know that

$$(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, w') = \mathcal{I}_{Q,\mathcal{T}}, \text{ where } Q = B_1 \sqcap \ldots \sqcap B_n.$$

By Lemma 6.8, it also implies that $w \in A^{\mathcal{I}_{Q,\mathcal{T}}}$ and $Q \sqsubseteq_{\mathcal{T}} \forall r.A$. By induction hypothesis, we know that $w' \in B_i^{\mathcal{I}_{X^k,\mathcal{T}}}$, for all $1 \leq i \leq n$. This means that $w' \in (\forall r.A)^{\mathcal{I}_{X^k,\mathcal{T}}}$ and $w \in A^{\mathcal{I}_{X^k,\mathcal{T}}}$ follows.

□

As a consequence of Lemma 6.9 and 6.10, we have the following lemma:

Lemma 6.11. *Let $k > 0$. $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^k,\mathcal{T}}$ iff for all $w \in N_R^*$ with $|w| \geq k$, it holds that w is label-synchronous in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and $\mathcal{I}_{X^k,\mathcal{T}}$.*

Proof: A consequence of Lemma 6.9 and 6.10.

□

Now, let $\mathcal{G}$ be the graph model of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ be the tree-unraveling of $\mathcal{G}$. By Lemma 4.4, we know that $\mathcal{G}$ and $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ are simulation-equivalent. Obviously, some elements of $\mathcal{G}$ may be also label-synchronous or label-asynchronous. It is written formally in the following definition.

Definition 6.12. *Let $[w] \in \Delta^{\mathcal{G}}$ and $Q = \bigsqcap_{\substack{[w] \in B^{\mathcal{G}}, \\ B \in N_C}} B$. $[w]$ is label-synchronous in $\mathcal{G}$ iff*

$$(\mathcal{G}, [w]) \simeq (\mathcal{J}_{Q,\mathcal{T}}, [\varepsilon])$$

As a consequence, we also have the following properties for all elements that are label-synchronous in $\mathcal{G}$.

Lemma 6.13.

1. *$w \in N_R^*$ is label-synchronous in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ iff $[w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \mathcal{G}$ is label-synchronous in $\mathcal{G}$.*
2. *If $[w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \mathcal{G}$ is label-synchronous in $\mathcal{G}$, then all its successors are also label-synchronous in $\mathcal{G}$.*

Proof:

1. $w \in N_R^*$ is label-synchronous in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$

   iff $(\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}, w) = \mathcal{I}_{Q,\mathcal{T}}$, where $Q = \bigsqcap_{\substack{w \in B^{\mathcal{I}}, \\ B \in N_C}} B$ (Definition 6.7)

   iff $(\mathcal{G}, [w]) \simeq (\mathcal{J}_{Q,\mathcal{T}}, [\varepsilon])$, where $Q = \bigsqcap_{\substack{[w] \in B^{\mathcal{G}}, \\ B \in N_C}} B$. (By Lemma 5.6 and 3.19).

   iff $[w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \mathcal{G}$ is label-synchronous in $\mathcal{G}$ (By Definition 6.12).

2. If $[w]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}} \in \mathcal{G}$ is label-synchronous in $\mathcal{G}$, then $w \in \mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is also label-synchronous in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ by Lemma 6.13. For the r-successor wr of w, for all $r \in N_R$, wr is also label-synchronous in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ by Lemma 6.8. As a consequence, the r-successor $[wr]$ of $[w]$, for all $r \in N_R$, is also label-synchronous in $\mathcal{G}$ by 6.13. □

If $\mathcal{G}$ has a cycle that contains an element $[u]$ which is label-asynchronous, then its successor $[v]$ is label-synchronous, then there will be a path in that cycle, starting from $[v]$ that goes back to $[u]$, which is a contradiction for the lemma above because $[u]$ is also a descendant of $[v]$ and $[u]$ has to be a label-synchronous element which is a contradiction for Lemma 6.8. It means that whenever we have a cycle in $\mathcal{G}$ that contains a label-asynchronous element, then all elements in the cycle are also label-asynchronous. This leads to the following characterization defined as the main theorem for the existence of the lcs.

Theorem 6.14. *Let $\mathcal{G}$ be the graph model of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$.*
The $lcs_{\mathcal{T}}(C, D)$ exists iff all cycles in $\mathcal{G}$ only contains label-synchronous elements.

Proof:

"⇐": If all cycles in $\mathcal{G}$ only contains label-synchronous elements, then if we unravel $\mathcal{G}$, then we obtain $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and all paths in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ have a finite prefix of label-asynchronous elements. Let $m-1$ be the length of the maximal finite prefix of label-asynchronous elements in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and from position m on only contains label-synchronous elements. It means that for all $w \in N_R^*$ with $|w| \geq m$, w is label-synchronous and it holds that $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^m,\mathcal{T}}$ by Lemma 6.11. Therefore, the $lcs_{\mathcal{T}}(C, D)$ exists.

"⇒": If the $lcs_{\mathcal{T}}(C, D)$ exists, then there is a $k \in \mathbb{N}$ such that $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^k,\mathcal{T}}$. By Lemma 6.11, it means that for all $w \in N_R^*$, with $|w| \geq k$, it holds that w is label-synchronous in $\mathcal{I}_{X^k,\mathcal{T}}$ and $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. By Lemma 4.4, we know that $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is the tree-unraveling of $\mathcal{G}$. Then, in order to show that all cycles in $\mathcal{G}$ only contains label-synchronous elements, we prove it by contradiction. Assume there is a cycle that contains label-asynchronous elements, it implies that all elements in this cycle are label-asynchronous. If we unravel $\mathcal{G}$, then we will obtain an infinite path, starting from ε, that only contains label-asynchronous elements. It implies that all elements $w \in N_R^*$, with $|w| \geq k$, are also label-asynchronous which is a contradiction to Lemma 6.11. □

Since we know the number of cycles in $\mathcal{G}$ is finite and it is enough to check whether all cycles in each cycle are label-synchronous, the theorem above provides us a decision procedure to solve the problem for the existence of the lcs in a finite time. The theorem above also implicitly addresses the claim in Corollary 5.8 to know how to obtain the depth k of the graph model, such that the lcs exists. It gives rise to a practical solution for obtaining the number k that we can use to measure the size or the role-depth of the lcs as well as to provide a solution for Problem III in Introduction. It is described in a more detail in the next section.

7 Upper Bound for the Role-Depth of the Least Common Subsumer

Let us recall Problem III in Introduction:

III. If the $lcs_{\mathcal{T}}(C, D)$ exists, then what is the lcs? And how big is the size of the lcs?

The problem above can be handled by the following lemma.

Lemma 7.1. *Let $\mathcal{G}$ be the graph of the intersection model $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$, $n = |\Delta^{\mathcal{G}}|$, and $N_{C,\mathcal{T}}$ be the set of concept names occurring in the TBox $\mathcal{T}$. It holds that*

1. *If the $lcs_{\mathcal{T}}(C, D)$ exists, then $(\mathcal{G}, [\varepsilon]) \simeq (\mathcal{J}_{X^{n+1},\mathcal{T}}, [\varepsilon])$.*
2. *$rd(lcs_{\mathcal{T}}(C, D)) \leq 2^{2\times|N_{C,\mathcal{T}}|+1}$*

Proof:

1. Since $\mathcal{G}$ is the graph model of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and both of them are simulation-equivalent, we can compute the $n+1$-characteristic concept X^{n+1} of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. We choose the number $n+1$ because if we unravel $\mathcal{G}$ to obtain $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$, then all finite paths of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$, with the length $n+1$, contain repeated elements. It is due to the fact that there are only n elements in $\mathcal{G}$. It also means that all elements on the depth $n+1$ in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ occur in the cycles of $\mathcal{G}$. Since the $lcs_{\mathcal{T}}(C, D)$ exists, we also know that all elements on the depth $n+1$ in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ are label-synchronous. By Lemma 6.11, it implies that $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}} = \mathcal{I}_{X^{n+1},\mathcal{T}}$. Finally, by Lemma 5.6, we can infer that $(\mathcal{G}, [\varepsilon]) \simeq (\mathcal{J}_{X^{n+1},\mathcal{T}}, [\varepsilon])$.

2. We have assumed that C and D are concept names occurring in $\mathcal{T}$. Because the elements of $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$ are sets of concept names occurring in $\mathcal{T}$, it implies that the number of elements in $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$ are bounded by $2^{|N_{C,\mathcal{T}}|}$, respectively. Next, we compute the product of $\mathcal{J}_{C,\mathcal{T}}$ and $\mathcal{J}_{D,\mathcal{T}}$ to obtain the subgraph $\mathcal{G}$ where all elements of $\mathcal{G}$ are reachable from $([\varepsilon]^{\sim_{\mathcal{I}_{C,\mathcal{T}}}}, [\varepsilon]^{\sim_{\mathcal{I}_{D,\mathcal{T}}}})$. It implies that $|\Delta^{\mathcal{G}}| \leq 2^{|N_{C,\mathcal{T}}|} \times 2^{|N_{C,\mathcal{T}}|}$. By Claim 1 of Lemma 7.1, we have to go one step further down in $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ in order to guarantee that $(\mathcal{G}, [\varepsilon]) \simeq (\mathcal{J}_{X^{n+1},\mathcal{T}}, [\varepsilon])$ and X^{n+1} is the $lcs_{\mathcal{T}}(C, D)$. Therefore, it implies that $rd(lcs_{\mathcal{T}}(C, D)) \leq 2^{2\times|N_{C,\mathcal{T}}|+1}$.

By considering two claims in this lemma, we finally have two answers for Problem III that X^{n+1} is the $lcs_{\mathcal{T}}(C, D)$ and the upper bound for the role-depth of $lcs_{\mathcal{T}}(C, D)$ is $2^{2\times|N_{C,\mathcal{T}}|+1}$.

□

8 Conclusions and Future Works

8.1 Conclusions

In this thesis, we studied a problem for the existence of the lcs w.r.t. general $\mathcal{FL}_0$-TBoxes. Accordingly, we presented characterizations for the existence of the lcs. Additionally, we devised a decision procedure to find the role-depth bounded lcs or the lcs, if it exists. In order to support the characterization for the existence and design a decision procedure to compute the lcs, the necessary notions are needed.

We assume that the inputs for all problems for the existence of the lcs and computing the lcs, if it exists are two concept names occurring in a TBox in PANF. We begin by computing the least functional models (LFMs) of input concepts w.r.t. a given TBox with infinitely many domain elements [Pen15], which eventually can be alternatively replaced by a graph model that only has finite number of domain elements. Since we follow the conditional setting for the existence of the lcs in $\mathcal{EL}$ [ZT13], we also introduced a special relation, namely simulation between functional interpretations or graph models. The subsumption problem w.r.t. general $\mathcal{FL}_0$-TBoxes, which is a basic inference for computing the lcs, can be characterized by means of a simulation.

Now we are ready to answer the three research questions mentioned in Introduction. Again, we always assume that the input concepts C and D are concept names occurring in a PANF TBox $\mathcal{T}$. The first question is described and handled as follows:

I. Let E be an $\mathcal{FL}_0$-concept. Is concept E the lcs of C and D w.r.t. $\mathcal{T}$?

We provide two approaches, which are actually semantically equivalent, for the problem above by means of LFMs and simulation as follows:

- An equality relationship between $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and $\mathcal{I}_{E,\mathcal{T}}$ provides us a characterization that E is the $lcs_{\mathcal{T}}(C, D)$.
- A simulation-equivalence between the corresponding graph models $\mathcal{G}$ and $\mathcal{J}_{E,\mathcal{T}}$ of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ and $\mathcal{I}_{E,\mathcal{T}}$, respectively, serves more conditions for the existence of lcs. Since only dealing with finite number of domain elements, this characterization offers any graph algorithm to make this problem decidable.

The approaches above provide us characterizations to address Problem II:

II. Does the lcs of C and D w.r.t. $\mathcal{T}$ exist?

The answer to check whether tbe $lcs_{\mathcal{T}}(C, D)$ exists is by looking for a $k \in \mathbb{N}$ such that $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ is equal to $\mathcal{I}_{X^k,\mathcal{T}}$, where X^k is the k-characteristic concept of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$. However, this characterization does not give us a decision problem since there are infinitely many k to check in finite time. Finally, the practical solution for the question above is by checking whether all cycles in the graph model $\mathcal{G}$ of $\mathcal{I}_{C,\mathcal{T}} \cap \mathcal{I}_{D,\mathcal{T}}$ only contains label-synchronous elements. It convinces us to see that the problem for existence of the $lcs_{\mathcal{T}}(C, D)$ is decidable. Then, it leads us to supply an algorithm to compute the $lcs_{\mathcal{T}}(C, D)$ and the size of the lcs, if the lcs exists, as well as to asnwer the last research question.

III. If the $lcs_{\mathcal{T}}(C, D)$ exists, then what is the lcs? and how big is the size of the lcs?

In order to solve this problem, we traverse the graph model $\mathcal{G}$ until we stop at a depth k such that all elements on this depth occur in the cycles of $\mathcal{G}$ and are label-synchronous. If this condition holds, then we know that the $lcs_{\mathcal{T}}(C, D)$ exists. Last, we show that the size of the lcs can be measured by the role-depth of the lcs that is always bounded by $2^{2\times|N_{C,\mathcal{T}}|+1}$, where $N_{C,\mathcal{T}}$ is the set of all concept names occurring in $\mathcal{T}$.

8.2 Future Works

As motivated in [PT11], the implementation for the results above by making a practical algorithm integrated with existing tools in OWL will be a promising work in the future. Moreover, on the theoretical side, the complexity problem for computing the upper bound of the role-depth of the lcs still be an open problem, which also leads us to investigate on how big the size of the lcs is in the size of input concepts and a given TBox. We also consider to compute generalizations with the same setting described in this thesis for a more expressive description logic language, such as $\mathcal{FLE}$. It was already initiated in [FRR99] which can be a basic foundation for deciding the existence of lcs, even though that work did not take any general $\mathcal{FLE}$-TBoxes into account.

Another computation of non-standard inferences, which is characterizing the existence of the most specific concept of an individual w.r.t. general $\mathcal{FL}_0$-TBoxes can be interestingly investigated in the future. It is due to the presence of cyclic ABoxes, in particular role assertions, that induces the occurrences of existential restrictions in a computed msc-concept, if it exists. Therefore, an alternative type of $\mathcal{FL}_0$-model, which is not functional anymore, has to be explored thoroughly in this work.

References

[Baa03] Baader, Franz: Least Common Subsumers and Most Specific Concepts in a Description Logic with Existential Restrictions and Terminological Cycles. In: Gottlob, Georg (Hrsg.) ; Walsh, Toby (Hrsg.): *Proceedings of the 18th International Joint Conference on Artificial Intelligence*, Morgan Kaufman, 2003, S. 319–324

[BBL05] Baader, Franz. ; Brandt, Sebastian. ; Lutz, Carsten.: Pushing the $\mathcal{EL}$ Envelope. In: *Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence IJCAI-05*. Edinburgh, UK : Morgan-Kaufmann Publishers, 2005

[BCM+03] Baader, Franz (Hrsg.) ; Calvanese, Diego (Hrsg.) ; McGuinness, Deborah L. (Hrsg.) ; Nardi, Daniele (Hrsg.) ; Patel-Schneider, Peter F. (Hrsg.): *The Description Logic Handbook: Theory, Implementation, and Applications*. New York, NY, USA : Cambridge University Press, 2003. – ISBN 0–521–78176–0

[BK98] Baader, Franz. ; Küsters, Ralf.: Computing the least common subsumer and the most specific concept in the presence of cyclic $\mathcal{ALN}$-concept descriptions. In: Herzog, O. (Hrsg.) ; Günter, A. (Hrsg.): *Proceedings of the 22nd Annual German Conference on Artificial Intelligence, KI-98* Bd. 1504. Bremen, Germany : Springer–Verlag, 1998 (Lecture Notes in Computer Science), S. 129–140

[BT01] Brandt, Sebastian. ; Turhan, Anni.-Yasmin. Using Non-standard Inferences in Description Logics — what does it buy me? In: *Proceedings of the KI-2001 Workshop on Applications of Description Logics (KIDLWS'01)*. Vienna, Austria : RWTH Aachen, September 2001 (CEUR-WS 44). – Proceedings online available from http://SunSITE.Informatik.RWTH-Aachen.DE/Publications/CEUR-WS/Vol-44/

[BTK03] Brandt, Sebastian ; Turhan, Anni-Yasmin ; Küsters, Ralf: Extensions of Non-standard Inferences to Description Logics with transitive Roles. In: Vardi, Moshe (Hrsg.) ; Voronkov, Andrei (Hrsg.): *Proceedings of the 10th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR 2003)*, Springer, 2003 (Lecture Notes in Computer Science)

[FRR99] Franz. Baader ; Ralf. Küsters ; Ralf. Molitor: Computing Least Common Subsumers in Description Logics with Existential Restrictions. In: T. Dean (Hrsg.): *Proceedings of the 16th International Joint Conference on Artificial Intelligence (IJCAI'99)*, Morgan Kaufmann, 1999, S. 96–101

[Gro04] Group, W3C OWL W.: *OWL Web Ontology Language*. `https://www.w3.org/TR/owl-features/`. Version: feb 2004

[Gro12] Group, W3C OWL W.: *OWL 2 Web Ontology Language*. https://www.w3.org/TR/owl2-overview/. Version: dec 2012

[HST99] Horrocks, Ian ; Sattler, Ulrike ; Tobies, Stephan: Practical Reasoning for Expressive Description Logics. In: *Proceedings of the 6th International Conference on Logic Programming and Automated Reasoning*. London, UK, UK : Springer-Verlag, 1999 (LPAR '99). – ISBN 3–540–66492–0, 161–180

[K98] Küsters, Ralf.: Characterizing the Semantics of Terminological Cycles in $\mathcal{ALN}$ using Finite Automata. In: *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, Morgan Kaufmann, 1998, S. 499–510

[KM00] Küsters, Ralf. ; Molitor, Ralf.: Computing Least Common Subsumers in $\mathcal{ALEN}$ / LuFG Theoretical Computer Science, RWTH Aachen. Germany, 2000 (00-07). – LTCS-Report. – See http://www-lti.informatik.rwth-aachen.de/Forschung/Reports.html.

[LPW10] Lutz, Carsten ; Piro, Robert ; Wolter, Frank: Enriching $\mathcal{EL}$-concepts with greatest fixpoints. In: *Proceedings of the 2010 Conference on ECAI*, 2010, S. 41–46

[LW10] Lutz, Carsten ; Wolter, Frank: Deciding Inseparability and Conservative Extensions in the Description Logic $\mathcal{EL}$. In: *J. Symb. Comput.* 45 (2010), Februar, Nr. 2, 194–228. http://dx.doi.org/10.1016/j.jsc.2008.10.007. – DOI 10.1016/j.jsc.2008.10.007. – ISSN 0747–7171

[Pen15] Pensel, Maximilian: *An Automata-Based Approach for Subsumption w.r.t. General Concept Inclusions in the Description Logic $\mathcal{FL}_0$*, Technische Universtät Dresden, Diplomarbeit, 2015. https://lat.inf.tu-dresden.de/research/mas/Pen-Mas-15.pdf

[PT11] Peñaloza, Rafael. ; Turhan, Anni.-Yasmin.: A Practical Approach for Computing Generalization Inferences in $\mathcal{EL}$. In: Grobelnik, Marko (Hrsg.) ; Simperl, Elena (Hrsg.): *Proceedings of the 8th European Semantic Web Conference (ESWC'11)*, Springer-Verlag, 2011 (Lecture Notes in Computer Science)

[ZT13] Zarriess, Benjamin ; Turhan, Anni-Yasmin: Most Specific Generalizations w.r.t. General $\mathcal{EL}$-TBoxes. In: *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI'13)*. Beijing, China : AAAI Press, 2013