TECHNISCHE UNIVERSITÄT DRESDEN

Fakultät Informatik Institut für Theoretische Informatik, Professur für Automatentheorie

Complexity and Expressive Power of Description Logics with Numerical Constraints

Filippo De Bortoli

Dissertation

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Dissertation

zur Erlangung des akademischen Grades

Doktor der Ingenieurwissenschaften (Dr.Ing-.)

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Eingereicht am: 8. Mai 2025
Verteidigt am: 11. August 2025

Abstract

Standard Description Logics (DLs) can encode numerical aspects of domain-specific knowledge using number restrictions and concrete domains. The former are used to compare the number of role successors of an individual described by a given concept with a fixed natural number. The latter are used to assign concrete values (e.g. numbers) to an individual, which can be referenced using features and constrained using predefined predicates (e.g. numerical comparisons).

Recently, number restrictions were extended using the quantifier-free fragment of Boolean Algebra with Presburger Arithmetic (QFBAPA). In the resulting DL, called $\mathcal{ALCSCC}$, reasoning is as complex as with number restrictions, in spite of the increase in expressive power. In this thesis, we develop a method to study the expressive power of $\mathcal{ALCSCC}$, which has a semantics based on finitely branching interpretations, using the locality properties satisfied by first-order logic (FOL) over certain restricted classes of models (such as finite and finitely branching models) rather than compactness, which fails in the finitely branching case. We thus generalize our earlier work, in which we introduced a variant $\mathcal{ALCSCC}^\infty$ defined over arbitrary interpretations and analyzed its expressive power using a bisimulation-based characterization.

Early research in DL considered cardinality restrictions (CRs) that compare the number of individuals described by a concept with a fixed natural number. Similarly to number restrictions, CRs have been generalized using QFBAPA, obtaining extended CRs defined over finite interpretations, increasing the expressive power without increasing the complexity of reasoning. We prove that the complexity of reasoning with extended CRs over arbitrary interpretations is the same w.r.t. the finite variant, and lift the notion of bisimulation used for $\mathcal{ALCSCC}^\infty$ to study their expressive power, with additional results based on 0–1 laws and model-theoretic properties used to differentiate the expressive power of different logics. We characterize the subsets of these logics that are FOL-definable and prove that neither of the logics is fully FOL-definable.

It is known that ω-admissible concrete domains $\mathfrak{D}$, such as Allen's interval algebra, the region connection calculus RCC8, and the rational numbers with ordering and equality, yield decidable extensions $\mathcal{ALC}(\mathfrak{D})$ of $\mathcal{ALC}$. If the constraint satisfaction problem (CSP) of $\mathfrak{D}$ is decidable in exponential time, we show that reasoning in $\mathcal{ALC}(\mathfrak{D})$ is ExpTime-complete. We then look at two notions of expressive power for logics with concrete domains. One enables the comparison of logics with (possibly different) concrete domains, and we analyze it by using a bisimulation that accounts for the presence of feature values and using a locality-based method as in the case of $\mathcal{ALCSCC}$. The other, called abstract expressive power, looks at the classes of first-order interpretations that can be expressed using extensions of FOL and DLs with concrete domains, compared to what their counterparts without concrete domains can express. If $\mathfrak{D}$ only has unary predicates, the abstract expressive power remains within FOL if we are allowed to introduce auxiliary symbols, and we obtain decidability results for some fragments of FOL extended with $\mathfrak{D}$ as a by-product. If we can state equality between elements of $\mathfrak{D}$, on the other hand, the abstract expressive power of most first-order fragments extended by $\mathfrak{D}$ is beyond that of FOL, and the two-variable fragment with concrete domains becomes undecidable. We find sufficient conditions on $\mathfrak{D}$ such that these extensions satisfy first-order properties such as compactness.

Finally, we study $\mathcal{ALCOSCC}(\mathfrak{D})$, a combination of $\mathcal{ALCSCC}$ and concrete domains where we can additionally refer to specific individuals by name and use so-called feature roles. We show that the consistency problem for this DL is ExpTime-complete, assuming that the CSP of $\mathfrak{D}$ is decidable in exponential time. We show that many natural extensions to this DL, including a tighter integration of the concrete domain and number restrictions, lead to undecidability.

Statement of authorship

I hereby certify that I have authored this document entitled *Complexity and Expressive Power of Description Logics with Numerical Constraints* independently and without undue assistance from third parties. No other than the resources and references indicated in this document have been used. I have marked both literal and accordingly adopted quotations as such. There were no additional persons involved in the intellectual preparation of the present document. I am aware that violations of this declaration may lead to subsequent withdrawal of the academic degree.

Dresden, 8th May 2025

Filippo De Bortoli

Acknowledgments

I want to start by thanking my supervisor, Franz Baader. In the acknowledgments of my master's thesis, which he also supervised, I wrote that "he helped me by hinting at possible directions of work, challenging my intuitions when I was wrong, giving positive feedback when I was right and sharing a fair amount of time staring at a whiteboard". After some more years spent working together, that still applies, though the world has *slightly* changed and we went from whiteboards to Zoom meetings and coffee meetings every now and then. Plus, I feel that I learnt a lot more about the joys and intricacies of the world of academic research. And I was given more than one chance to keep my research going, for which I am extremely grateful. Thank you for all of this and for all that you taught me.

There are many people with whom I have shared the excitement and the frustration of the research process, either through conversations in our office, at the canteen or while drinking (too much) coffee. A heartfelt thank you to all the members of the logic-adjacent groups at TU Dresden and my teammates at the Chair of Automata Theory. In particular, I thank my QuantLA fellows and former officemates Jakub and Willi, and my colleagues and scientific collaborators Patrick and Stefan.

I want to thank the people that assisted me with the administrative and bureaucratic processes and especially Kerstin Achtruth and Petra Humann, who helped me countless times with a lot of kindness and while patiently deciphering my spoken and written German.

I am grateful to Johannes Möbius and the conversations we had, which allowed me to see things differently and face the challenges that I faced during this journey.

The start of my doctoral studies coincided with the beginning of my passion for bouldering, which later brought me to vertical and outdoors climbing. I want to thank all the people that shared this wonderful activity with me, and especially Ale, Max and Lucía for encouraging me to climb a lot and outdoors: *"der grüner Mann, der so laut schreit"* is grateful for that.

I am deeply grateful to my family for supporting me from far away, with the provvidential food packages and all the encouragement. Thank you Mamma, Papà, Stefy, Diego and of course thank you Brandy, Ginger and Whiskey.

Then comes the second family, the one that was born in Trento and that eventually spread all over Europe. I am happy that you are still part of my life, and thank you especially to Anna, Zambo and Jimmy for listening to my rants over the phone. We can do it!

...and then comes the third family, the one that was built over the years spent in Dresden, which gave me the opportunity to know some of the most wholesome humans on Earth. La

Marti, Ivan, Jack, Ele, Fabri, Cate, Anto, Monci, Albi, Irina, who were there first and filled my days with joy and laughter, and everybody who came afterwards and kept the good vibes rolling. I apologize for not writing all of your names, but there would be no space left for the thesis otherwise. Thank you from the bottom of my heart.

A huge thank you to all my friends in my hometown Verona and all over Italy and Europe: I am happy whenever I get to meet you, and I am happy we are still in touch after so many years.

Finally, I want to thank Julia. I am grateful that you are part of my life, that you support me, that you understand me, that we get to share a lot of nice and memorable experiences, and that we still share snacks every day. That's pretty romantic.

Funding. The work contained in this thesis was partially supported by the German Research Foundation (DFG) within the Research Training Group 1763 (QuantLA) and by the German Federal Ministry of Education and Research (BMBF, SCADS22B) and the Saxon State Ministry for Science, Culture and Tourism (SMWK) by funding the competence center for Big Data and AI "ScaDS.AI Dresden/Leipzig".

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1 Introduction

Description logics (DLs) [22, 13] are a well-investigated family of logic-based knowledge representation languages, which can be used to formalize the terminological knowledge of an application domain in a machine-processable way. They are designed to offer good tradeoffs between expressive power and complexity of reasoning and form the theoretical backbone of the Web ontology language OWL 2.[1] Successful applications of DLs to ontology formalization can be found in domains such as biology and medicine [60]. For instance, large medical ontologies such as SNOMED CT[2] and Galen[3] have been developed using appropriate DLs.

A key feature of DLs is the ability to construct descriptions of complex concepts (i.e., sets of individuals sharing certain properties) by combining *concept names* (unary predicates) and *role names* (binary predicates) using appropriate concept constructors. For example, the concept of a parent can be described as $\mathsf{Human} \sqcap \exists \mathsf{child}.\mathsf{Human}$, where the concept name Human and the role name child are combined using concept *conjunction* ($\sqcap$) and existential *role restriction* ($\exists r.C$). Knowledge about the relationship between concepts can then be expressed using concept inclusions (CIs), such as

$$\mathsf{Human} \sqcap \exists \mathsf{child}.\mathsf{Human} \sqsubseteq \exists \mathsf{eligible}.\mathsf{TaxBreak},$$

which says that parents are eligible for a tax break.

Such purely qualitative statements are not always sufficient to express quantitative information (e.g. the number of children required for a tax break) that is relevant for an application domain. To accommodate diverse application domains, the DL community has developed logics with different constructors, whose *expressive power* [6, 73] is tailored towards what is needed in these domains while leaving reasoning decidable. In many cases, however, the added expressive power turns out to be useful also in other applications. *Qualified number restrictions* [33, 62, 61] that constrain the number of role successors belonging to a certain concept by a fixed natural number can be employed in DLs to express such quantitative information. The CI $\mathsf{Human} \sqcap (\geqslant 3\, \mathsf{child}.\mathsf{Human}) \sqsubseteq \exists \mathsf{eligible}.\mathsf{TaxBreak}$ says that a tax break is available if one has at least three children. To express quantitative information about the whole domain of discourse we may use *cardinality restrictions (CRs)* [12, 96] which enable us e.g. to state that there are[4] at

[1] `https://www.w3.org/TR/owl2-overview/`
[2] `https://www.snomed.org/`
[3] `https://bioportal.bioontology.org/ontologies/GALEN`
[4] `https://ourworldindata.org/grapher/landline-internet-subscriptions`, last accessed 10/09/2024.

least thirty-seven million subscriptions to landline Internet in Germany:

$$|\mathsf{LegalGermanEntity} \sqcap \exists \mathsf{subscribedTo}.\mathsf{LandlineInternet}| \geqslant 37000000$$

Concrete domain reasoning [20] can represent a different type of quantitative information, where concrete objects such as numbers or strings can be assigned to individuals using partial functions called *features*. For example, a tax break might only be available if the annual salary is not too high. The CI $\mathsf{Human} \sqcap (\geq 3\,\mathsf{child}.\mathsf{Human}) \sqcap \exists \mathsf{salary}.<_{100{,}000} \sqsubseteq \exists \mathsf{eligible}.\mathsf{TaxBreak}$ specifies at least three children and an annual salary of less than 100,000 € as eligibility criteria for a tax break. The study of concrete domain restrictions was motivated by a mechanical engineering application [21]. Due to their usefulness in many application domains, they are included in the OWL 2 standard, albeit in the restricted form of unary concrete domains (called datatypes), where all predefined predicates have arity one [63].

We can readily show that by adding any of the quantitative constructors presented in the previous paragraph, we obtain an extension of $\mathcal{ALC}$ and of CIs. One way to prove this is to show that a given DL $\mathcal{L}_1$ can be expressed by another DL $\mathcal{L}_2$ using the same concept and role names by providing a semantic-preserving translation of $\mathcal{L}_1$ concept descriptions into $\mathcal{L}_2$, so that for example the $\mathcal{ALC}$ concept $\exists \mathsf{child}.\mathsf{Human}$ can be expressed by the concept $(\geqslant 1\,\mathsf{child}.\mathsf{human})$ in the DL $\mathcal{ALCQ}$ that extends $\mathcal{ALC}$ with qualified number restrictions. This same technique can be used to show that a DL, such as $\mathcal{ALC}$ or $\mathcal{ALCQ}$, is a fragment of first-order logic (FOL).

Proving that each of these extensions is strictly more expressive, on the other hand, is more challenging. The first formal investigation of the expressive power of DLs was performed in [5, 6], but in a rather ad hoc manner. More fundamental characterizations of the expressive power of various concept description languages up to the DL $\mathcal{ALC}$ based on the model-theoretic notion of *bisimulation* are given in [73]. This approach, pioneered by van Benthem [101] for the modal logic K (which is a syntactic variant of $\mathcal{ALC}$), characterizes a given DL as the fragment of FOL that is invariant under an appropriate notion of bisimulation. The notion of bisimulation thus provides a formal way to prove that one DL is more expressive than another, when using the same concept and role names.

Aside from an increase in terms of expressive power, the introduction of qualified number restrictions, cardinality restrictions and concrete domain restrictions in DL research created considerable algorithmic challenges. For $\mathcal{ALCQ}$, the extension of the basic DL $\mathcal{ALC}$ with qualified number restrictions, it was open for a decade whether the increase in expressivity also increases the complexity of reasoning if numbers in number restrictions are assumed to be represented in binary, until Tobies [99, 96] was able to show that this complexity is unchanged w.r.t. $\mathcal{ALC}$ (PSpace without and ExpTime with CIs). On the other hand, it turned out that the unrestricted use of transitive roles within number restrictions can cause undecidability [64]. The addition of CRs also increases the complexity of reasoning: for $\mathcal{ALCQ}$, consistency w.r.t. CIs is ExpTime-complete, but consistency w.r.t. CRs is NExpTime-complete if the numbers occurring in the CRs are assumed to be encoded in binary [96]. With unary coding of numbers, consistency stays ExpTime-complete even w.r.t. CRs [96]. Note that, using unary coding of numbers, the number n is assumed to contribute n to the size of the input, whereas with binary coding the size of the number n is $\log n$. Thus, for large numbers, assuming binary coding (or coding w.r.t. any base larger than 1) is more realistic. It should be noted that both number restrictions and CRs can be expressed in $\mathcal{C}^2$, the two-variable fragment of FOL with counting quantifiers [54, 87], whose satisfiability problem is known to be NExpTime-complete [90].

The original decidability result for $\mathcal{ALC}(\mathfrak{D})$, i.e., $\mathcal{ALC}$ extended with an admissible concrete domain $\mathfrak{D}$, in [20] did not take CIs into account. In the presence of CIs, integrating even rather simple concrete domains into the DL $\mathcal{ALC}$ may cause undecidability [78, 23]. In [79], it was proved that integrating a so-called ω-admissible concrete domain into $\mathcal{ALC}$ leaves reasoning decidable also in the presence of CIs. There, Allen's interval algebra [2] and RCC8 [91] are proved to be ω-admissible. Using well-known notions and results from model theory, additional ω-admissible concrete domains were exhibited in [23, 24], for example the rational numbers with comparisons $\mathfrak{Q} := (\mathbb{Q}, <, =, >)$. Decidability results for $\mathcal{ALC}(\mathfrak{D})$ in the presence of CIs for concrete domains $\mathfrak{D}$ that are not ω-admissible, such as integers with ordering and equality or strings with lexicographic orderings, can be found in [36, 74, 43]. A simpler, but considerably more restrictive way of achieving decidability is to use unary concrete domains [63].

Our goal is to look at extensions of DLs with qualified number restrictions, cardinality restrictions or concrete domain restrictions and study the complexity of reasoning with the obtained DLs as well as the expressive power of these languages. Since many of the introduced languages lack practical support for reasoning, it is essential to understand what capabilities cannot be expressed by less expressive DLs that are already supported by existing reasoners. On the other hand, studying the complexity of reasoning in the newly obtained DLs is equally crucial: knowing that reasoning with a certain DL is decidable or even has tight complexity bounds paves the way for further research into practical ways to introduce these languages in existing ontologies or improve existing reasoners with new algorithmic insights.

Beyond qualified number restrictions. The classical number restrictions available in $\mathcal{ALCQ}$ can only be used to compare the number of role successors of an individual with a *fixed* natural number. They cannot compare the numbers of different kinds of role successors to each other without relating them to a fixed number. To overcome this deficit, $\mathcal{ALCQ}$ has been extended by allowing the statement of constraints on role successors using the quantifier-free fragment of Boolean Algebra with Presburger Arithmetic (QFBAPA) [72], in which one can express Boolean combinations of set constraints and numerical constraints comparing the cardinalities of finite sets. In the resulting DL, called $\mathcal{ALCSCC}$ [7], we can describe humans that have exactly as many cars as children as

$$\mathsf{Human} \sqcap \mathsf{succ}(|\mathsf{own} \cap \mathsf{Car}| = |\mathsf{child} \cap \mathsf{Human}|)$$

without having to specify the exact numbers of cars and children. The complexity of reasoning in $\mathcal{ALCSCC}$ is the same as in $\mathcal{ALC}$ and $\mathcal{ALCQ}$, where concept satisfiability is PSpace-complete without a TBox and ExpTime in the presence of a TBox [7]. While the PSpace result also follows from previous work [41] on modal logics with Presburger constraints, the ExpTime result was new. The same complexity results hold for $\mathcal{ALCSCC}^\infty$ [16], a variant of $\mathcal{ALCSCC}$ where constraints on role successors are evaluated over an infinite variant QFBAPA$^\infty$ of QFBAPA. We review the logics QFBAPA and QFBAPA$^\infty$, as well as the definitions of $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ and their complexity results shown in [7, 16], in the first part of Chapter 3.

The reason we introduced $\mathcal{ALCSCC}^\infty$ in [16] was to ease the comparison of its expressive power with standard DLs, since $\mathcal{ALCSCC}$ is defined w.r.t. finitely branching interpretations only. Indeed, adding cardinality constraints based on QFBAPA strictly extends the expressive power of $\mathcal{ALCQ}$. In [7] it is shown that the constraint $\mathsf{succ}(|r| = |s|)$, which describes individuals that have the same number of r-successors as s-successors, cannot be expressed in $\mathcal{ALCQ}$. We were able to show in [16] that $\mathcal{ALCSCC}^\infty$ is not a fragment of FOL and characterized the first-order fragment of this logic ($\mathcal{ALCCQU}$ or equivalently $\mathcal{ALCQ}t$) using a form of counting bisimulation [80].

In Chapter 4 we prove the same results for $\mathcal{ALCSCC}$, where only finitely branching interpretations are available. The proof techniques used in [16], which were inspired by the ones in [80], cannot be employed in this setting since they depend on compactness of FOL, which does not hold for the restriction of FOL to finitely branching interpretations. Instead, we employ a proof technique inspired by [92, 84], which utilizes locality properties of FOL rather than compactness. Interestingly, this approach can deal with arbitrary interpretations, finitely branching interpretations, and finite interpretations in a uniform way.

Beyond cardinality restrictions. Just like classical qualified number restrictions, CRs can only relate the cardinality of a concept to a *fixed* natural number. In [19], the authors introduced and investigated an expressive class of constraints on the cardinalities of concepts in *finite* interpretations, encompassing CRs and CBoxes, called *extended cardinality restrictions*. Again, the main idea was to use QFBAPA to formulate and combine these constraints. An example of a constraint expressible this way, but not expressible using CRs is

$$20 \cdot |\mathsf{Car} \sqcap \exists \mathsf{ownedBy}.\mathsf{German} \sqcap \exists \mathsf{fueledBy}.\mathsf{Electricity}| \leqslant |\mathsf{Car} \sqcap \exists \mathsf{ownedBy}.\mathsf{German}|$$

which states that at most five percent of the cars driven in Germany are electric[5]. In [19] it is shown that, in the DL $\mathcal{ALC}$, the complexity of reasoning w.r.t. extended CRs is the same as for reasoning w.r.t. CRs for binary coding. In addition, the paper introduces a restricted version of this formalism called RCBox, which can express CIs, but not CRs, and shows that this way the complexity can be lowered to ExpTime. The NExpTime upper bound for the general case may be derived from the NExpTime upper bound in [102] for a more expressive logic with n-ary relations and function symbols, but the ExpTime result for RCBoxes was new.

In [8, 9] the work of [7] and [19] was combined by considering extended cardinality constraints in $\mathcal{ALCSCC}$. This turned out to be non-trivial since the local cardinality constraints of $\mathcal{ALCSCC}$ may interact with the global ones in the extended cardinality constraints. Nevertheless, it was shown that the complexity results (NExpTime-complete in general, and ExpTime-complete in the restricted case) hold not only for $\mathcal{ALC}$, but also for $\mathcal{ALCSCC}$. Finally, [10] shows that the ExpTime upper bound can be extended from RCBoxes to *ERCBoxes*, which partially cover CRs by using positive Boolean combinations of *semi-restricted CRs*.

We present the mentioned complexity results in Chapter 3 and extend them to $\mathcal{ALCSCC}^\infty$ and arbitrary interpretations. Additionally, we analyze the complexity of the entailment problem for knowledge bases that are built using extended CRs. While for most settings this problem can be reduced to consistency checking without an increase in complexity, we show that for semi-restricted CRs the complexity may change depending on what coefficients are used. In Chapter 5 we look into the expressive power of TBoxes, CRs, and extended cardinality CRs that use $\mathcal{ALCSCC}^\infty$ concepts, by adapting methods and ideas from [80]. Here, we do not restrict the class of interpretations under consideration and we can therefore resort to the formal properties of FOL mentioned above, such as compactness.

Concrete domains, ω-admissibility and complexity. As mentioned above, extensions of $\mathcal{ALC}$ by ω-admissible concrete domains were investigated in [79] and later in [23, 24]. In [79] the concept satisfiability problem w.r.t. an $\mathcal{ALC}(\mathfrak{D})$ TBox was shown to be decidable if $\mathfrak{D}$ is ω-admissible and only contains binary relations, while [23, 24] generalized this result to cases

[5] `https://ourworldindata.org/electric-car-sales`, last accessed 25/04/2025.

where $\mathfrak{D}$ has predicates of arbitrary arity. It was further conjectured that "it is possible to prove ExpTime-completeness of satisfiability in $\mathcal{ALC}(\mathfrak{D})$ provided that satisfiability in $\mathfrak{D}$ can be decided in ExpTime" [79]. In Chapter 6 we verify this conjecture, by providing an ExpTime upper bound for the consistency problem for $\mathcal{ALC}(\mathfrak{D})$ *ontologies* consisting of a TBox $\mathcal{T}$ and an *ABox* $\mathcal{A}$, which is a set of *concept* and *role assertions* of the form $C(a)$ and $r(a, b)$ where a and b are *named individuals*.

Abstract expressive power of logics with concrete domains. As mentioned above, many DL [22] are decidable fragments of FOL, but there are also decidable DLs whose KBs cannot always be expressed by an FOL sentence. A case in point are DLs with concrete domains, at least at first sight. In such DLs, the abstract interpretation domain is complemented by the concrete domain, and partial functions can be used to assign values in the concrete domain to abstract objects, which can then be constrained using the predefined predicates of the concrete domain. For example, assume that we want to model physical objects, collected in a concept PO, which can be decomposed into their proper parts using a role hpp for "has proper part." If we want to take the weight of such objects into account, it makes sense to assign a number for its weight to every physical object using a feature (i.e., partial function) w, and to state that this weight is positive and that proper parts are physical objects that have a smaller weight than the whole. Using the syntax employed above and in [79, 24], these conditions can be expressed with the help of universal CD-restrictions w.r.t. an appropriate concrete domain by the following CI:

$$\mathsf{PO} \sqsubseteq \forall \mathsf{hpp}.\, \mathsf{PO} \sqcap \exists w.\, (x_1 > 0) \sqcap \forall w, \mathsf{hpp}\, w.\, {>}(x_1, x_2). \tag{1.1}$$

Depending on what kind of decomposition into proper parts we have in mind, we can use the rational numbers or the integers as concrete domain. The former would be more appropriate for settings like cutting a cake, where a given piece can always be cut into even smaller parts, whereas the latter is more appropriate for settings where physical objects are composed of finitely many atomic parts that cannot be divided any further.

If we employ the integers, then for any element of PO there is a positive integer such that the length of all hpp-chains issuing from it are bounded by this number. Using this fact, it is easy to show that the logic at hand is not compact, i.e., there may be unsatisfiable infinite sets of sentences for which all finite subsets are satisfiable. In particular, this implies that the *abstract expressive power* of this logic, which considers only the abstract domain and the interpretation of concept and role names, but ignores the feature values, cannot be contained in FOL. For the rational numbers, instead, the extensions of $\mathcal{ALC}$ or FOL with this concrete domain share the compactness and downward Löwenheim-Skolem properties with FOL. The reason is that the rational numbers with $>$ are *homomorphism ω-compact*, which means that a countable set of constraints is solvable iff all its finite subsets are solvable. We can, however, prove that the abstract expressive power of these logics is nevertheless not contained in FOL, though we cannot use a compactness argument to show this.

In Chapter 7, we investigate the impact of the choice of concrete domain $\mathfrak{D}$ on the formal properties satisfied by $\mathcal{ALC}(\mathfrak{D})$ and the extension FOL($\mathfrak{D}$) of FOL by concrete domain predicates. Our first main result is that FOL($\mathfrak{D}$) (i.e., FOL extended with the concrete domain $\mathfrak{D}$) shares compactness, the downward Löwenheim-Skolem property, and the Craig interpolation property with FOL if the employed concrete domain satisfies some reasonable model-theoretic assumptions. On the DL side, the extension $\mathcal{ALC}(\mathfrak{D})$ of the well-known DL $\mathcal{ALC}$ with a concrete domain $\mathfrak{D}$ fulfilling the model-theoretic assumptions mentioned above satisfies compactness, the downward Löwenheim-Skolem property, and also the upward Löwenheim-Skolem property. Additionally, we establish that if $\mathfrak{D}$ is strongly positive, homomorphism ω-compact and the finite unsatisfiable constraint systems for $\mathfrak{D}$ are recursively enumerable, then the unsatisfiable sentences of $\mathsf{FOL}(\mathfrak{D})$ are recursively enumerable. We provide sufficient conditions for which the abstract expressive power of this DL is (not) contained in that of first-order logic. As a by-product, we obtain on the one hand decidability results for several fragments of FOL extended with *unary* concrete domains, and on the other hand show that even decidable fragments of FOL can yield undecidable extensions, if the concrete domain can express equality of its elements.

Comparing logics with concrete domains. While the notion of abstract expressive power allows us to compare logics with and without concrete domains, it only provides limited insights when comparing logics extended by different concrete domains. In Chapter 8 we delve into the expressive power of DLs with concrete domains, by introducing a notion of *concrete bisimulation* which also accounts for the presence of feature names. We show how this notion can be used to separate the expressive power of DLs extended by domains over the same set but with different relations, or DLs with the same concrete domain but different kinds of CD-restrictions. Finally, in the spirit of Chapter 4 we find sufficient conditions on $\mathfrak{D}$ such that $\mathcal{ALC}(\mathfrak{D})$ is the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is invariant under concrete bisimulation, even when restricted to the classes of finite and finitely branching interpretations.

Concrete domains meet cardinality constraints. Finally, Chapter 9 is dedicated to the combination of local cardinality constraints and concrete domain reasoning. We introduce $\mathcal{ALCOSCC}(\mathfrak{D})$, a combination of the DLs $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$ with ω-admissible concrete domains $\mathfrak{D}$ as well as nominals ($\mathcal{O}$). This DL goes beyond a pure combination of number restrictions and concrete domains by additionally allowing for their interaction. For a numerical concrete domain, it seems natural to use the values of concrete features directly in the QF-BAPA constraints of $\mathcal{ALCSCC}$, e.g. to describe people that own more books than their age. We show, however, that this unrestricted combination easily leads to undecidability. Instead, we use concrete domain constraints to define *feature roles*, which can then be employed within QF-BAPA constraints. For example, the feature role ($\mathsf{salary} < \mathtt{next}\,\mathsf{salary}$) connects an individual to all individuals that have a higher salary. One can use this to describe all persons that have a lower salary than at least half of their children with $\mathsf{succ}(|\mathsf{child} \cap (\mathsf{salary} < \mathtt{next}\,\mathsf{salary})| > |\mathsf{child} \cap (\mathsf{salary} \geqslant \mathtt{next}\,\mathsf{salary})|)$. However, we show that the unrestricted use of such concrete roles also leads to undecidability. Hence, we additionally restrict them to pairs of individuals that are already connected by a role name.

Regarding the expressive power of this DL, we combine the work done in Chapters 4 and 8 and introduce the notion of *concrete Preburger bisimulation*, which we employ to show that the DL obtained by removing feature roles from $\mathcal{ALCOSCC}(\mathfrak{D})$ cannot express the feature roles introduced in $\mathcal{ALCOSCC}(\mathfrak{D})$. The main result, concerning complexity, is that reasoning in $\mathcal{ALCOSCC}(\mathfrak{D})$ stays in ExpTime if the complexity of reasoning in $\mathfrak{D}$ is in ExpTime. There are few results in the literature that determine the exact complexity of reasoning in DLs with concrete domains [79, 74, 43], and we extend the results obtained in Chapter 6 for ExpTime-ω-admissible concrete domains from $\mathcal{ALC}(\mathfrak{D})$ to $\mathcal{ALCOSCC}(\mathfrak{D})$. Apart from the aforementioned undecidability results, we show that adding transitive roles also makes reasoning undecidable, even under strong syntactic restrictions.

Publications

The following publications constitute the majority of the work contained in this thesis:

[14] Baader, F., De Bortoli, F.: Description Logics That Count, and What They Can and Cannot Count. In: Kovacs, L., Korovin, K., Reger, G. (eds.) ANDREI-60. Automated New-era Deductive Reasoning Event in Iberia. EPiC Series in Computing, pp. 1–25. EasyChair (2020). https://doi.org/10.29007/ltzn

[15] Baader, F., De Bortoli, F.: Logics with Concrete Domains: First-Order Properties, Abstract Expressive Power, and (Un)Decidability. SIGAPP Applied Computing Review **24**(3), 5–17 (2024). https://doi.org/10.1145/3699839.3699840

[17] Baader, F., De Bortoli, F.: The Abstract Expressive Power of First-Order and Description Logics with Concrete Domains. In: Proceedings of the 39th ACM/SIGAPP Symposium on Applied Computing. SAC '24, pp. 754–761. ACM, New York, NY, USA (2024). https://doi.org/10.1145/3605098.3635984

[11] Baader, F. *et al.*: Concrete Domains Meet Expressive Cardinality Restrictions in Description Logics. In: Barrett, C., Waldmann, U. (eds.) Automated Deduction – CADE 30. LNAI,. Vol. 15943, pp. 676–695. Springer, Heidelberg (2025). https://doi.org/10.1007/978-3-031-99984-0_35

[34] Borgwardt, S., De Bortoli, F., Koopmann, P.: The Precise Complexity of Reasoning in $\mathcal{ALC}$ with ω-Admissible Concrete Domains. In: Giordano, L., Jung, J.C., Ozaki, A. (eds.) Proceedings of the 37th International Workshop on Description Logics (DL'24). CEUR Workshop Proceedings. CEUR-WS, Bergen, Norway (2024)

Additionally, the content of Chapters 4 and 8 is being prepared for a conference submission.

Awards. The conference paper [17] was awarded the Best Paper Award in the Information Systems Area at the 39th ACM/SIGAPP Symposium on Applied Computing (SAC '24). The conference paper [16], which is based on the author's master's thesis and constitutes a starting point for further work developed in Chapters 4 and 5 and partially published in [14], was awarded with the Best Student Paper Award at the 12th International Symposium on Frontiers of Combining Systems (FroCoS '19).

2 Preliminaries

We devote this chapter to the preliminaries. Here, we briefly introduce the notions of *first-order logic (*FOL*)*, *constraint satisfaction problems (CSP)*, *description logics (DLs)* and well-established extensions with *qualified number restrictions*, *cardinality restrictions* and *concrete domain reasoning*, as well as basic notions and results on the *expressive power* of FOL and DLs. Readers that are already familiar with these topics may skip this chapter and refer to it occasionally.

First-Order Logic

In this section, we recall the syntax and semantics of *first-order logic (*FOL*)*, which will serve as the basis for defining several notions of interest in the remainder of the thesis.

Definition 2.1 (Syntax of FOL). *Let σ be a set, called* signature*, that contains countably many* relation symbols P_1, P_2, *...and countably many* function symbols f_1, f_2, ..., *so that each relation and function symbol is associated with a natural number $k \in \mathbb{N}$ called its* arity*. Function symbols of arity 0 in σ are also called* constants*. Given a countable set* Var *of* variables*, we inductively define the set of σ-*terms *as follows:*

- *all elements of* Var *and all constants of σ are σ-terms;*
- *if t_1, ..., t_k are σ-terms and $f \in \sigma$ is a k-ary function symbol, then $f(t_1, \dots, t_k)$ is a σ-term.*

The set FOL$[\sigma]$ *of σ-*formulae *is defined inductively, according to the following rules:*

Atomic formulae *if x, y are σ-terms then $x = y$ is a σ-formula, and if t_1, ..., t_k are σ-terms and $P \in \sigma$ is a k-ary relation symbol, then $P(t_1, \dots, t_k)$ is a σ-formula;*

Boolean connectives *if ϕ and ψ are σ-formulae, then so are $\neg\phi$ (negation), $\phi \wedge \psi$ (conjunction) and $\phi \vee \psi$ (disjunction);*

Quantified formulae *if ϕ is a σ-formula and $x \in$ Var a variable, then $\exists x.\phi$ (existential quantification) and $\forall x.\phi$ (universal quantification) are σ-formulae.*

We also define $\phi \rightarrow \psi := \neg\phi \vee \psi$ (implication) and $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \psi)$ (equivalence).

Definition 2.2 (Free variables and sentences). *We recursively define the set* free(ϕ) *of* free variables *of a* FOL$[\sigma]$ *formula ϕ as follows:*

- *if* ϕ *is an atomic formula, then* $\mathrm{free}(\phi)$ *is the set of all variables occurring in terms of* ϕ*;*
- $\mathrm{free}(\neg\phi) := \mathrm{free}(\phi)$ *and* $\mathrm{free}(\phi \wedge \psi) := \mathrm{free}(\phi) \cup \mathrm{free}(\psi)$ *(the definition for all other binary Boolean connectives is equal to that of conjunction);*
- $\mathrm{free}(\exists x.\phi) = \mathrm{free}(\forall x.\phi) := \mathrm{free}(\phi) \setminus \{x\}$.

A sentence *is a formula with no free variables.*

The syntax of FOL introduced in Definition 2.1 describes which symbolic expressions correspond to well-defined formulae of the language. The *semantics* of FOL, on the other hand, explains how to assign a meaning to well-defined formulae.

Definition 2.3 (Semantics of FOL). *An* interpretation $\mathcal{I}$ *of a signature* σ *consists of a non-empty set* $\Delta^{\mathcal{I}}$ *together with a mapping* $\cdot^{\mathcal{I}}$ *that assigns*

- *to each constant* $c \in \sigma$ *an element* $c^{\mathcal{I}} \in \Delta^{\mathcal{I}}$*,*
- *to each* k*-ary function symbols* $f \in \sigma$ *a function* $f^{\mathcal{I}} \colon (\Delta^{\mathcal{I}})^k \to \Delta^{\mathcal{I}}$ *and*
- *to each* k*-ary relation symbol* $P \in \sigma$ *a* k*-ary relation* $P^{\mathcal{I}} \subseteq (\Delta^{\mathcal{I}})^k$*.*

A variable assignment *over* $\mathcal{I}$ *is a function* $w\colon \mathrm{Var} \to \Delta^{\mathcal{I}}$*, which is extended to a* term assignment *by defining*

- $w(c) := c^{\mathcal{I}}$ *for all constants* $c \in \sigma$ *and*
- $w(f(t_1, \dots, t_k)) := f^{\mathcal{I}}(w(t_1), \dots, w(t_k))$ *for all* k*-ary function symbols* $f \in \sigma$*.*

If w *is a variable assignment, we denote with* $w\{x \mapsto d\}$ *the assignment* w' *such that* $w'(x) = d$ *and* $w'(y) = w(y)$ *for every other variable in* Var.

Given an interpretation $\mathcal{I}$ *of* σ*, a term assignment* w *and a* FOL$[\sigma]$ *formula* ϕ*, we recursively define the* satisfiability relation $\mathcal{I}, w \models \phi$ *as follows:*

- $\mathcal{I}, w \models t_1 = t_2$ *iff* $w(t_1) = w(t_2)$*;*
- $\mathcal{I}, w \models P(t_1, \dots, t_k)$ *iff* $(w(t_1), \dots, w(t_k)) \in P^{\mathcal{I}}$*;*
- $\mathcal{I}, w \models \neg\phi$ *iff not* $\mathcal{I}, w \models \phi$ *(also written* $\mathcal{I}, w \not\models \phi$*);*
- $\mathcal{I}, w \models \phi \wedge \psi$ *iff* $\mathcal{I}, w \models \phi$ *and* $\mathcal{I}, w \models \psi$*;*
- $\mathcal{I}, w \models \phi \vee \psi$ *iff* $\mathcal{I}, w \models \phi$ *or* $\mathcal{I}, w \models \psi$*;*
- $\mathcal{I}, w \models \exists x.\phi$ *iff* $\mathcal{I}, w\{x \mapsto d\} \models \phi$ *for some* $d \in \Delta^{\mathcal{I}}$*;*
- $\mathcal{I}, w \models \forall x.\phi$ *iff* $\mathcal{I}, w\{x \mapsto d\} \models \phi$ *for all* $d \in \Delta^{\mathcal{I}}$*.*

We say that $\mathcal{I}$ *is a* model *of the sentence* ϕ*, in symbols* $\mathcal{I} \models \phi$*, if* $\mathcal{I}, w \models \phi$ *holds for some (and thus all) assignments* w*.*[1] *A first-order sentence* ϕ *over a signature* σ *is* satisfiable *if it has a model and it is* valid *if every interpretation of* σ *is a model of* ϕ*. A first-order sentence* ψ *is a* logical consequence *of* ϕ*, in symbols* $\phi \models \psi$*, if every model of* ϕ *is also a model of* ψ*; we also say that* ϕ entails ψ*.*

[1]Since a sentence contains no free variable, its truth value is the same, no matter what assignment is used.

We can study satisfiability, validity and entailment of first-order sentences interchangeably, since a sentence ϕ is valid iff its negation $\neg\phi$ is unsatisfiable and ψ is a logical consequence of ϕ is the implication $\phi \to \psi$ is valid, i.e. the sentence $\phi \land \neg\psi$ is unsatisfiable. In particular, we focus on the following *decision problem* for FOL:

given a sentence ϕ, does ϕ have a model?

An algorithm that solves this problem is said to be *sound* if it recognizes a satisfiable sentence and *complete* if it recognizes an unsatisfiable sentence. A *decision procedure* is a sound and complete algorithm that solves this decision problem and that terminates on every input ϕ. The famous completeness theorem of Gödel states that FOL is *semidecidable*: there exists a sound and complete algorithm to enumerate all the sentences that are unsatisfiable [50], thus all the valid sentences and all the entailments that hold between FOL sentences. However, the decision problem is *undecidable*, as proved by Church [39] and Turing [100], which means that there is no decision procedure to check satisfiability of a first-order sentence.

Two formal properties derived from the proof of the semidecidability of FOL that we will consider throughout the thesis are *compactness* and *recursive enumerability* [45, 48, 59]. These properties can be specified as follows:

(Countable) Compactness: Let Φ be an at most countable set of FOL sentences. If every finite subset of Φ is satisfiable, then Φ is satisfiable;

Recursive enumerability: The set of unsatisfiable sentences in FOL is *recursively enumerable* (r.e.), i.e. there is an algorithm that enumerates all unsatisfiable FOL sentences.

Decidable Fragments of First-Order Logic

One way to regain decidability of the satisfiability problem is to restrict the attention to *fragments* of FOL, i.e. subsets that are defined according to some syntactic criteria. A family of fragments that has been thorougly studied is obtained by restricting the number of distinct variables that can occur in a formula. Among the fragments that are obtained in this way, we highlight the one-variable, the two-variable and the three-variable fragment, respectively denoted with FOL_1, FOL_2 and FOL_3. If we assume that the signature σ contains no function symbol, then the satisfiability problem for FOL_2 is decidable in NExpTime even with equality predicates [53], and it additionally enjoys the *finite model property* [83], that is, every satisfiable formula in this language has a model whose domain is finite. In contrast, the satisfiability problem for FOL_3 is undecidable even in the absence of both function symbols and equality predicates [66]. As soon as we allow for function symbols in the signature, the satisfiability problem is already undecidable for FOL_1 with equality, by reduction from the word problem for finitely generated groups [31].

Another way to obtain decidable fragments of FOL is to restrict the syntax of the formulae that occur in the scope of a quantified variable. The most notable example, in this setting, is given by the *guarded fragment* [4]. Given a set of variables Var and a signature σ that is *relational*, i.e. that only contains relation symbols, the set of *guarded* formulae (over Var and σ) is defined inductively:

Atomic formulae every expression $x = y$ with $x, y \in \mathrm{Var}$ and $R(x_1, \dots, x_k)$ with $R \in \sigma$ a k-ary relation and $x_1, \dots, x_k \in \mathrm{Var}$ is a guarded formula;

Boolean connectives if ϕ, ψ are guarded formulae then $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$ are guarded formulae;

Quantified formulae if G is an atomic guarded formula, ϕ is a guarded formula such that $\text{free}(\phi) \subseteq \text{free}(G)$ and $x_1, \dots, x_k \in \text{Var}$ then $\exists x_1. \dots \exists x_k.\ G \wedge \phi$ and $\forall x_1.\ \dots \forall x_k.G \rightarrow \phi$ are guarded formulae.

We notice that if $\phi(x)$ is a guarded formula, then $\exists x.\phi(x)$ and $\forall x.\phi(x)$ are also guarded formulae, since we can use the tautological expression $x = x$ to guard ϕ when quantifying.

The satisfiability problem for the guarded fragment of FOL, denoted with GF, is 2ExpTime-complete and becomes ExpTime-complete by bounding the number of variables or the arity of the predicates involved [52]. As a consequence, the satisfiability problem for GF_2, the two-variable guarded fragment of FOL, is ExpTime-complete. This fragment is especially interesting in that it captures the expressive power of several description logics, as discussed later in this chapter.

First-Order Logic and Counting Quantifiers

A syntactic variant of FOL is obtained by considering *counting quantifiers* of the form $\exists_{\leqslant n} x$ or $\exists_{\geqslant n} x$ where n is a natural number; the intuitive semantics of these quantifiers corresponds to "there are at least/at most n x's such that...". Clearly, these quantifiers can be expressed using the quantifiers that are already available in FOL. Indeed, $\exists_{\geqslant n} x.\phi(x)$ is equivalent to

$$\exists x_1. \cdots .\exists x_n.((\bigwedge\nolimits_{1 \leqslant i < j \leqslant n} x_i \neq x_j) \wedge \bigwedge_{i=1}^{n} \phi(x_i)).$$

As done before, we can study the complexity of the satisfiability problem for subsets of FOL extended with counting quantifiers. For $\mathcal{C}^2$, the two-variable fragment of FOL with counting quantifiers, the satisfiability problem is decidable [54, 87] and in particular NExpTime complete [90]. The guarded fragment of $\mathcal{C}^2$, denoted with GC_2, is decidable and its satisfiability problem is ExpTime-complete [67]. Considering the syntax of the guarded fragment that we introduced above, the fragment GC_2 is obtained by extending GF_2 with quantified formulae of the form $\exists_{\geqslant n} x.G \wedge \phi$ and $\exists_{\leqslant n} x.G \wedge \phi$. In this case, though, formulae of the form $\exists_{\geqslant n} x.\phi(x)$ or $\exists_{\leqslant n} x.\phi(x)$ are specifically disallowed.

Constraint Satisfaction Problems

Let $\mathfrak{D}$ be a *relational structure*, that is, a set D called the *domain* of $\mathfrak{D}$ endowed with a countable set of *relations* σ, such that each relation $P \in \sigma$ is associated with a number $k \in \mathbb{N}$, called its *arity*, and is interpreted as a set $P^D \subseteq D^k$. As an example, each tuple $(\mathbb{K}, <, =, >)$ where $\mathbb{K}$ is either $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ provides a relational structure where the ordering and equality relations in the tuple are interpreted according to their standard definition over each of these sets of numbers.

We define a *constraint system* over a relational structure $\mathfrak{D}$ as a set Γ of *constraints* of the form $P(x_1, \dots, x_k)$ where P is a k-ary relation of $\mathfrak{D}$ and each x_i is a variable. The constraint system Γ is *satisfiable* if there is a mapping h, called *homomorphism* or *solution*, that assigns each variable in Γ to an element of D such that $(h(x_1), \dots, h(x_k)) \in P^D$ holds for every constraint $P(x_1, \dots, x_k)$ in Γ. The *constraint satisfaction problem (CSP)* for $\mathfrak{D}$, denoted $\text{CSP}(\mathfrak{D})$, asks if an input *finite* constraint system Γ over $\mathfrak{D}$ is satisfiable.

Example 2.4. *Let us consider the relational structure $\mathfrak{Q} := (\mathbb{Q}, <, =, >)$ of the rational numbers with the standard ordering and equality relations, which we write infix. The constraint system $\Gamma := \{x_1 > x_2, x_2 > x_3, x_3 > x_1\}$ provides an instance of* CSP($\mathfrak{Q}$) *that is unsatisfiable: a solution h to Γ would satisfy $h(x_1) > h(x_1)$, which contradicts the interpretation of $>$ over $\mathbb{Q}$. We observe that* CSP($\mathfrak{Q}$) *can be decided in polynomial time: given any finite constraint system Γ, we can derive a directed graph G where we identify all variables x, y for which $x = y$ occurs in Γ and introduce a directed edge $x \to y$ if $x < y$ or $y > x$ are in Γ. Then, Γ has a solution iff G is acyclic.*

A k-ary relation P is *first-order definable* on a relational structure $\mathfrak{D}$ with set of relations σ if there exists a first-order formula $\phi(x_1, \dots, x_k)$ over the signature σ such that

$$(d_1, \dots, d_k) \in P^D \text{ iff } \mathfrak{D}, \{x_i \mapsto d_i \mid i = 1, \dots, k\} \models \phi(x_1, \dots, x_k),$$

where we consider $\mathfrak{D}$ as an interpretation of σ and the satisfaction relation $\models$ is the one defined for FOL; in this case, $\phi(x_1, \dots, x_k)$ is a *definition* of P.

Constraint systems are not allowed to contain negated constraints and can only express positive information. On certain structures, we can simulate negated constraints by leveraging the way in which relations are interpreted. For example, in every constraint system Γ over the structure $\mathfrak{Q}$ defined in Example 2.4 we can encode the negated constraint $\neg(x = y)$ by either adding $x < y$ or $y < x$ to Γ. Here, we used the fact that the complement of the equality relation has a first-order definition that is positive and quantifier-free, since $\neg(x = y)$ is equivalent to $x < y \lor y < x$.

Definition 2.5. *A relational structure $\mathfrak{D}$ with relation set σ is* strongly positive *if for every k-ary relation $P \in \sigma$ there is a quantifier-free, equality-free formula $\phi_{\neg P}(x_1, \dots, x_k)$ over σ without negation that defines the complement of P over $\mathfrak{D}$, that is,*

$$(d_1, \dots, d_k) \notin P^D \textit{ iff } \mathfrak{D}, \{x_i \mapsto d_i \mid i = 1, \dots, k\} \models \phi_{\neg P}(x_1, \dots, x_k).$$

We say that $\mathfrak{D}$ is weakly closed under negation (WCUN) *if for every k-ary relation P over $\mathfrak{D}$ the formula $\phi_{\neg P}(x_1, \dots, x_k)$ above is a disjunction of atomic formulae of $P_1(x_1, \dots, x_k)$, ..., $P_n(x_1, \dots, x_k)$ with P_1, ..., P_n k-ary relations of $\mathfrak{D}$.*

As seen above, $\mathfrak{Q}$ is both strongly positive and WCUN. In fact, every structure that is WCUN is strongly positive. Among all relational structures that are WCUN and thus strongly positive, we single out two classes where the definition of the complement is further restricted and allows to reduce instances of the CSP with negated constraints to instances without negation. A strongly positive structure $\mathfrak{D}$ with relation set σ is

- *closed under negation* if the complement of any k-ary relation $P \in \sigma$ is defined by a k-ary relation $P_c \in \sigma$, and
- *jointly exhaustive and pairwise disjoint* (JEPD) if for all $k \geqslant 1$, either σ has no k-ary relation, or D^k is partitioned by all k-ary relations in σ.

An example of a JEPD structure is given by the structure $\mathfrak{Q}$ considered in Example 2.4.

The following property motivates the usage of the term "strongly positive" above: every first-order definable relation in a structure of this kind is positively definable.

Proposition 2.6. *Every first-order definable relation on a strongly positive relational structure $\mathfrak{D}$ has a positive definition. If a relation has a quantifier-free definition on $\mathfrak{D}$, then it has a quantifier-free positive definition.*

While the CSP of a structure is concerned with *finite* constraint systems, there are situations in which we are interested in establishing if infinite constraint systems over a relational structure have a solution, in particular countable constraint systems. The following definition introduces a notion of *compactness* for constraint systems that is similar to the notion of compactness for countable sets of FOL sentence described in the previous section.

Definition 2.7. *A relational structure $\mathfrak{D}$ is* homomorphism ω-compact *if for every countable constraint system Γ over $\mathfrak{D}$, we have that Γ is satisfiable iff every finite subset of Γ is satisfiable.*

Example 2.8. *The structure $\mathfrak{Q}$ in Example 2.4 is homomorphism ω-compact [23, 24]. Indeed, the reduction we described to check the satisfiability of a constraint system over $\mathfrak{Q}$ tells us that a countable constraint system Γ is unsatisfiable iff the induced directed graph G contains a cycle $x_1 \rightarrow \ldots x_n \rightarrow x_1$, which is itself induced by a finite subset of Γ. In contrast, the relational structure $\mathfrak{Z} := (\mathbb{Z}, <, =, >)$ is not homomorphism ω-compact: to witness this failure, we consider the constraint system $\Gamma := \{x_q > x_r \mid q, r \in \mathbb{Q}, q > r\}$. We notice that Γ is countable and unsatisfiable in $\mathfrak{Z}$, since any solution h requires the interval $[h(x_0), h(x_1)] \subseteq \mathbb{Z}$ to be dense, contradicting the fact that $\mathbb{Z}$ is not dense; however, any finite subset of Γ is clearly satisfiable.*

Description Logics

Description Logics (DLs) are a prominent family of knowledge representation languages used to formalize ontologies, and they are the main object of investigation of this work. In this section, we introduce the DL $\mathcal{ALC}$, define its syntax and semantics and describe the decision problems typically studied in DLs. We then discuss the complexity of the decision problems associated to $\mathcal{ALC}$ and the relationship between this DL and FOL. We refer the reader to Chapter 2 and 5 of [22] for further details.

We begin by introducing the syntax of the DL $\mathcal{ALC}$.

Definition 2.9. *Given countable, disjoint sets $\mathsf{N_C}$ and $\mathsf{N_R}$ of* concept *and* role names*, we inductively define the set of $\mathcal{ALC}$* concept descriptions *(or simply* concepts*) as follows:*

Concept names, top and bottom *every concept name in $\mathsf{N_C}$ is a concept description, and so are the symbols $\top$ (top) and $\bot$ (bottom);*

Boolean constructors *if C, D are concept descriptions, then so are $C \sqcap D$ (conjunction), $C \sqcup D$ (disjunction) and $\neg C$ (negation);*

Role restrictions *if C is a concept description and $r \in \mathsf{N_R}$ a role name, then $\exists r.C$ (existential restriction) and $\forall r.C$ (value restriction) are concept descriptions.*

A TBox *is a finite set of* concept inclusions (CIs) *$C \sqsubseteq D$ where C, D are concept descriptions.*

As seen in Chapter 1, an example of a CI written using $\mathcal{ALC}$ concept descriptions is

$$\mathsf{Human} \sqcap \exists \mathsf{child}.\mathsf{Human} \sqsubseteq \exists \mathsf{eligible}.\mathsf{TaxBreak},$$

where concept names Human and TaxBreak and the role names child and eligible are combined using concept conjunction and existential role restrictions.

The semantics of $\mathcal{ALC}$ is defined in terms of *interpretations* of $\mathsf{N_C}$ and $\mathsf{N_R}$, which are similar to the interpretations used to define the semantics of FOL in the previous section.

Definition 2.10. *An* interpretation $\mathcal{I}$ *of* $\mathsf{N_C}$ *and* $\mathsf{N_R}$ *consists of a non-empty set* $\Delta^\mathcal{I}$ *and a mapping* $\cdot^\mathcal{I}$ *that assigns a set* $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ *to* $A \in \mathsf{N_C}$ *and a set* $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ *to* $r \in \mathsf{N_R}$. *Given an* individual $d \in \Delta^\mathcal{I}$ *we denote with* $r^\mathcal{I}(d)$ *the set of its* r-successors*, that is,*

$$r^\mathcal{I}(d) := \{e \in \Delta^\mathcal{I} \mid (d, e) \in r^\mathcal{I}\}.$$

We define the interpretation of $\mathcal{ALC}$ *concepts recursively by setting* $\top^\mathcal{I} := \Delta^\mathcal{I}$, $\bot^\mathcal{I} := \emptyset$ *and*

$$(\neg C)^\mathcal{I} := \Delta^\mathcal{I} \setminus C^\mathcal{I}, \quad (C \sqcap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I}, \quad (C \sqcup D)^\mathcal{I} := C^\mathcal{I} \cup D^\mathcal{I}$$

for the Boolean constructors; lastly, we define the interpretation of role restrictions as

$$(\exists r.C)^\mathcal{I} := \{d \in \Delta^\mathcal{I} \mid \text{there exists } e \in r^\mathcal{I}(d) \text{ such that } e \in C^\mathcal{I}\}$$
$$(\forall r.C)^\mathcal{I} := \{d \in \Delta^\mathcal{I} \mid \text{if } e \in r^\mathcal{I}(d) \text{ then } e \in C^\mathcal{I}\}$$

An interpretation $\mathcal{I}$ *is a* model *of the TBox* $\mathcal{T}$ *if* $C^\mathcal{I} \subseteq D^\mathcal{I}$ *holds for every inclusion* $C \sqsubseteq D$ *in* $\mathcal{T}$. *A concept description* C *is* satisfiable *if there is an interpretation* $\mathcal{I}$ *such that* $C^\mathcal{I} \neq \emptyset$*, and it is* satisfiable w.r.t. the TBox $\mathcal{T}$ *if there is a model* $\mathcal{I}$ *of* $\mathcal{T}$ *such that* $C^\mathcal{I} \neq \emptyset$.

Reasoning with $\mathcal{ALC}$ concepts and TBoxes

The decision problems that we consider for $\mathcal{ALC}$ and the DLs that we will introduce throughout the thesis are the following:

Satisfiability is an $\mathcal{ALC}$ concept C satisfiable w.r.t. an $\mathcal{ALC}$ TBox $\mathcal{T}$?

Consistency does an $\mathcal{ALC}$ TBox $\mathcal{T}$ have a model?

Entailment is every model of an $\mathcal{ALC}$ TBox $\mathcal{T}$ also a model of the $\mathcal{ALC}$ CI $C \sqsubseteq D$?[2]

For $\mathcal{ALC}$ and other expressive DL, we can reduce these tasks to one another in polynomial time, that is, we can reformulate each of these reasoning tasks as a different task, where the input is obtained by applying a polynomial time procedure to the original input. In particular, the following relationships exist between the reasoning tasks listed above. The proof of this theorem can be found in [22] (Theorem 2.17), except for the first item; we will prove a similar reduction in Chapter 3 and thus omit the proof here.

Theorem 2.11. *Let* C, D *be* $\mathcal{ALC}$ *concept descriptions and* $\mathcal{T}$ *be an* $\mathcal{ALC}$ *TBox. Then:*

- C *is satisfiable w.r.t.* $\mathcal{T}$ *iff the TBox* $\mathcal{T} \cup \{\top \sqsubseteq \exists r.C\}$ *is consistent, where* r *is a fresh role name occurring neither in* C *nor in* $\mathcal{T}$*;*
- C *is satisfiable w.r.t.* $\mathcal{T}$ *iff* $\mathcal{T}$ *does not entail the CI* $C \sqsubseteq \bot$*;*
- $\mathcal{T}$ *is consistent iff* $\top$ *is satisfiable w.r.t.* $\mathcal{T}$*;*
- $\mathcal{T}$ *is consistent iff it does not entail the CI* $\top \sqsubseteq \bot$*;*
- $\mathcal{T}$ *entails* $C \sqsubseteq D$ *iff* $C \sqcap \neg D$ *is unsatisfiable w.r.t* $\mathcal{T}$*;*

[2] The entailment problem is also known in the literature as the *subsumption* problem (cf. [22]).

As a consequence of these relationships, we can transfer results concerning the complexity of performing one reasoning task to the other tasks; this is not always the case, and we will see one example where this is not applicable in Section 3.3. The following well-known result establishes the complexity of the reasoning tasks listed above for $\mathcal{ALC}$.

Theorem 2.12 ([95, 94]). *The concept satisfiability problem for $\mathcal{ALC}$ is PSpace-complete if no TBox is considered and ExpTime-complete in the presence of a TBox. The consistency and subsumption problems for $\mathcal{ALC}$ are ExpTime-complete.*

The relationship with first-order logic

We can establish the decidability of the concept satisfiability problem for $\mathcal{ALC}$ by reducing this problem to the satisfiability problem for a decidable fragment of FOL. Given two distinct free variables x and y, we define two mutually recursive translation functions π_x and π_y that assign to each $\mathcal{ALC}$ concept description C a FOL formula $\pi_x(C)$ and $\pi_y(C)$. We define π_x as follows:

$$
\begin{aligned}
&\pi_x(A) := A(x) && \pi_x(C \sqcup D) := \pi_x(C) \vee \pi_x(D)\\
&\pi_x(\neg C) := \neg \pi_x(C) && \pi_x(\exists r.C) := \exists y.(r(x,y) \wedge \pi_y(C))\\
&\pi_x(C \sqcap D) := \pi_x(C) \wedge \pi_x(D) && \pi_x(\forall r.C) := \forall y.(r(x,y) \to \pi_y(C)).
\end{aligned}
$$

The definition of π_y is obtained by swapping all occurrences of x and y in the formulae above, so that e.g. $\pi_y(\exists r.C) := \exists x.(r(y,x) \wedge \pi_x(C))$. Using these mappings, we associate to every $\mathcal{ALC}$ TBox $\mathcal{T}$ a FOL sentence $\pi(\mathcal{T})$ that is the conjunction of sentences $\forall x.(\pi_x(C) \to \pi_x(D))$ for every concept inclusion $C \sqsubseteq D$ in $\mathcal{T}$. The following holds (cf. Theorem 2.23 in [22]).

Theorem 2.13. *An $\mathcal{ALC}$ TBox $\mathcal{T}$ is satisfiable iff the* FOL *sentence $\pi(\mathcal{T})$ is satisfiable.*

Since the obtained sentence is in particular a GF_2 sentence and it is obtained in polynomial time w.r.t. the size of $\mathcal{T}$, we obtain an ExpTime upper bound for the satisfiability problem of $\mathcal{ALC}$ as a corollary of Theorem 2.13 (see the previous section for a discussion of the decidability of GF_2). Moreover, this translation provides us with a view of $\mathcal{ALC}$ concepts descriptions and TBoxes as fragments of FOL over the signature $\mathsf{N_C} \cup \mathsf{N_R}$, namely those of formulae $\pi_x(C)$ or $\pi(\mathcal{T})$ where C is an $\mathcal{ALC}$ concept and $\mathcal{T}$ an $\mathcal{ALC}$ TBox [32]. Later we show a different characterization of $\mathcal{ALC}$, not in terms of the syntax of its translation into FOL, but rather as the set of formulae whose truth is preserved by a certain relation between interpretations called *bisimulation*. Using this characterization, we can prove that a certain extension of $\mathcal{ALC}$ adds more expressive power to the resulting DL, justifying its deployment in a knowledge base.

Extensions of $\mathcal{ALC}$

In this section, we look at extensions of $\mathcal{ALC}$ and concept inclusions that put an emphasis on quantitative aspects of modelling. For all these extensions we define syntax, semantics and show complexity results for concept satisfiability and consistency.

Qualified number restrictions

The most prominent use case for constraints on the cardinalities of sets of objects in the context of description logics is found in so-called *(qualified) number restrictions* [61], already introduced in the early days of the field [33, 62]. Using number restrictions, we can for example define parents that are eligible for a tax break as those who have at least three children:

$$\mathsf{Human} \sqcap (\geqslant 3\,\mathsf{child}.\mathsf{Human}) \sqsubseteq \exists \mathsf{eligible}.\mathsf{TaxBreak}.$$

Definition 2.14 (Syntax and Semantics of $\mathcal{ALCQ}$). *Given countable and disjoint sets* $\mathsf{N_C}$, $\mathsf{N_R}$ *of concept and role names, we inductively define the set of* $\mathcal{ALCQ}$ *concept descriptions by extending Definition 2.9 with the following rule:*

Qualified number restrictions *if* $n \in \mathbb{N}$, $r \in \mathsf{N_R}$ *is a role name and* C *is an* $\mathcal{ALCQ}$ *concept description, then* $(\geqslant n\, r.C)$ *and* $(\leqslant n\, r.C)$ *are* $\mathcal{ALCQ}$ *concept descriptions.*

We define the interpretation of $\mathcal{ALCQ}$ *concept descriptions recursively, by extending Definition 2.10 to qualified number restrictions as follows:*

$$(\geqslant n\, r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \textit{the set } r^{\mathcal{I}}(d) \cap C^{\mathcal{I}} \textit{ contains at least } n \textit{ elements } \},$$
$$(\leqslant n\, r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \textit{the set } r^{\mathcal{I}}(d) \cap C^{\mathcal{I}} \textit{ contains at most } n \textit{ elements } \}.$$

In the previous section we showed how to translate $\mathcal{ALC}$ concepts and TBoxes to GF_2, whose satisfiability problem is known to be ExpTime-complete. Similarly, we define a translation function that assigns $\mathcal{ALCQ}$ concepts to GC_2 formulae, by extending the mapping $\pi_x(C)$ defined in the previous section to qualified number restrictions, as follows:

$$\pi_x((\geqslant n\, r.C)) := \exists_{\geqslant n} y.(r(x, y) \wedge \pi_y(C)),$$
$$\pi_x((\leqslant n\, r.C)) := \neg \pi_x((\geqslant n + 1\, r.C)).$$

Since the satisfiability problem for GC_2 is ExpTime-complete [67], we are able to deduce that consistency of $\mathcal{ALCQ}$ TBoxes – and thus concept satisfiability w.r.t. a $\mathcal{ALCQ}$ TBox – is decidable in exponential time. Coupled with the fact that this problem generalizes the satisfiability problem for $\mathcal{ALC}$ and is thus ExpTime-hard, we conclude that it is ExpTime-complete. This result was already known prior to [67] establishing the complexity of reasoning in GC_2. In particular, the complexity of reasoning in $\mathcal{ALCQ}$ has been shown to be the same as for $\mathcal{ALC}$ for both concept satisfiability without a TBox [97] and w.r.t. a TBox [98], independently of whether the numbers n occurring in the number restrictions are encoded in unary or binary. Under the assumption of *unary coding* each number n contributes linearly w.r.t. n to the size of the input, whereas under *binary coding* the contribution is logarithmic w.r.t. n. An assumption of binary coding (or coding in any base larger than 1) better reflects reality, where numbers are typically represented as sequences of bits or using decimal digits.

Theorem 2.15 ([97, 98]). *The concept satisfiability problem for* $\mathcal{ALCQ}$ *is PSpace-complete if no TBox is considered and ExpTime-complete in the presence of a TBox.*

Cardinality restrictions

To formulate numerical constraints on the extensions of concepts in an interpretation we can use *cardinality restrictions* (CRs) [12, 96].

Definition 2.16 (Cardinality restrictions). *A* cardinality restriction (CR) *is an expression of the form* $|C| \leqslant n$ *or* $|C| \geqslant n$ *where* $n \in \mathbb{N}$ *is a natural number and* C *is a concept description. A* CBox *is a finite set of CRs. An interpretation* $\mathcal{I}$ *is a* model *of the CBox* $\mathcal{C}$*, in symbols* $\mathcal{I} \models \mathcal{C}$*, if* $|C^{\mathcal{I}}| \geqslant n$ *holds for every CR* $|C| \geqslant n$ *in* $\mathcal{C}$ *and similarly* $|C^{\mathcal{I}}| \leqslant n$ *if* $|C| \leqslant n$ *is in* $\mathcal{C}$.

For example, the CRs[3]

$$|\mathsf{LegalGermanEntity} \sqcap \exists \mathsf{subscribedTo.LandlineInternet}| \geqslant 37000000$$

state that there are at least thirty-seven million subscriptions to landline Internet in Germany.[4] The main difference between qualified number restrictions and CRs is that constraints of the former kind are evaluated over the set of role successors of a certain individual, whereas restrictions of the latter type range over the whole domain of an interpretation.

Relationship with FOL. We have seen that $\mathcal{ALC}$ and $\mathcal{ALCQ}$ can be seen as subsets of the fragments GF_2 and GC_2 of FOL, using a translation function $\pi_x(C)$ that assigns to each concept in one of these DLs a FOL formula in one free variable x in the target fragment of FOL. By extending this translation function to CRs, we are able to encode CBoxes by means of FOL sentences, and in particular as sentences of the logic $\mathcal{C}^2$:

$$\pi(|C| \geqslant n) := \exists_{\geqslant n} x.\ \pi_x(C),$$
$$\pi(|C| \leqslant n) := \exists_{\leqslant n} x.\ \pi_x(C)$$

We define $\pi(\mathcal{C})$ as the conjunction of the translations of each CR occurring in the CBox $\mathcal{C}$.

Reasoning with CRs. We can reduce reasoning w.r.t an $\mathcal{ALC}$ TBox to reasoning w.r.t. an $\mathcal{ALC}$ CBox, since the CI $C \sqsubseteq D$ is in fact equivalent to the CR $|C \sqcap \neg D| \leqslant 0$; therefore, the consistency problem for CBoxes is ExpTime-hard. The exact complexity of the consistency problem for $\mathcal{ALCQ}$ CBoxes has been established in [96] and depends on the coding of the numbers occurring in CRs. Consistency of $\mathcal{ALCQ}$ CBoxes was shown to be NExpTime-complete in [96] if binary coding of numbers is used, whereas for unary coding it stays in ExpTime.

Theorem 2.17 ([96]). *The consistency problem for $\mathcal{ALCQ}$ CBoxes is NExpTime-complete under the assumption of binary coding and ExpTime-complete under the assumption of unary coding.*

Concrete Domains

One of the shortcomings of $\mathcal{ALC}$ is that it does not provide an interface to refer to the attributes of individuals, nor does it allow for the use of relations between these attributes to constrain their value. *Concrete domains* address these issues and offer the opportunity to specify restrictions over the values of certain *features* associated to individuals.

Formally, a *concrete domain* is a fixed relational structure $\mathfrak{D}$. To enable the reference to values of $\mathfrak{D}$ within concept descriptions, we introduce a countable set $\mathsf{N_F}$ of *feature names*, disjoint from $\mathsf{N_C}$ and $\mathsf{N_R}$. We relate individuals, their role successors and values of a concrete domain using *feature paths*. Hereafter, a *feature path* is either a feature name $f \in \mathsf{N_F}$ or an expression rf where $r \in \mathsf{N_R}$ is a role name and f is a feature name. Using feature paths and the relations of $\mathfrak{D}$, we are ready to define the DL $\mathcal{ALC}(\mathfrak{D})$.

[3]Note that the syntax we use here for CRs differs from the one introduced in [12] to make it more similar to the syntax used later on for our extensions of CRs.

[4]See `https://ourworldindata.org/grapher/landline-internet-subscriptions`, last accessed September 10, 2024.

Definition 2.18 (Syntax and semantics of $\mathcal{ALC}(\mathfrak{D})$). *Given countable, disjoint sets* $\mathsf{N_C}$, $\mathsf{N_R}$ *and* $\mathsf{N_F}$ *of concept, role and feature names, we inductively define the set of* $\mathcal{ALC}(\mathfrak{D})$ *concept descriptions by extending Definition 2.9 with the following rule:*

CD-restrictions *if C is a concept description, P is a k-ary relation of $\mathfrak{D}$ and p_1, ..., p_k are feature paths, then $\exists p_1, \dots, p_k.P$ (existential CD-restriction) and $\forall p_1, \dots, p_k.P$ (universal CD-restriction) are concept descriptions.*

An interpretation of disjoint sets $\mathsf{N_C}$, $\mathsf{N_R}$ *and* $\mathsf{N_F}$ *is an interpretation $\mathcal{I}$ of* $\mathsf{N_C}$ *and* $\mathsf{N_R}$ *as in Definition 2.10 extended with a set of* partial *functions $f^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \rightharpoonup D$. Given an individual $d \in \Delta^{\mathcal{I}}$, we denote with $f^{\mathcal{I}}(d)$ the value associated by $f^{\mathcal{I}}$ to d, if it is defined, and define*

$$p^{\mathcal{I}}(d) := \begin{cases} \{f^{\mathcal{I}}(d)\} & p = f, \\ \{f^{\mathcal{I}}(e) \mid e \in r^{\mathcal{I}}(d)\} & p = rf. \end{cases}$$

We define the interpretation of $\mathcal{ALC}(\mathfrak{D})$ concept descriptions recursively, by extending Definition 2.10 to CD-restrictions as follows:

$$(\exists p_1, \dots, p_k.P)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \textit{there exist } v_i \in p_i^{\mathcal{I}}(d) \textit{ for } i = 1, \dots, k \textit{ s.t. } (v_1, \dots, v_k) \in P^D\},$$
$$(\forall p_1, \dots, p_k.P)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \textit{if } v_i \in p_i^{\mathcal{I}}(d) \textit{ for } i = 1, \dots, k \textit{ then } (v_1, \dots, v_k) \in P^D\}.$$

The presence of restrictions related to a concrete domain $\mathfrak{D}$ can easily render concept satisfiability w.r.t. an $\mathcal{ALC}(\mathfrak{D})$ TBox undecidable [78, 23]. This is true even if the CSP associated to $\mathfrak{D}$ is decidable. For example, the relational structure $\mathfrak{Q}_+$ with a ternary predicate $+(x, y, z)$ interpreted as the standard sum $x + y = z$ over the set $\mathbb{Q}$ has a decidable CSP. However, one can reduce the halting problem for two-register machines, which is known to be undecidable [82], to concept satisfiability w.r.t. an $\mathcal{ALC}(\mathfrak{Q}_+)$ TBox, obtaining the following result.

Proposition 2.19 ([23]). *Concept satisfiability w.r.t. TBoxes in $\mathcal{ALC}(\mathfrak{Q}_+)$ is undecidable.*

Thus, we need stronger conditions than simple decidability of the CSP of the concrete domain $\mathfrak{D}$ to regain decidability of concept satisfiability w.r.t. general $\mathcal{ALC}(\mathfrak{D})$ TBoxes. One of these conditions regards the compositionality of solutions of constraint systems over $\mathfrak{D}$ and is derived from the *amalgamation property* (AP) used in model theory (cf. [24]):

AP if Γ, Γ' are constraint systems over $\mathfrak{D}$ and

$$P(x_1, \dots, x_k) \in \Gamma \text{ iff } P(x_1, \dots, x_k) \in \Gamma'$$

holds for all variables $x_1, \dots, x_k$ occurring in both Γ, Γ' and for all k-ary predicates P over $\mathfrak{D}$, then Γ and Γ' are satisfiable iff the constraint system $\Gamma \cup \Gamma'$ is satisfiable.

Another condition that is used to establish decidability is concerned with the ability to express equality of two individuals within a constraint system, where only relations of the concrete domain are available. A concrete domain that satisfies this property is called *jointly diagonal* (JD) and is formally defined as:

JD there is a quantifier-free, equality-free first-order formula $\phi_=(x, y)$ over the signature of $\mathfrak{D}$ that defines the equality relation = between two elements of $\mathfrak{D}$.

These conditions have been combined with the JEPD and *homomorphism ω-compactness* properties introduced earlier, yielding the following class of concrete domains [79, 23]. Here, a *patchwork* is a concrete domain $\mathfrak{D}$ that is JEPD, JD and satisfies AP. If $\mathfrak{D}$ is a patchwork, we call a constraint system $\mathfrak{C}$ *complete* if, for all $k \in \mathbb{N}$, either $\mathfrak{D}$ has no k-ary predicates, or for all $v_1, \dots, v_k \in V(\mathfrak{C})$ there is exactly one k-ary predicate P over $\mathfrak{D}$ such that $P(v_1, \dots, v_k) \in \mathfrak{C}$.

Definition 2.20. *A concrete domain $\mathfrak{D}$ is ω-*admissible *if*

- *$\mathfrak{D}$ has a finite signature,*
- *$\mathfrak{D}$ is a patchwork,*
- *$\mathfrak{D}$ is homomorphism ω-compact, and*
- CSP($\mathfrak{D}$) *is decidable.*

Requiring the signature of $\mathfrak{D}$ to be finite is necessary to ensure decidability of $\mathcal{ALC}(\mathfrak{D})$. Without this assumption, one can find instances of $\mathfrak{D}$ that satisfy all the other conditions of Definition 6.1 such that reasoning in $\mathcal{ALC}(\mathfrak{D})$ is undecidable. One such example is given by the concrete domain $(\mathbb{Z}, \{+_m \mid m \in \mathbb{Z}\})$ where the binary relation $+_m$ relates those integers whose difference is equal to m [24]. The conditions of Definition 6.1 are satisfied by Allen's interval algebra [2], the region connection calculus RCC8 [91] and $\mathfrak{Q}$ [79, 24].

If $\mathfrak{D}$ has finitely many k-ary relations for all $k \in \mathbb{N}$ and satisfies all the conditions of Definition 2.20 except for finiteness of its signature, we can obtain an ω-admissible domain by fixing a natural number d and only considering the relations of $\mathfrak{D}$ that have arity at most d. This is not so restrictive, as for every $\mathcal{ALC}(\mathfrak{D})$ concept and TBox we can easily find such a bound d.

We also notice that the original notion of ω-admissibility in [79] did not consider JD, which was added later in [23]. The following result was shown in [79] for concrete domains with binary predicates and later generalized in [23] for arbitrary arities.

Theorem 2.21 ([79, 23]). *Let $\mathfrak{D}$ be a ω-admissible concrete domain. Then, satisfiability of a concept w.r.t. an $\mathcal{ALC}(\mathfrak{D})$ TBox is decidable.*

The Expressive Power of Logics

In the previous sections we have described several extensions of the DL $\mathcal{ALC}$ and TBoxes. Some of these extensions increase the complexity of reasoning, while others retain a similar complexity but are not supported by existing reasoners. It is thus natural to ask: can we encode any of these extensions in $\mathcal{ALC}$? Or do they effectively increase the expressive power of $\mathcal{ALC}$?

We adhere to the definition of *expressive power* of knowledge representation languages presented in [6, 73]. According to this view, the expressive power of a concept description, a TBox or a CBox is represented by their models, and is thus tightly related to the semantics assigned to each of these objects. This view is also adopted in *model theory*, a research area of mathematics which studies the relationships between the syntax of formulae in a logical language, typically first-order logic, and the properties of the models that each of these formulae defines [44, 59].

We may want to impose restrictions on the classes of models considered while studying the expressive power. For example, the DL $\mathcal{ALCSCC}$ introduced in Chapter 3 is defined w.r.t. finitely branching interpretations; to compare the expressive power of another DL with $\mathcal{ALCSCC}$, we must either change the semantics of $\mathcal{ALCSCC}$ to account for arbitrary interpretations, or compare the two DLs only w.r.t. finitely branching interpretations. Another class of models with interesting properties is that of finite models, studied in *finite model theory* [44]. Many properties of first-order sentences that hold w.r.t. arbitrary interpretations, such as compactness, fail when restricting to the class of finite interpretations.

We can however use other properties of first-order formulae that hold even if we restrict to finite or finitely branching interpretation to study the expressive power of a logical language. One such property, presented below, is related to the notion of *quantifier depth* of a FOL formula over the signature $\mathsf{N_C} \cup \mathsf{N_R}$.

Definition 2.22. *The* quantifier depth depth(ϕ) *of a* FOL *formula* ϕ *over the signature* $\mathsf{N_C} \cup \mathsf{N_R}$ *is defined as*

- $\text{depth}(A(x)) := 0$ *and* $\text{depth}(r(x, y)) = 0$ *for all* $A \in \mathsf{N_C}$ *and* $r \in \mathsf{N_R}$*;*
- $\text{depth}(\neg\phi) := \text{depth}(\phi)$*;*
- $\text{depth}(\phi \wedge \psi) := \max(\text{depth}(\phi), \text{depth}(\psi))$ *(and similarly for* $\phi \vee \psi$, $\phi \rightarrow \psi$ *and* $\phi \leftrightarrow \psi$*);*
- $\text{depth}(\exists x.\phi) := \text{depth}(\phi) + 1$ *and* $\text{depth}(\forall x.\phi) := \text{depth}(\phi) + 1$.

A fundamental result in finite model theory is given by the *Ehrenfeucht-Fraïsse method*, which provides specific algebraic conditions to ensure that two interpretations satisfy the same FOL formulae of quantifier depth q [44]. In particular, this method relates satisfiability of formulae with depth at most q with the existence of q-isomorphisms, defined as follows (see [44], Definition 1.2.1 of *partial isomorphism* and Definition 1.3.1 of q*-isomorphism*).

Definition 2.23. *A* partial isomorphism *between interpretations* $\mathcal{I}$, $\mathcal{J}$ *of* $\mathsf{N_C}$, $\mathsf{N_R}$ *is an injective partial function* $p \colon \Delta^{\mathcal{I}} \rightharpoonup \Delta^{\mathcal{J}}$ *s.t.* $d \in A^{\mathcal{I}}$ *iff* $p(d) \in A^{\mathcal{J}}$ *and* $(d, d') \in r^{\mathcal{I}}$ *iff* $(p(d), p(d')) \in r^{\mathcal{J}}$ *holds for all* $d, d' \in \Delta^{\mathcal{I}}$ *for which* p *is defined,* $A \in \mathsf{N_C}$ *and* $r \in \mathsf{N_R}$*. Given* $q \in \mathbb{N}$*, a* q-isomorphism *between* $d \in \Delta^{\mathcal{I}}$ *and* $e \in \Delta^{\mathcal{J}}$ *is a sequence* $I_0, \dots, I_q$ *of non-empty sets of partial isomorphisms between* $\mathcal{I}$ *and* $\mathcal{J}$ *with* $\{d \mapsto e\} \in I_q$ *satisfying the following properties for all* $0 \leqslant i < q$*:*

i-***forth*** *if* $p \in I_{i+1}$ *and* $d' \in \Delta^{\mathcal{I}}$ *then there exists* $p' \in I_i$ *that extends* p *and such that* $p'(d') = e'$ *for some* $e' \in \Delta^{\mathcal{J}}$*;*

i-***back*** *if* $p \in I_{i+1}$ *and* $e' \in \Delta^{\mathcal{J}}$ *then there exists* $p' \in I_i$ *that extends* p *and such that* $e' = p'(d')$ *for some* $d' \in \Delta^{\mathcal{I}}$*.*

We say that d, e *are* q-isomorphic *if there is a* q*-isomorphism between* d *and* e*.*

The Ehrenfeucht-Fraïsse method relates the existence of a q-isomorphism between two individuals and the satisfiability of FOL formulae of quantifier depth q w.r.t. these individuals (see [44], Theorem 1.3.2).

Theorem 2.24 (Ehrenfeucht-Fraïsse). *The individuals* d *and* e *are* q*-isomorphic iff they satisfy the same* FOL *formulae* $\phi(x)$ *of quantifier depth at most* q*.*

Hereafter, we write $\mathbb{C}$ to denote a class of interpretations of $\mathsf{N_C}$ and $\mathsf{N_R}$ and introduce the following classes of interpretations:

- the class $\mathbb{C}_{\mathsf{all}}$ of arbitrary interpretations,

- the class $\mathbb{C}_{\mathsf{fb}}$ of *finitely branching interpretations*, i.e. where every individual has finitely many role successors, and
- the class $\mathbb{C}_{\mathsf{fin}}$ of finite interpretations.

Clearly, $\mathbb{C}_{\mathsf{fin}} \subseteq \mathbb{C}_{\mathsf{fb}} \subseteq \mathbb{C}_{\mathsf{all}}$ holds.

Definability and equivalence. The comparison of the expressive power of the DLs $\mathcal{L}$ and $\mathcal{L}'$ defined over $\mathsf{N_C}$ and $\mathsf{N_R}$ as *concept languages* is based on the notion of *equivalence* between an $\mathcal{L}$ concept C and an $\mathcal{L}'$ concept C'.

Definition 2.25. *Let $\mathbb{C}$ be a class of interpretations of $\mathsf{N_C}$ and $\mathsf{N_R}$, and $\mathcal{L}$, $\mathcal{L}'$ two DLs defined over $\mathsf{N_C}$ and $\mathsf{N_R}$. Then, the $\mathcal{L}$ concept C is* $\mathbb{C}$-equivalent *to the $\mathcal{L}'$ concept C' if $C^{\mathcal{I}} = C'^{\mathcal{I}}$ holds for all $\mathcal{I} \in \mathbb{C}$. We say that $\mathcal{L}'$* can be expressed in $\mathcal{L}$ w.r.t. $\mathbb{C}$ *if for every $\mathcal{L}'$ concept C' there is an $\mathcal{L}$ concept C that is $\mathbb{C}$-equivalent to C'. We say that $\mathcal{L}'$* is more expressive than $\mathcal{L}$ w.r.t. $\mathbb{C}$ *if $\mathcal{L}$ can be expressed in $\mathcal{L}'$ but $\mathcal{L}'$ cannot be expressed in $\mathcal{L}$ w.r.t. $\mathbb{C}$.*

In Chapter 2 we provided a first-order translation of $\mathcal{ALC}$ concepts into FOL formulae $\phi(x)$ that is equisatisfiable. While not explicitly stated, this translation shows that $\mathcal{ALC}$ is a fragment of FOL, in that $\pi_x(C)$ provides a *first-order definition* of C as follows. Here, we compare an $\mathcal{L}$ concept C and an FOL order formula $\phi(x)$ over the signature $\mathsf{N_C} \cup \mathsf{N_R}$.

Definition 2.26. *Let $\mathbb{C}$ be a class of interpretations of $\mathsf{N_C}$ and $\mathsf{N_R}$, and $\mathcal{L}$ a DL defined over $\mathsf{N_C}$ and $\mathsf{N_R}$. The first-order formula $\phi(x)$ is* $\mathbb{C}$-equivalent *to the $\mathcal{L}$ concept C if $\phi^{\mathcal{I}} = C^{\mathcal{I}}$ for all $\mathcal{I} \in \mathbb{C}$, where $\phi^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \mathcal{I} \models \phi(d)\}$. We say that C is* FOL-definable w.r.t. $\mathbb{C}$ *if it is $\mathbb{C}$-equivalent to some* FOL *formula $\phi(x)$, and that $\mathcal{L}$ is a* first-order fragment *if every $\mathcal{L}$ concept is* FOL*-definable.*

According to Definition 2.26 both $\mathcal{ALC}$ and $\mathcal{ALCQ}$ (as concept languages) are first-order fragments. It is also clear that $\mathcal{ALC}$ can be expressed in $\mathcal{ALCQ}$ w.r.t. $\mathbb{C}_{\mathsf{fin}}$, $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{all}}$. To show that $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALC}$, we can use the notion of *bisimulation* [73, 92].

Definition 2.27 ($\mathcal{ALC}$ bisimulation). *Let $\mathcal{I}$ and $\mathcal{J}$ be interpretations of $\mathsf{N_C}$ and $\mathsf{N_R}$. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is an $\mathcal{ALC}$* bisimulation *between $\mathcal{I}$ and $\mathcal{J}$ if for all $A \in \mathsf{N_C}$ and all role names $r \in \mathsf{N_R}$ the following three properties are satisfied:*

Atomic *$(d, e) \in \rho$ implies $d \in A^{\mathcal{I}}$ iff $e \in A^{\mathcal{J}}$;*

Forth *if $(d, e) \in \rho$ and d' is an r-successor of d, then there is an r-successor e' of e such that $(d', e') \in \rho$;*

Back *if $(d, e) \in \rho$ and e' is an r-successor of e, then there is an r-successor d' of d such that $(d', e') \in \rho$;*

Two individuals $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$ are called $\mathcal{ALC}$ bisimilar *if there is an $\mathcal{ALC}$ bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$ such that $(d, e) \in \rho$.*

In the context of expressive power, the most important property of an $\mathcal{ALC}$ bisimulation is that the elements that it relates satisfy the same $\mathcal{ALC}$ concepts and thus cannot be distinguished using an $\mathcal{ALC}$ concept description. Formally, we say that an $\mathcal{ALC}$ concept C or an FOL formula $\phi(x)$ is $\mathbb{C}$-*invariant under $\mathcal{ALC}$ bisimulation* if for all $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ and for every $\mathcal{ALC}$ bisimulation ρ that relates $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$ it holds that $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$ (or $d \in \phi^{\mathcal{I}}$ iff $e \in \phi^{\mathcal{J}}$).

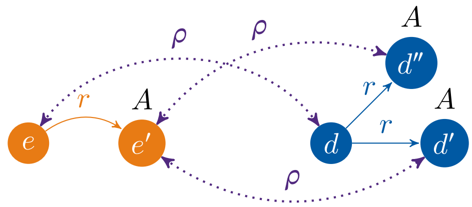


Figure 2.1: A bisimulation ρ between the interpretations $\mathcal{I}$ and $\mathcal{J}$.

Theorem 2.28 ([22]). *Every $\mathcal{ALC}$ concept description is $\mathbb{C}_{\mathsf{all}}$-invariant under $\mathcal{ALC}$ bisimulation.*

Using this fact, we are able to show that $\mathcal{ALCQ}$ cannot be expressed in $\mathcal{ALC}$ *over the same signature sets* $\mathsf{N_C}$ *and* $\mathsf{N_R}$. We present a strategy that will be adopted throughout the thesis: to show that a concept C cannot be expressed w.r.t. a class of interpretations $\mathbb{C}$ in a DL $\mathcal{L}$ for which we have a notion of $\mathcal{L}$ bisimulation, we show how to find two interpretations $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ and individuals $d \in C^{\mathcal{I}}$ and $e \notin C^{\mathcal{J}}$ that are $\mathcal{ALC}$ bisimilar. Combined with a result of the form of Theorem 2.28, this is sufficient to conclude that C is not $\mathbb{C}$-equivalent to any concept description in $\mathcal{L}$.

Proposition 2.29. *There is an $\mathcal{ALCQ}$ concept that is not $\mathbb{C}_{\mathsf{all}}$-equivalent to any $\mathcal{ALC}$ concept.*

Proof. Using the strategy outlined above, we prove by contradiction that the $\mathcal{ALCQ}$ concept $C := (\geqslant 2\, r.A)$ is not $\mathbb{C}_{\mathsf{all}}$-equivalent to any $\mathcal{ALC}$ concept over $\mathsf{N_C} := \{A\}$ and $\mathsf{N_R} := \{r\}$. If C was $\mathbb{C}_{\mathsf{all}}$-equivalent to an $\mathcal{ALC}$ concept D, then the fact that the individuals d and e in the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted in Figure 2.1 are related by the $\mathcal{ALC}$ bisimulation ρ implies, together with Theorem 2.28, that $d \in D^{\mathcal{I}}$ iff $e \in D^{\mathcal{J}}$. This leads to a contradiction, since $d \in C^{\mathcal{I}}$, but $e \notin C^{\mathcal{J}}$. We conclude that C and D cannot be $\mathbb{C}_{\mathsf{all}}$-equivalent. □

Since the interpretations $\mathcal{I}$ and $\mathcal{J}$ used in the proof of Proposition 2.29 are both finite and finitely branching, the result holds also w.r.t. the classes $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$. Similarly to what done for concept languages, we can compare the expressive power of TBoxes or CBoxes defined over the same or different DLs, as well as first-order sentences ϕ.

Definition 2.30. *Let $\mathbb{C}$ be a class of interpretations of $\mathsf{N_C}$ and $\mathsf{N_R}$. Then, the TBoxes $\mathcal{T}$ and $\mathcal{T}'$ are* $\mathbb{C}$-equivalent *if for all $\mathcal{I} \in \mathbb{C}$ it holds that $\mathcal{I} \models \mathcal{T}$ iff $\mathcal{I} \models \mathcal{T}'$. The notion of equivalence for CBoxes is defined similarly. The TBox $\mathcal{T}$ is* FOL-definable w.r.t. $\mathbb{C}$ *if there is a sentence ϕ over the signature $\mathsf{N_C} \cup \mathsf{N_R}$ such that $\mathcal{I} \models \mathcal{T}$ iff $\mathcal{I} \models \phi$ for all $\mathcal{I} \in \mathbb{C}$.*

We show that $\mathcal{ALC}$ CBoxes are more expressive than $\mathcal{ALC}$ TBoxes, i.e. that there is an $\mathcal{ALC}$ CBox that is not equivalent to any $\mathcal{ALC}$ TBox, by showing that all $\mathcal{ALC}$ TBoxes are invariant under (finite) disjoint unions, defined as follows.

Definition 2.31 (Disjoint unions). *Given a (possibly infinite) index set $\mathbb{I}$ and a family of interpretations $(\mathcal{I}_\nu)_{\nu\in\mathbb{I}} \subseteq \mathbb{C}$, their* disjoint union *$\mathcal{I}$ is defined by:*

$$
\begin{aligned}
\Delta^{\mathcal{I}} &:= \{(d,\nu) \mid \nu \in \mathbb{I} \text{ and } d \in \Delta^{\mathcal{I}_\nu}\},\\
A^{\mathcal{I}} &:= \{(d,\nu) \mid \nu \in \mathbb{I} \text{ and } d \in A^{\mathcal{I}_\nu}\} \text{ for all } A \in \mathsf{N_C},\\
r^{\mathcal{I}} &:= \{((d,\nu),(e,\nu)) \mid \nu \in \mathbb{I} \text{ and } (d,e) \in r^{\mathcal{I}_\nu}\} \text{ for all } r \in \mathsf{N_R}.
\end{aligned}
$$

A FOL *formula* $\phi(x)$ *is* $\mathbb{C}$-invariant under (finite) disjoint unions *if for all (finite) families of interpretations* $(\mathcal{I}_\nu)_{\nu\in\mathbb{I}} \subseteq \mathbb{C}$, $\nu \in \mathbb{I}$ *and* $d \in \Delta^{\mathcal{I}_\nu}$ *it holds that* $\mathcal{I}_\nu \models \phi(d)$ *iff* $\mathcal{I} \models \phi((d,\nu))$. *The notion of* $\mathbb{C}$*-invariance under (finite) disjoint unions for concepts* C *is defined similarly.*

A TBox $\mathcal{T}$ *is* $\mathbb{C}$-invariant under (finite) disjoint unions *if, using the notation above,* $\mathcal{I}$ *is a model of* $\mathcal{T}$ *iff every* $\mathcal{I}_\nu$ *with* $\nu \in \mathbb{I}$ *is a model of* $\mathcal{T}$. *Finally, a class* $\mathbb{C}$ *of interpretations is* closed under (finite) disjoint unions *if* $\mathcal{I} \in \mathbb{C}$ *for all* $(\mathcal{I}_\nu)_{\nu\in\mathbb{I}} \subseteq \mathbb{C}$ *(where* $\mathbb{I}$ *is finite).*

For $\mathcal{ALC}$ concepts, invariance under disjoint unions can be shown to hold as a consequence of invariance under $\mathcal{ALC}$ bisimulations [22].

Theorem 2.32. *Let* $\mathbb{C}$ *be closed under (finite) disjoint unions. Then, every* $\mathcal{ALC}$ *concept description is* $\mathbb{C}$*-invariant under (finite) disjoint unions.*

Proof. Using the notation of Definition 2.31, we observe that the relation

$$\rho := \{(d,(d,\nu)) \mid d \in \Delta^{\mathcal{I}_\nu}, \nu \in \mathbb{I}\} \tag{2.1}$$

is an $\mathcal{ALC}$ bisimulation between $\mathcal{I}_\nu$ and $\mathcal{I}$ for every $\nu \in \mathbb{I}$. Thanks to Theorem 2.28 we conclude that invariance under disjoint unions holds for every $\mathcal{ALC}$ concept description. □

Using this result, we are able to show that $\mathcal{ALC}$ TBoxes are also $\mathbb{C}$-invariant under (finite) disjoint unions, provided that $\mathbb{C}$ is closed under (finite) disjoint unions.

Corollary 2.33. *Let* $\mathbb{C}$ *be closed under (finite) disjoint unions. Then, every* $\mathcal{ALC}$ *TBox is* $\mathbb{C}$*-invariant under (finite) disjoint unions.*

Proof. Given an $\mathcal{ALC}$ TBox $\mathcal{T}$, we define the $\mathcal{ALC}$ concept $C_\mathcal{T} := \bigsqcap\{\neg C \sqcup D \mid C \sqsubseteq D \in \mathcal{T}\}$. Here, we use the fact that an interpretation $\mathcal{J}$ is a model of $\mathcal{T}$ iff $C_\mathcal{T}^\mathcal{J} = \Delta^\mathcal{J}$ and apply Theorem 2.32 to this concept and to every individual in $\mathcal{I}$ and $\mathcal{I}_\nu$ to prove that the corollary holds. □

We apply Corollary 2.33 to show that $\mathcal{ALC}$ CBoxes are strictly more expressive than $\mathcal{ALC}$ TBoxes over any class of interpretations $\mathbb{C}$ that contains $\mathbb{C}_{\mathsf{fin}}$ and is closed under (finite) disjoint unions. In particular, there is no $\mathcal{ALC}$ TBox that is $\mathbb{C}$-equivalent to the CR $|A| \leqslant 1$. If, by contradiction, such a TBox $\mathcal{T}$ existed, then the finite interpretation $\mathcal{I}$ containing a single individual belonging to A would be a model of $|A| \leqslant 1$ and thus of $\mathcal{T}$. Using Corollary 2.33, we obtain that the disjoint union $\mathcal{J}$ of $\mathcal{I}$ with itself is a model of $\mathcal{T}$ and thus a model of $|A| \leqslant 1$, which is a contradiction since $A^\mathcal{J}$ contains two elements. Therefore, we conclude that $|A| \leqslant 1$ is not $\mathbb{C}$-equivalent to any $\mathcal{ALC}$ TBox.

Useful model transformations and model properties

Another formal property of $\mathcal{ALC}$ concept descriptions that is especially important in the context of reasoning is the *tree model property*: if a concept is satisfiable w.r.t. a TBox, then it has a model that is a tree in the sense of the following definition.

Definition 2.34. *A* path *of length* ℓ *in an interpretation* $\mathcal{I}$ *is a sequence* $p := \langle d_0 \cdots d_\ell\rangle$ *such that* d_{i+1} *is a role successor of* d_i *for* $0 \leqslant i < \ell$, *and whose* endpoint *is* $\mathsf{end}(p) := d_\ell$. *An interpretation* $\mathcal{I}$ *is a* tree *of depth* ℓ *if there exists* $d \in \Delta^\mathcal{I}$, *called the* root *of* $\mathcal{I}$, *such that every other element in* $\mathcal{I}$ *is connected to* d *by exactly one path of length at most* ℓ *and* d *is not the endpoint of a path. A concept* C *has a* tree model *w.r.t. a TBox* $\mathcal{T}$ *if there is a model* $\mathcal{I}$ *of* C *w.r.t.* $\mathcal{T}$ *that is a directed tree whose root* $d \in \Delta^\mathcal{I}$ *belongs to* $C^\mathcal{I}$.

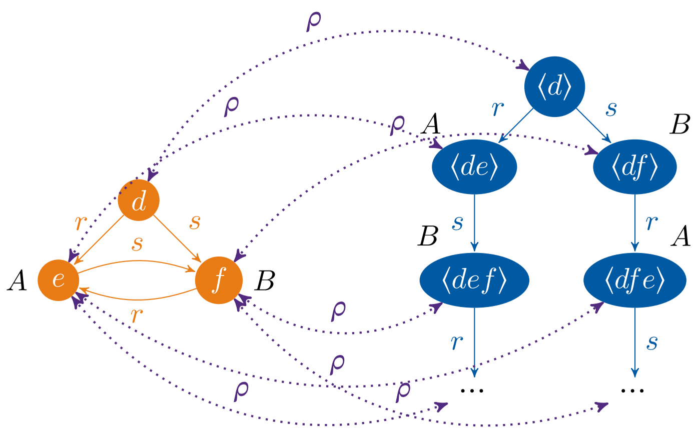


Figure 2.2: An interpretation $\mathcal{I}$ on the left and its unravelling at d on the right. These two interpretations are related by an $\mathcal{ALC}$ bisimulation, that relates d and $\langle d \rangle$.

A well-established technique to turn an interpretation into a tree while preserving satisfiability is that of *unravelling*, which is based on the idea of showing which path was used to reach an individual d' from a given individual d. Using paths of starting point $d \in \Delta^{\mathcal{I}}$ as the domain set, we are ready to formally define the unravelling of d.

Definition 2.35 (unravelling). *Given an interpretation $\mathcal{I}$, the* unravelling *of $d \in \Delta^{\mathcal{I}}$ is the interpretation $\mathrm{u}^{\mathcal{I}}(d)$ whose domain $\Delta^{\mathrm{u}^{\mathcal{I}}(d)}$ is the set of paths in $\mathcal{I}$ with starting point d, that is defined over $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$ as follows:*

$$A^{\mathrm{u}^{\mathcal{I}}(d)} := \{p \in \Delta^{\mathrm{u}^{\mathcal{I}}(d)} \mid \mathsf{end}(p) \in A^{\mathcal{I}}\}$$
$$r^{\mathrm{u}^{\mathcal{I}}(d)} := \left\{(p,p') \in \Delta^{\mathrm{u}^{\mathcal{I}}(d)} \times \Delta^{\mathrm{u}^{\mathcal{I}}(d)} \mid p' = p\langle e \rangle \text{ and } (\mathsf{end}(p), e) \in r^{\mathcal{I}}\right\}.$$

Figure 2.2 partially shows the unravelling of the individual d; the obtained unravelling (on the right) is in particular infinite, since the interpretation $\mathcal{I}$ (on the left) contains a cycle that is reachable from d. Using the properties of $\mathcal{ALC}$ bisimulation, we use the following lemma to show that $\mathcal{ALC}$ has the tree model peoperty.

Lemma 2.36. *Let $\mathcal{I}$ be an interpretation of $\mathsf{N_C}$ and $\mathsf{N_R}$ and $d \in \Delta^{\mathcal{I}}$. Then, there exists an $\mathcal{ALC}$ bisimulation between $\mathcal{I}$ and the unravelling of d that relates d and the root $\langle d \rangle$ of its unravelling.*

Proof. It is shown in [22] that the relation

$$\rho := \{(e,p) \in \Delta^{\mathcal{I}} \times \mathrm{u}^{\mathcal{I}}(d) \mid e = \mathsf{end}(p)\} \tag{2.2}$$

is an $\mathcal{ALC}$ bisimulation between $\mathcal{I}$ and the unravelling of d. □

Corollary 2.37 (Tree model property). *If $\mathcal{T}$ is an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept that is satisfiable w.r.t. $\mathcal{T}$, then C has a tree model w.r.t. $\mathcal{T}$.*

One last interesting formal property of $\mathcal{ALC}$ is that it satisfies the *finite model property*, which states that if a concept is satisfiable w.r.t. a TBox, then it has a *finite model* w.r.t. that TBox, i.e. a model with a finite domain (see [22] for a detailed proof).

Theorem 2.38 (Finite model property of $\mathcal{ALC}$). *Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept. If C has a model with respect to $\mathcal{T}$, then it has a finite model w.r.t. $\mathcal{T}$.*

3 Reasoning with Expressive Cardinality Constraints

In this chapter, we introduce the logics QFBAPA and QFBAPA$^\infty$ [72, 16] and use them to define the DLs $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ [7, 16], as well as *extended cardinality restrictions* (ECRs) over finite and infinite interpretations. We present existing results on concept satisfiability w.r.t. $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ TBoxes [7, 16] and on consistency of knowledge bases with extended CRs over finite interpretations [19, 9], and extend the latter results to $\mathcal{ALCSCC}^\infty$ and arbitrary interpretations. Additionally, we establish the complexity for the entailment problem w.r.t. knowledge bases that use extended CRs and $\mathcal{ALCSCC}$ or $\mathcal{ALCSCC}^\infty$ concepts.

Aside from previous publications, the work contained in this chapter is based on the paper:

[14] Baader, F., De Bortoli, F.: Description Logics That Count, and What They Can and Cannot Count. In: Kovacs, L., Korovin, K., Reger, G. (eds.) ANDREI-60. Automated New-era Deductive Reasoning Event in Iberia. EPiC Series in Computing, pp. 1–25. EasyChair (2020). `https://doi.org/10.29007/ltzn`

The results concerning the consistency for ERCBoxes in Section 3.3 have been adapted from previously published results on RCBoxes, while the complexity results for the entailment problem are unpublished.

3.1 Quantifier-free Boolean Algebra with Presburger Arithmetic

In this logic one can build *set terms* by applying Boolean operations — intersection ($\cap$), union ($\cup$), and complement ($\cdot^c$) — to set variables as well as the constants $\emptyset$ (empty set) and $\mathcal{U}$ (set universe). Set terms s, t can then be used to state inclusion ($s \subseteq t$) and equality constraints ($s = t$) between sets. *PA expressions* k, ℓ are built from non-negative integer constants, PA variables, and set cardinalities $|s|$ using addition as well as multiplication with a non-negative integer constant. They can be used to form numerical constraints of the form $k = \ell$ and $k < \ell$ and n dvd ℓ where k, ℓ are PA expressions and n is a non-negative integer constant. A *formula* of QFBAPA is a Boolean combination of set and numerical constraints.

The semantics of set terms and set constraints is defined using *substitutions* σ that assign a *finite* set $\sigma(\mathcal{U})$ to $\mathcal{U}$ and subsets of $\sigma(\mathcal{U})$ to set variables. The evaluation of set terms and set constraints by such a substitution is defined in the obvious way, using the standard notions of intersection, union, complement[1], inclusion, and equality for sets. PA expressions are evaluated over the natural numbers $\mathbb{N}$. Thus, substitutions additionally assign elements of $\mathbb{N}$ to PA variables and cardinality expressions $|s|$ are evaluated under σ as the cardinality of $\sigma(s)$. When evaluating PA expressions w.r.t. a substitution σ, we employ the usual way of adding, multiplying, and comparing integers.

A *solution* σ of a QFBAPA formula ϕ is a substitution that evaluates ϕ to true, using the above rules for evaluating set and numerical constraints — keeping in mind that n dvd k evaluates to true under σ if $\sigma(k)$ is divisible by n — and the usual interpretation of the Boolean operators occurring in ϕ. The formula ϕ is *satisfiable* if it has a solution.

A variant of QFBAPA that allow sets to be infinite, called QFBAPA$^\infty$, has been introduced in [16] . In this variant, we disallow the usage of divisibility constraints and evaluate PA expressions over $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, i.e., the non-negative integers extended with a symbol for infinity. A substitution σ evaluates $|s|$ as the cardinality of $\sigma(s)$ if such set is finite and as ∞ if $\sigma(s)$ is not finite.[2] We extend the evaluation of PA expressions to $\mathbb{N}^\infty$ using the following rules ranging over $N \in \mathbb{N}$:

1. $\infty + N = N + \infty = \infty = \infty + \infty$,
2. if $N \neq 0$ then $N \cdot \infty = \infty = \infty \cdot N$, else $0 \cdot \infty = 0 = \infty \cdot 0$,
3. $N < \infty$ and $\infty \not< N$, as well as $\infty = \infty$ and $\infty \not< \infty$.

Note that, in QFBAPA$^\infty$, we can enforce infinity of a set even though we do not allow the use of ∞ as a constant. For instance, $|s| = \infty$ is not an admissible numerical constraint, but it is easy to see that the constraint $|s| + 1 = |s|$ can only be satisfied by a substitution that assigns an infinite set to the set term s.

A normal form for set and cardinality constraints. Set constraints $s \subseteq t$ are equivalent to the numerical constraint $|s \cap t^c| \leqslant 0$ in both QFBAPA and QFBAPA$^\infty$ and so we can dispense with them. Divisibility constraints n dvd ℓ in QFBAPA can be replaced with a numerical constraint $\ell = n \cdot |x_s|$ where x_s is a fresh set variable. Finally, we can rewrite equivalently $k < \ell$ as $k \leqslant \ell + 1$. Therefore, we can assume without loss of generality that every QFBAPA and QFBAPA$^\infty$ formula is a Boolean combination of *atomic formulae* of the form

$$\alpha_0 + \alpha_1 t_1 + \cdots + \alpha_k t_k \leqslant \beta_0 + \beta_1 u_1 + \cdots + \beta_\ell u_\ell \tag{3.1}$$

where each expression t_i, u_j is either a set cardinality $|s|$ or a PA variable and all coefficients α_i, β_j are natural numbers.

Satisfiability in QFBAPA and QFBAPA$^\infty$. The satisfiability problem for the logics QFBAPA and QFBAPA$^\infty$ is NP-complete. NP-hardness in both settings is implied by the presence of Boolean constructors in the language, which allow a reduction from e.g. propositional satisfiability, which is NP-complete. Deriving a tight upper bound for this decision problem, on the other hand, is non-trivial. For QFBAPA, satisfiability in non-deterministic polynomial time has first been shown in [72] through the usage of a "sparse solution" lemma (see Fact 1

[1]The complement is defined w.r.t. $\sigma(\mathcal{U})$, that is, $\sigma(s^c) = \sigma(\mathcal{U}) \setminus \sigma(s)$.

[2]Note that we do not distinguish between different infinite cardinalities, such as countably infinite, uncountably infinite, etc.

in [72] and Lemma 3 in [7]), which was later used to derive complexity upper bounds for reasoning in $\mathcal{ALCSCC}$ [7]. We showed in [16] that an analogous "sparse solution" lemma also holds for QFBAPA$^\infty$, implying that satisfiability of QFBAPA$^\infty$ formulae is also decidable in non-deterministic polynomial time.

The notion of "sparse solution" is based on a decomposition of set terms in a QFBAPA or QFBAPA$^\infty$ formula using so-called Venn regions. If ϕ is a formula containing the pairwise distinct set variables $X_1,\ldots,X_k$, a *Venn region* for ϕ is a set term of the form

$$X_1^{c_1} \cap \ldots \cap X_k^{c_k},$$

where c_i is either empty or c for $i = 1, \ldots, k$. For a Venn region v for ϕ and a set variable X in ϕ, we write $X \in v$ to indicate that X occurs without complement in v, and $X \notin v$ if X^c occurs in v.

It is easy to see that we can express every set term in ϕ as a disjoint union of certain Venn regions, which implies that the cardinality of that set term equals the sum of the cardinalities of the used Venn regions. With this decomposition, we can effectively reduce the satisfiability of a set of cardinality constraints to that of a set of linear inequalities over $\mathbb{N}$ for QFBAPA and $\mathbb{N}^\infty$ for QFBAPA$^\infty$. However, a naive encoding could lead to an exponential blowup, since the number of Venn regions is exponential w.r.t. the size of ϕ. This is where the "sparse solution" lemma comes into play (cf. Lemma 3 in [7], Theorem 1 in [16]).

Lemma 3.1 ([7, 16]). *For every QFBAPA (resp. QFBAPA$^\infty$) formula ϕ, one can compute in polynomial time a number N whose value is polynomial in the size of ϕ such that the following holds for every solution σ of ϕ: there is a solution σ' of ϕ such that*

(i) $|\{v \text{ Venn region } s.t.\ \sigma'(v) \neq \emptyset\}| \leq N$, *and*

(ii) $\{v \text{ Venn region } s.t.\ \sigma'(v) \neq \emptyset\} \subseteq \{v \text{ Venn region } s.t.\ \sigma(v) \neq \emptyset\}$.

Using this property of QFBAPA and QFBAPA$^\infty$, we can devise a non-deterministic polynomial time decision procedure to check formula satisfiability and obtain the following complexity result.

Theorem 3.2 ([72, 7, 71, 16]). *The satisfiability problem for QFBAPA and QFBAPA$^\infty$ formulae is NP-complete.*

3.2 The DLs $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$

In this section, we introduce the DLs $\mathcal{ALCSCC}$ [7] and $\mathcal{ALCSCC}^\infty$ [16] which extend $\mathcal{ALC}$ and $\mathcal{ALCQ}$ with constraints over sets of role successors of an individual. We recall their syntax and semantics, and complexity results for concept satisfiability (w.r.t. a TBox) obtained in [7, 16].

Definition 3.3. *Given finite, disjoint sets $\mathsf{N_C}$ of concept names and $\mathsf{N_R}$ of role names, we inductively define the set of $\mathcal{ALCSCC}$ concept descriptions over the signature $(\mathsf{N_C}, \mathsf{N_R})$ by extending Definition 2.9 with the following rule:*

Role successor constraint *if* $\mathtt{con}$ *is a set or numerical constraint of QFBAPA using role names and already defined $\mathcal{ALCSCC}$ concept descriptions over $(\mathsf{N_C}, \mathsf{N_R})$ as (set) variables, then* $\mathsf{succ}(\mathtt{con})$ *is an $\mathcal{ALCSCC}$ concept description over* $(\mathsf{N_C}, \mathsf{N_R})$.

For example, the $\mathcal{ALCSCC}^\infty$ concept description $\mathsf{Human} \sqcap \mathsf{succ}(|\mathsf{own} \cap \mathsf{Car}| = |\mathsf{child} \cap \mathsf{Human}|)$ describes all persons that own as many cars as the number of children they have, without specifying the exact quantities. Of course, successor constraints can also be nested, as in the concept description $\mathsf{Human} \sqcap \mathsf{succ}(\mathsf{child} \cap \mathsf{Human} \subseteq \mathsf{succ}(|\mathsf{own} \cap \mathsf{Car}| = 0))$, which describes all individuals whose children own no cars.

Definition 3.4. *We define the interpretation of $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ concept descriptions recursively, by extending Definition 2.10 to role successor constraints as follows. For $\mathcal{ALCSCC}$ concepts, we only consider finitely branching interpretations $\mathcal{I}$ and define for $d \in \Delta^\mathcal{I}$ the substitution σ_d that assigns to $\mathcal{U}$ as the set $\mathsf{ars}^\mathcal{I}(d) := \bigcup\{r^\mathcal{I}(d) \mid r \in \mathsf{N_R}\}$ of all role successors of d, to $\emptyset$ the empty set, to each role name r occurring in $\mathtt{con}$ the set $r^\mathcal{I}(d)$ and to concept descriptions D the set $D^\mathcal{I} \cap \mathsf{ars}^\mathcal{I}(d)$ (i.e., the set of role successors of d that belong to D).[3] Then $d \in \mathsf{succ}(\mathtt{con})^\mathcal{I}$ iff σ_d is a solution of the QFBAPA formula $\mathtt{con}$.*

For $\mathcal{ALCSCC}^\infty$ concepts we consider arbitrary interpretations and define the semantics similarly, except that $\mathtt{con}$ is now evaluated as a QFBAPA$^\infty$ formula.

The concept satisfiability problem is PSpace-complete in the absence of a TBox and ExpTime-complete w.r.t. a TBox for both $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$. For $\mathcal{ALCSCC}$, this has been shown in [7], and the same ideas were adapted to show the result for $\mathcal{ALCSCC}^\infty$ in [16]. We observe that for $\mathcal{ALCQ}$ and $\mathcal{ALC}$, the complexity of the concept satisfiability problem restricted to finite and finitely branching interpretations is derived directly from the general case, since both DLs satisfy the finite model property. In contrast, $\mathcal{ALCSCC}^\infty$ contains concepts that have an infinite model but neither a finitely branching nor a finite model. For example, the successor constraint $\mathsf{succ}(|r| + 1 = |r|)$ is satisfiable by an interpretation containing an element with infinitely many r-successors; however, the fact that $n + 1 \neq n$ for all $n \in \mathbb{N}$ implies that there cannot be a model of this concept that is finitely branching or finite. Nevertheless, the two problems have the same complexity.

Theorem 3.5 ([7, 16]). *Concept satisfiability in $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ is PSpace-complete without a TBox and ExpTime-complete in the presence of a TBox.*

3.3 Extended Cardinality Restrictions

So far, we have considered knowledge bases that are purely terminological, in the form of TBoxes, or that contain simple quantitative information, in the form of CBoxes. The cardinality restrictions that we may use in a CBox allow us to compare the cardinality of the interpretation of a certain concept description with a fixed natural number n, but do not enable other forms of comparison, for instance, one cannot use a CBox to compare the cardinalities of the interpretations of different concept descriptions.

In the same spirit of the previous section, where we defined $\mathcal{ALCSCC}$ as a generalization of $\mathcal{ALCQ}$ where we can define constraints over sets of role successors using a QFBAPA formula, we generalize TBoxes and CBoxes by introducing more expressive cardinality restrictions based on QFBAPA. Some extensions that are defined here allow for Boolean combinations of cardinality restrictions: to ease the comparison of the expressive power of terminological and quantitative knowledge bases, we also consider Boolean variants of TBoxes and CBoxes.

[3] Note that, by induction, the sets $D^\mathcal{I}$ are well-defined.

Definition 3.6. *Given a DL language $\mathcal{L}$, we say that a* Boolean $\mathcal{L}$ TBox *is a Boolean combination of $\mathcal{L}$ TBoxes, and that a* Boolean $\mathcal{L}$ CBox *is a Boolean combination of $\mathcal{L}$ CBoxes (cf. Definition 2.16). An* extended CR *is an inequality of the form*

$$n_0 + n_1|C_1| + \cdots + n_k|C_k| \leqslant m_0 + m_1|D_1| + \cdots + m_\ell|D_\ell|, \tag{3.2}$$

where the C_i, D_j are $\mathcal{L}$ concept descriptions and the n_i, m_j are natural numbers. A semi-restricted CR *is an extended CR where $m_0 = 0$, and a* restricted CR *is one that additionally satisfies $n_0 = 0$. Then:*

- *an $\mathcal{L}$* ECBox *is a Boolean combination of extended CRs,*
- *an $\mathcal{L}$* ERCBox *is an ECBox consisting of positive Boolean combination of semi-restricted CRs and*
- *an $\mathcal{L}$* RCBox *is an ERCBox consisting of a conjunction of restricted CRs.*

A finite interpretation $\mathcal{I}$ is a model *of the extended CR* (3.2) *if*

$$n_0 + n_1|C_1^\mathcal{I}| + \cdots + n_k|C_k^\mathcal{I}| \leqslant m_0 + m_1|D_1^\mathcal{I}| + \cdots + m_\ell|D_\ell^\mathcal{I}| \tag{3.3}$$

is true, when evaluated as a QFBAPA formula. Extending this condition, we define the notions of model and (in)consistency for Boolean TBoxes, Boolean CBoxes, ECBoxes, ERCBoxes and RCBoxes in the usual way. Further, we extend the notion of model and the related definitions to arbitrary interpretations $\mathcal{I}$, by evaluating (3.2) *as a QFBAPA$^\infty$ formula.*

For a given class $\mathbb{C}$ of interpretations, an ECBox $\mathcal{E}$ $\mathbb{C}$-entails *an extended CR* (3.2) *if every model $\mathcal{I} \in \mathbb{C}$ of $\mathcal{E}$ satisfies* (3.3)*; the notion of $\mathbb{C}$-entailment of a (semi)-restricted CR w.r.t. an ERCBox and an RCBox is defined similarly.*

Extended Cardinality Constraints and Boolean CBoxes

The consistency problem for $\mathcal{ALC}$ and $\mathcal{ALCSCC}$ ECBoxes w.r.t. finite interpretations is known to be NExpTime-complete for binary coding of numbers [19, 8, 9]. The upper bound is obtained by a reduction from consistency of an ECBox $\mathcal{E}$ to satisfiability of a QFBAPA formula $\delta_\mathcal{E}$ which is exponentially larger than $\mathcal{E}$. Since the satisfiability problem for QFBAPA is NP-complete for binary coding of numbers, as mentioned in Section 3.1, this yields the sought upper bound for $\mathcal{ALC}$ and $\mathcal{ALCSCC}$. We can easily transfer this upper bound to $\mathcal{ALCSCC}^\infty$ ECBoxes evaluated w.r.t. arbitrary interpretations by using the same translation $\delta_\mathcal{E}$ for $\mathcal{E}$ and test its satisfiability in QFBAPA$^\infty$, whose satisfiability problem is also NP-complete as argued in Section 3.1

NExpTime-hardness of the consistency problem for ECBoxes and Boolean CBoxes in $\mathcal{ALCQ}$, $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ under binary coding of numbers is a clear consequence of the complexity of reasoning with $\mathcal{ALCQ}$ CBoxes shown in Theorem 2.17.

Theorem 3.7 ([96, 19, 8, 16]). *If $\mathcal{DL} \in \{\mathcal{ALCQ}, \mathcal{ALCSCC}, \mathcal{ALCSCC}^\infty\}$, then consistency of $\mathcal{DL}$ Boolean CBoxes and ECBoxes is NExpTime-complete w.r.t. finite and arbitrary interpretations if numbers are encoded in binary. For ECBoxes, NExpTime-hardness already holds for unary coding of numbers.*

The reason why the coding of numbers is irrelevant in the presence of ECBoxes is that we can succinctly represent coefficients in the ECBoxes using PA expressions (see [19] for a more detailed argument). In particular, a positive number n can be represented in QFBAPA using a term s_n of size at most $2\log(n)$ where $s_0 := |\emptyset|$, $s_{2m} := 2 \cdot s_m$ and $s_{2m+1} := s_{2m} + 1$ for $m \in \mathbb{N}$. Thus, we can simulate binary coding of numbers even in the unary case.

Boolean TBoxes

We have seen in Chapter 2 that reasoning w.r.t. an $\mathcal{ALCQ}$ TBox is an ExpTime-complete problem, and in Section 3.2 we argued that the same complexity is attained for reasoning w.r.t. a TBox in $\mathcal{ALCSCC}$ [7] and $\mathcal{ALCSCC}^\infty$ [16]. This complexity result holds when we generalize to Boolean TBoxes in $\mathcal{ALCQ}$, $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$.

To prove the exponential time upper bound, we notice that consistency of a Boolean TBox $\mathcal{T}$ can be reduced in exponential time to checking consistency of an exponential number of TBoxes w.r.t. the size of $\mathcal{T}$. First, bring $\mathcal{T}$ into an equivalent Boolean TBox in disjunctive normal form; then, we replace every negated CI $\neg(C \sqsubseteq D)$ with a non-negated CI $\top \sqsubseteq \exists r.(C \sqcap \neg D)$ where r is a fresh role name. We prove that this second step yields a correct reduction.

Proposition 3.8. *Let C and D be concept descriptions over the signature $(\mathsf{N_C}, \mathsf{N_R})$ where $\mathsf{N_R}$ is non-empty. Then, the negated CI $\neg(C \sqsubseteq D)$ is satisfiable if and only if the CI $\top \sqsubseteq \exists r.(C \sqcap \neg D)$ with r a fresh role name is satisfiable.*

Proof. If $\mathcal{I}$ is a model of $\neg(C \sqsubseteq D)$, then there exists $d \in (C \sqcap \neg D)^\mathcal{I}$. Then, the interpretation $\mathcal{I}'$ obtained by adding d as r-successor of every individual in $\Delta^\mathcal{I}$ satisfies the CI $\top \sqsubseteq \exists r.(C \sqcap \neg D)$. Vice versa, it is clear that every model of $\top \sqsubseteq \exists r.(C \sqcap \neg D)$ is also a model of $\neg(C \sqsubseteq D)$. □

The conversion described above runs in exponential time w.r.t. the size of $\mathcal{T}$ and produces a disjunction of at most exponentially many TBoxes, each of which can be tested for consistency in exponential time in any of the aforementioned DLs. Since $\mathcal{T}$ is consistent if and only if at least one of the obtained disjuncts is consistent, we deduce that $\mathcal{T}$ can be tested for consistency in exponential time.

Theorem 3.9. *Consistency of Boolean TBoxes in $\mathcal{ALCQ}$, $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ is ExpTime-complete, both for unary and binary encoding of numbers. Consequently, entailment of a CI $C \sqsubseteq D$ w.r.t. a (Boolean) TBox in these DLs is ExpTime-complete, both for unary and binary encoding of numbers.*

Proof. Hardness of this problem follows from the fact that TBoxes are a special instance of Boolean TBoxes, and the consistency problem for TBoxes in the mentioned DLs is ExpTime-hard. The exponential time upper bound is described above.

For the second point, we notice that $\mathcal{T}$ entails $C \sqsubseteq D$ iff the Boolean TBox $\mathcal{T} \wedge \neg(C \sqsubseteq D)$ is consistent, and since this reduction is polynomial the exponential time upper bound is preserved. ExpTime-hardness of the entailment problem follows from the fact that $\mathcal{T}$ is inconsistent iff it entails the CI $\top \sqsubseteq \bot$. □

(Semi-)restricted Cardinality Constraints

Restricted cardinality constraints over $\mathcal{ALC}$ concepts were first introduced in [19], as a way to reduce the complexity of reasoning w.r.t. ECBoxes. Indeed, it was shown there that the consistency problem for $\mathcal{ALC}$ RCBoxes is ExpTime-complete. Later, the same problem was shown to be ExpTime-complete for $\mathcal{ALCSCC}$ [8, 9], using an approach based on type elimination. Meanwhile, ERCBoxes were introduced in [93] and there it was shown that (finite) consistency of knowledge bases of this kind is ExpTime-complete for a DL that subsumes $\mathcal{ALCQ}$ but neither $\mathcal{ALCSCC}$ nor $\mathcal{ALCSCC}^\infty$. Later, this complexity result was shown to hold for $\mathcal{ALCSCC}$ [10], using a type elimination algorithm to derive the upper bound.

While it would be possible to adapt the type elimination approach to show that the exponential time upper bounds for checking consistency of $\mathcal{ALCSCC}^\infty$ RCBoxes and ERCBoxes hold, we give here a simpler proof of this result for $\mathcal{ALCSCC}^\infty$. Recall that an $\mathcal{ALCSCC}^\infty$ ERCBox $\mathcal{R}$ is a positive Boolean combination of inequalities of the form

$$n_0 + n_1|C_1| + \cdots + n_k|C_k| \leqslant n_{k+1}|C_{k+1}| + \cdots + n_{k+\ell}|C_{k+\ell}|, \tag{3.4}$$

where each C_i is a $\mathcal{ALCSCC}^\infty$ concept description and each n_i is a natural number. In Chapter 5 we show that ERCBoxes are preserved under disjoint unions, that is, if $\mathbb{I}$ is a set of models of $\mathcal{R}$, then their disjoint union is also a model of $\mathcal{R}$. Using this fact, it is straightforward to see that systems of inequalities (3.4) are satisfiable iff they have a solution where every concept C_i is assigned to a set that is either empty or infinite, since we can take the disjoint union of countably infinitely many copies of a solution of (3.4) and still obtain a solution. We use this last property to reduce consistency of $\mathcal{ALCSCC}^\infty$ ERCBoxes to consistency of Boolean $\mathcal{ALCSCC}^\infty$ TBoxes. First, we transform $\mathcal{R}$ to an equivalent ERCBox in disjunctive normal form, i.e. a disjunction of ERCBoxes $\mathcal{R}_1$, ..., $\mathcal{R}_m$ where m is at most exponentially large w.r.t. $\mathcal{R}$ and each $\mathcal{R}_i$ is of polynomial size w.r.t. $\mathcal{R}$. For each of these ERCBoxes, we consider a TBox $\mathcal{T}_i$ that is initialized as $\mathcal{T}_i := \emptyset$ for $i = 1, \dots, m$. Then, we set $c := 1$ and proceed with the following steps:

1. Check if the $\mathcal{ALCSCC}^\infty$ TBox $\mathcal{T}_c$ is consistent. If it is inconsistent, proceed to step (3). Otherwise, for all concepts C_j occurring in a term $n_j|C_j|$ in an inequality of $\mathcal{R}_c$, check if C_j is satisfiable w.r.t. $\mathcal{T}_c$, and if not add the CI $C_j \sqsubseteq \bot$ to $\mathcal{T}_c$. Then, go to step (2).

2. For all inequalities (3.4) in $\mathcal{R}_c$ such that $C_{k+j} \sqsubseteq \bot$ belongs to $\mathcal{T}_c$ for $j = 1, \dots, \ell$, check if $n_0 = 0$. If $n_0 \neq 0$, proceed to step (3). Otherwise, add $C_i \sqsubseteq \bot$ to $\mathcal{T}_c$ for $i = 1, \dots, k$. If no new CI has been added to $\mathcal{T}_c$, then return consistent. Otherwise, go to step (1).

3. If $c = m$, return inconsistent, otherwise increment c by 1 and continue with step (1).

This approach has been used in [14] to prove that the consistency problem for $\mathcal{ALCSCC}^\infty$ RCBoxes is decidable in exponential time. Here, we use it to provide an exponential time upper bound for checking consistency of $\mathcal{ALCSCC}^\infty$ ERCBoxes.

Lemma 3.10. *The algorithm terminates after an exponential number of iterations, and it returns consistent iff the ERCBox $\mathcal{R}$ is consistent.*

Proof. Termination after an exponential number of iterations is an immediate consequence of the fact that m is exponential in the size of $\mathcal{R}$, for $i = 1, \dots, m$ the size of $\mathcal{R}_i$ is polynomial w.r.t. $\mathcal{R}$ and only polynomially many CIs of the form $C_j \sqsubseteq \bot$ can be added to $\mathcal{T}_i$, since the concepts C_j for which such a CI can be added must occur in an inequality in $\mathcal{R}_i$.

Now, assume that $\mathcal{R}$ and hence its disjunctive normal form is consistent, and let $\mathcal{I}$ be a model of $\mathcal{R}_c$ for some $1 \leqslant c \leqslant m$. By an induction on the number of iterations, it is easy to show that we must have $C^\mathcal{I} = \emptyset$ for all CIs added to $\mathcal{T}_c$ during the run of the algorithm. Consequently, the algorithm cannot reach Step 3 for the selected value of c, since $\mathcal{I}$ is a model of $\mathcal{T}$. Since the algorithm always terminates, it must thus return consistent.

Next, assume that the algorithm returns consistent, let $1 \leqslant c \leqslant m$ be the value for which the algorithm terminated and let $\mathcal{T}_c$ be the corresponding TBox. Then $\mathcal{T}_c$ is consistent, and for every concept C occurring in an inequality of $\mathcal{R}_c$ such that $C \sqsubseteq \bot$ does not belong to $\mathcal{T}_c$, there is a model $\mathcal{I}_C$ of $\mathcal{T}$ such that $C^{\mathcal{I}_C} \neq \emptyset$. Later we will show that $\mathcal{ALCSCC}^\infty$ TBoxes are invariant under disjoint unions. As a consequence, this implies that there is an interpretation $\mathcal{I}_\infty$ such that the following holds for all concepts C occurring in an inequality of $\mathcal{R}_c$:

- if $C \sqsubseteq \bot$ belongs to $\mathcal{T}_c$, then $C^{\mathcal{I}_\infty} = \emptyset$;
- if $C \sqsubseteq \bot$ does not belong to $\mathcal{T}_c$, then the cardinality of $C^{\mathcal{I}_\infty}$ is infinite.

It remains to shows that $\mathcal{I}_\infty$ is a model of $\mathcal{R}_c$. Thus, consider an inequality (3.4) in $\mathcal{R}_c$. If there is a j with $k+1 \leqslant j \leqslant k+\ell$ such that $C_j^{\mathcal{I}_\infty}$ is infinite, then this inequality is clearly satisfied by $\mathcal{I}_\infty$. Otherwise, $C_j \sqsubseteq \bot$ belongs to $\mathcal{T}_c$ for all $k+1 \leqslant j \leqslant k+\ell$, and thus also $C_i \sqsubseteq \bot$ belongs to $\mathcal{T}_c$ for all i with $1 \leqslant i \leqslant k$. This shows that, again, the inequality is solved. We conclude that $\mathcal{I}_\infty$ is a model of $\mathcal{R}_c$ and thus of $\mathcal{R}$. □

Given that consistency of $\mathcal{ALCSCC}^\infty$ TBoxes can be tested in exponential time, as seen above, the algorithm we described above runs in exponential time. Combined with the fact that CIs $C \sqsubseteq D$ are equivalent to inequalities $|C \sqcap \neg D| \leqslant 0$ (and, over finite models, to $|C| \leqslant |C \sqcap D|$) and thus reasoning with ERCBoxes is as hard as reasoning with TBoxes, we obtain the following result for $\mathcal{ALCSCC}^\infty$ and sublogics. The analogous result for $\mathcal{ALCSCC}$ has been proved in [10].

Proposition 3.11. *The consistency problem for ERCBoxes in $\mathcal{ALCQ}$, $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ is ExpTime-complete, both for unary and binary encoding of numbers.*

Regarding the entailment problem for RCBoxes and ERCBoxes, we prove that the complexity of the problem changes depending on the coefficients values occurring in the knowledge base, as follows.

Theorem 3.12. *Checking if an ERCBox $\mathcal{R}$ entails an ERCBox $\mathcal{R}'$ in $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ is an ExpTime-complete problem if $\mathcal{R}'$ only contains inequalities* (3.4) *with $n_0 = 0$ or $n_0 = 1$, and coNExpTime-complete otherwise.*

Proof. First, we notice that the negation of a semi-restricted cardinality constraint (3.4) with $n_0 = 0$ or $n_0 = 1$ is also a semi-restricted cardinality constraint. If we assume that all inequalities in $\mathcal{R}'$ are of this form, then by negating $\mathcal{R}'$ we obtain another ERCBox $\mathcal{R}''$. In particular, $\mathcal{R}$ entails $\mathcal{R}'$ iff $\mathcal{R} \wedge \mathcal{R}''$ is inconsistent, which can be checked in exponential time. Hardness of this problem can be derived by reducing entailment of a CI w.r.t. a TBox to entailment of a semi-restricted cardinality constraint with $n_0 = 0$ w.r.t. a ERCBox or non-entailment of a semi-restricted cardinality constraint with $n_0 = 1$ w.r.t. a ERCBox.

In the general case, we are able to reduce the consistency problem for $\mathcal{ALCO}$ ERCBoxes (see Chapter 9 for a definition of $\mathcal{ALCO}$), which is known to be NExpTime-complete [10], to non-entailment of a semi-restricted cardinality constraint w.r.t. an $\mathcal{ALC}$ ERCBox. Let $\mathcal{R}$ be an $\mathcal{ALCO}$ ERCBox containing the nominals $\{o_1\}, \dots, \{o_n\}$. We transform $\mathcal{R}$ into an $\mathcal{ALC}$ ERCBox $\mathcal{R}'$ by first replacing each occurrence of $\{o_i\}$ in $\mathcal{R}$ with a fresh concept name A_i and then, if the unique name assumption is in place, by adding the semi-restricted cardinality constraints $|A_i| \geqslant 1$ and $|A_i \sqcap A_j| \leqslant 0$ for $1 \leqslant i < j \leqslant n$. We observe that $\mathcal{R}$ is consistent iff $\mathcal{R}'$ does not entail the semi-restricted cardinality constraint $|A_1| + \dots + |A_n| \geqslant n+1$. If $\mathcal{R}$ has a model $\mathcal{I}$, we construct a model $\mathcal{I}'$ of $\mathcal{R}'$ by setting $A_i^{\mathcal{I}'} := \{o_i\}^{\mathcal{I}}$ for $i = 1, \dots, n$. Clearly, this interpretation does not satisfy $|A_1| + \dots + |A_n| \geqslant n+1$, and we conclude that $\mathcal{R}'$ does not entail this semi-restricted cardinality constraint. On the other hand, assume that $\mathcal{R}'$ does not entail $|A_1| + \dots + |A_n| \geqslant n+1$. Then, there is a model $\mathcal{I}'$ of $\mathcal{R}'$ that satisfies $|A_1| + \dots + |A_n| \leqslant n$, and since $\mathcal{I}'$ satisfies $|A_i| \geqslant 1$ we deduce that $A_i^{\mathcal{I}'}$ contains a single individual d_i for $i = 1, \dots, n$. In particular, d_i and d_j are distinct if $i \neq j$. Then, we obtain a model $\mathcal{I}$ of $\mathcal{R}$ by setting $\{o_i\}^{\mathcal{I}} := \{d_i\}$

for $i = 1, \ldots, n$. We conclude that the entailment problem in consideration is coNExpTime-hard. To prove membership in coNExpTime, it is sufficient to notice that the negation of an ERCBox is an ECBox, and therefore we can reduce non-entailment of an ERCBox w.r.t. to another ERCBox to consistency of an $\mathcal{ALCSCC}$ or $\mathcal{ALCSCC}^\infty$ ECBoxes, which is known to be decidable in non-deterministic exponential time. □

Summary

We started this chapter by introducing the DLs $\mathcal{ALCSCC}$ [7] and $\mathcal{ALCSCC}^\infty$ [16] and several generalizations of TBoxes and CBoxes based on *(semi-)restricted* or *extended cardinality constraints* [19, 93, 8, 10]. We reviewed the existing results on the complexity of concept satisfiability w.r.t. an $\mathcal{ALCSCC}$ or $\mathcal{ALCSCC}^\infty$ TBox, and generalized the results concerning the consistency problem for $\mathcal{ALCSCC}$ ECBoxes and subclasses thereof w.r.t. finite interpretations, lifting them to the setting where arbitrary interpretations are allowed. Then, we looked at the complexity of the entailment problem, which can be reduced to consistency or concept satisfiability in most cases, with the exception of the case of Boolean combinations of (semi-)restricted cardinality constraints. There, reducibility to consistency depends on the employed coefficients, and these coefficients influence the complexity of checking entailment. We summarize the results of this chapter in the following table, where we assume binary coding of numbers.

		$\mathcal{ALCSCC}$	$\mathcal{ALCSCC}^\infty$
Concept satisfiability	with no TBox	PSpace-c. [7]	PSpace-c. [16]
	w.r.t. a TBox	ExpTime-c. [7]	ExpTime-c. [16]
Boolean TBox consistency		ExpTime-c.	ExpTime-c.
RCBox consistency		ExpTime-c. [8]	ExpTime-c.
ERCBox	consistency	ExpTime-c. [10]	ExpTime-c.
	entailment	coNExpTime-c.	coNExpTime-c.
CBox consistency		NExpTime-c. [8]	NExpTime-c.
ECBox consistency		NExpTime-c. [8]	NExpTime-c.

Complexity of entailment is only indicated where it differs from consistency, and each entry without citation corresponds to a novel contribution of this thesis.

4 Expressive Power of $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ over Restricted Classes Of Models

In this chapter, we investigate the expressive power of the DL $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ over the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$ of arbitrary, finitely branching and finite interpretations. We introduce the notion of *Presburger (Pr) bisimulation* and show that $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ concepts are invariant under Pr bisimulation. Then, we introduce the notion of Pr (q,ℓ)-bisimulation with $q, \ell \in \mathbb{N}$ and show that for FOL formulae $\phi(x)$ invariance under Pr bisimulation is equivalent to invariance under Pr (q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$. This result is used to prove that there are $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ concepts that are not FOL-definable. Finally, we introduce the DL $\mathcal{ALCQ}t$ and prove that it corresponds to the FOL-definable fragment of $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$, and in particular it is the fragment of FOL that is invariant under Pr bisimulation.

The work contained in this chapter is based on the paper:

[18] Baader, F., De Bortoli, F.: The Expressive Power of Description Logics with Numerical Constraints over Restricted Classes of Models. In: Thiemann, R., Weidenbach, C. (eds.) Proceedings of the 5th International Symposium on Frontiers of Combining Systems (FroCoS '25). LNAI,. Vol. 15979. Springer, Heidelberg (2025). https://doi.org/10.1007/978-3-032-04167-8_2

4.1 Presburger Bisimulation

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be w.l.o.g. finite. This assumption is not too restrictive, as it is encountered every time we study the expressive power of a given concept or ontology. We base our inquiry into the expressive power of $\mathcal{ALCSCC}$ concepts on the notion of *safe role type*. A *safe role type* (w.r.t. $\mathsf{N_R}$) is a subset $\tau \neq \emptyset$ of $\mathsf{N_R}$. Alternatively, we can view τ as a set term of QFBAPA, obtained as the intersection of one or more role names $r \in \mathsf{N_R}$ and all literals r^c for all the remaining role names in $\mathsf{N_R}$. For example, if $\mathsf{N_R} = \{r, s, t\}$, then $r \cap s \cap t^c$ is a safe role type, while $r^c \cap s^c \cap t^c$, $r \cap s$ and $r \cap s \cap t^c \cap r^c$ are not. Safe role types τ are interpreted w.r.t. an interpretation $\mathcal{I}$ as the relation

$$\tau^\mathcal{I} := (\bigcap_{r\in\tau} r^\mathcal{I} \setminus (\bigcup_{r\in\mathsf{N_R}\setminus\tau} r^\mathcal{I})),$$

which is always a subset of $\bigcup_{r\in\mathsf{N_R}} r^\mathcal{I}$ and thus always relating elements that are connected by some role. Then, $\tau^\mathcal{I}(d) := \{e \in \Delta^\mathcal{I} \mid (d,e) \in \tau^\mathcal{I}\}$ is a finite set for $d \in \Delta^\mathcal{I}$ if $\mathcal{I}$ is finitely branching, and $e \in \mathsf{ars}^\mathcal{I}(d)$ iff $e \in \tau^\mathcal{I}(d)$ for exactly one safe role type τ. For $\mathcal{ALCSCC}^\infty$ we showed in [16] that each set term s in a cardinality constraint $\mathfrak{con}$ within a succ-restriction can be rewritten as the disjoint union of terms of the form $\tau \cap C$ where τ is a safe role type and C an $\mathcal{ALCSCC}^\infty$ concept. Adapting the proof in [16], we obtain that this property holds for $\mathcal{ALCSCC}$ concepts.

Proposition 4.1. *Let $\mathbb{C} \subseteq \mathbb{C}_\mathsf{fb}$ be a class of interpretations. Then, every $\mathcal{ALCSCC}$ concept of the form* $\mathsf{succ}(\mathfrak{con})$ *is $\mathbb{C}$-equivalent to a concept of the form* $\mathsf{succ}(\mathfrak{con}')$ *where $\mathfrak{con}'$ only contains set terms of the form $\tau \cap C$, where τ is a safe role type and C an $\mathcal{ALCSCC}$ concept.*

Proof. Every set term s occurring in a cardinality expression $|s|$ in $\mathfrak{con}$ uses role names in $\mathsf{N_R}$ and $\mathcal{ALCSCC}$ concept descriptions as set variables. By the semantics of $\mathcal{ALCSCC}$, we obtain that $\sigma_d(s) = \sigma_d(s) \cap \sigma_d(\mathcal{U}) = \sigma_d(s \cap \mathcal{U}) = \sigma_d(s \cap (\bigcup \mathsf{N_R}))$ holds for all $\mathcal{I} \in \mathbb{C}$ and $d \in \Delta^\mathcal{I}$. Therefore, we can replace s with the set term $s \cap (\bigcup \mathsf{N_R})$. By bringing this set term into "disjunctive normal form" using the distributivity of set intersection over set union, we obtain an equivalent set term t that is a union of set terms of the form $t' \cap C$, where t' is an intersection of role literals containing at least one role positively and C is a concept description. Clearly, t' can be expressed as the union of one or more safe role types. After rewriting t' this way and using distributivity again as above, we can aggregate the resulting terms of t over the same safe role type τ into a unique term, since $(\tau \cap C_1) \cup (\tau \cap C_2)$ is equivalent to $(\tau \cap (C_1 \sqcup C_2))$. Therefore, t is equivalent to a disjoint union of terms of the form $\tau \cap C$. □

Proposition 4.1 suggests the following definition of *bisimulation*. We adapt the notion of *counting bisimulation* [80] to account for safe role types and define invariance w.r.t. a class $\mathbb{C}$ of interpretations.

Definition 4.2. *Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be finite and $\mathbb{C}$ a class of interpretations. The binary relation $\rho \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{J}$ is a* Presburger (Pr) bisimulation *between the interpretations $\mathcal{I}$ and $\mathcal{J}$ if for all $A \in \mathsf{N_C}$ and all safe role types τ over $\mathsf{N_R}$ the following properties are satisfied:*

Atomic *$(d,e) \in \rho$ implies $d \in A^\mathcal{I}$ iff $e \in A^\mathcal{J}$;*

Forth *if $(d,e) \in \rho$ and $D \subseteq \tau^\mathcal{I}(d)$ is finite, then there is a set $E \subseteq \tau^\mathcal{J}(e)$ such that ρ contains a bijection between D and E;*

Back *if $(d,e) \in \rho$ and $E \subseteq \tau^\mathcal{J}(e)$ is finite, then there is a set $D \subseteq \tau^\mathcal{I}(d)$ such that ρ contains a bijection between D and E.*

We call $d \in \Delta^\mathcal{I}$ and $e \in \Delta^\mathcal{J}$ Pr bisimilar *if $(d,e) \in \rho$ for some Pr bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$. A concept C is* $\mathbb{C}$-invariant *under Pr bisimulation if $d \in C^\mathcal{I}$ iff $e \in C^\mathcal{J}$ holds for all Pr bisimilar individuals $d \in \Delta^\mathcal{I}$, $e \in \Delta^\mathcal{J}$ with $\mathcal{I}, \mathcal{J} \in \mathbb{C}$.*

The notion of counting bisimulation is obtained by replacing safe role types τ in Definition 4.2 with role names r. In [16] we proved that $\mathcal{ALCSCC}^\infty$ concepts are $\mathbb{C}_\mathsf{all}$-invariant under Pr bisimulation. A similar property holds for $\mathcal{ALCSCC}$ concepts: here, we only need to check that invariance holds w.r.t. finitely branching interpretations, and the proof used for $\mathcal{ALCSCC}^\infty$ in [16] applies with minimal modifications. Since finite interpretations are finitely branching, this implies $\mathbb{C}_\mathsf{fin}$-invariance of $\mathcal{ALCSCC}$ concepts under Pr bisimulation.

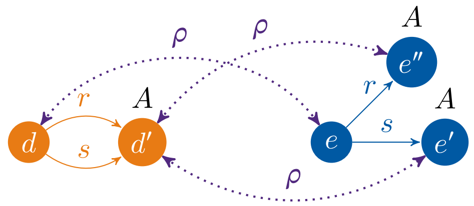


Figure 4.1: A representation of a counting bisimulation ρ which is not a Pr bisimulation.

Theorem 4.3. *Every $\mathcal{ALCQ}$ concept is $\mathbb{C}_{\mathsf{all}}$-invariant under counting bisimulation, every $\mathcal{ALCSCC}^\infty$ concept is $\mathbb{C}_{\mathsf{all}}$-invariant under Pr bisimulation and every $\mathcal{ALCSCC}$ concept is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation.*

Proof. The first two results have been proved respectively in [80] and [16]. For the last claim, we proceed by structural induction over an $\mathcal{ALCSCC}$ concept C. The cases where C is a concept name, a conjunction of concepts or the negation of a concept are illustrated in later results for other notions of bisimulation, so we omit them here. We focus on the case $C = \mathsf{succ}(\mathfrak{con})$, where we inductively assume that every subconcept of C is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation. Let $\mathcal{I}, \mathcal{J} \in \mathbb{C}_{\mathsf{fb}}$ and ρ a Pr bisimulation relating $d \in \Delta^\mathcal{I}$ and $e \in \Delta^\mathcal{J}$. We recall that $\mathfrak{con}$ is of the form (3.1) (but contains no PA variable), and provide for every $\mathcal{ALCSCC}$ concept C and safe role type τ over $\mathsf{N_R}$ an injective mapping from $D := \tau^\mathcal{I}(d) \cap C^\mathcal{I}$ to $E := \tau^\mathcal{J}(e) \cap C^\mathcal{J}$ and vice versa, thus proving that these sets have the same size and thus that $\mathfrak{con}$ is evaluated equally w.r.t. d and e. This implies that $d \in C^\mathcal{I}$ iff $e \in C^\mathcal{J}$. Since $\mathcal{I}$ and $\mathcal{J}$ are finitely branching, the sets D and E are both finite. Thanks to the forth property, we find a set $E' \subseteq \tau^\mathcal{J}(e)$ such that ρ contains a bijection between D and E'. By our inductive hypothesis, the concept C is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation, so we obtain that $E' \subseteq C^\mathcal{J}$. Then, $E' \subseteq E$ holds, and the bijection between D and E' is the sought injective mapping from D to E. Using the back property, we similarly prove that there is an injective mapping from E to D.

Together with the other cases, this concludes our proof, thus we conclude that every $\mathcal{ALCSCC}$ concept is $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr bisimulation. □

Comparing DLs using counting and Presburger bisimulations

First, we show that $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ are more expressive than $\mathcal{ALCQ}$, using the notion of counting bisimulation in the same spirit of Proposition 2.29.

Corollary 4.4 ([16]). *Let $\mathsf{N_R} = \{r, s\}$ and $\mathsf{N_C} = \{A\}$. There is no $\mathcal{ALCQ}$ concept description C such that C is $\mathbb{C}_{\mathsf{fb}}$-equivalent to the $\mathcal{ALCSCC}$ concept description* $\mathsf{succ}(|r \cap s \cap A| \geqslant 1)$.

In fact, if $\mathsf{succ}(|r \cap s \cap A| \geqslant 1)$ was $\mathbb{C}_{\mathsf{fb}}$-equivalent to an $\mathcal{ALCQ}$ concept description, then it would need to be $\mathbb{C}_{\mathsf{fb}}$-invariant under $\mathcal{ALCQ}$ bisimulation. However, Fig. 4.1 shows two finitely branching interpretations in which the individuals d and e are counting bisimilar, but whereas d belongs to $\mathsf{succ}(|r \cap s \cap A| \geqslant 1)$, the individual e does not.

We can also use Pr bisimulations to compare $\mathcal{ALCSCC}$ with other DLs with expressive counting constraints. In [10], we introduced the logic $\mathcal{ALCSCC}^{++}$ where we replace the restrictions $\mathsf{succ}(\mathfrak{con})$ of $\mathcal{ALCSCC}$ with extended ones of the form $\mathsf{sat}(\mathfrak{con})$. The semantics of this DL is defined w.r.t. finite interpretations $\mathcal{I}$ and restrictions $\mathsf{sat}(\mathfrak{con})$ are interpreted using a QFBAPA assignment σ_d as in $\mathcal{ALCSCC}$, with the difference that here $\mathcal{U}$ is mapped to $\sigma_d(\mathcal{U}) := \Delta^\mathcal{I}$. We show that the newly introduced restrictions cannot be expressed in $\mathcal{ALCSCC}$.

Theorem 4.5. *There are $\mathcal{ALCSCC}^{++}$ concepts that are not $\mathbb{C}_{\mathsf{fin}}$-equivalent to any $\mathcal{ALCSCC}$ concept.*

Proof. Assume, by contradiction, that there is an $\mathcal{ALCSCC}$ concept D that is $\mathbb{C}_{\mathsf{fin}}$-equivalent to $C := \mathsf{sat}(|A| \leqslant 1)$. Let $\mathcal{I}$ be the interpretation consisting of a single individual d with $d \in A^\mathcal{I}$, and let $\mathcal{J}$ consist of two individuals e, e' with both $e, e' \in A^\mathcal{J}$. Clearly, $d \in C^\mathcal{I}$ holds while $e, e' \notin C^\mathcal{J}$. By the assumption of $\mathbb{C}_{\mathsf{fin}}$-equivalence, we obtain that $d \in D^\mathcal{I}$ and $e, e' \notin D^\mathcal{J}$. However, the relation $\rho := \{(d, e), (d, e')\}$ is a Pr bisimulation. This leads to a contradiction, since by Theorem 4.3 it must hold that $d \in D^\mathcal{I}$ iff $e, e' \in D^\mathcal{J}$. Therefore, we conclude that C and D cannot be $\mathbb{C}_{\mathsf{fin}}$-equivalent. □

To compare the expressive power of DLs with and without concrete domains, we introduce in Chapter 7 the notion of *abstract expressive power* based on *abstract models*, obtained from models $\mathcal{I}$ that interpret $\mathsf{N_F}$ by "forgetting" the interpretation of $\mathsf{N_F}$. For DLs without concrete domains, such as $\mathcal{ALCSCC}$, models and abstract models coincide. Then, two concepts C and D are *abstractly* $\mathbb{C}$*-equivalent* if the abstract models of C in $\mathbb{C}$ coincide with those of D. Using Theorem 4.3 we show that for some instances of $\mathfrak{D}$ we can find $\mathcal{ALC}(\mathfrak{D})$ concepts whose abstract expressive power cannot be captured in $\mathcal{ALCSCC}$.

Theorem 4.6. *There is an $\mathcal{ALC}(\mathfrak{Q})$ concept that is not abstractly $\mathbb{C}_{\mathsf{fb}}$-equivalent to any $\mathcal{ALCSCC}$ concept.*

Proof. We show that $D := \exists f, rf. <$ is the sought $\mathcal{ALC}(\mathfrak{D})$ concept. Assume, by contradiction, that there exists an $\mathcal{ALCSCC}$ concept C such that the finitely branching abstract models of D coincide with the models of C. Let $\mathcal{I}$ be the interpretation of $\mathsf{N_C}$ and $\mathsf{N_R}$ with $\Delta^\mathcal{I} := \{a\}$ and $r^\mathcal{I} := \{(a, a)\}$ with $r \in \mathsf{N_R}$. Let $\mathcal{J}$ be the interpretation of $\mathsf{N_C}$ and $\mathsf{N_R}$ whose domain is $\mathbb{N}$ and where $n + 1$ is an r-successor of n for $n \in \mathbb{N}$. The relation $\rho := \{a\} \times \mathbb{N}$ is then a Pr bisimulation, and by Theorem 4.3 it follows that $a \in C^\mathcal{I}$ iff $n \in C^\mathcal{J}$ for $n \in \mathbb{N}$.

Clearly, $\mathcal{J}$ is an abstract model of D: by using $f^\mathcal{J}(n) := n$ for $n \in \mathbb{N}$ as interpretation of $f \in \mathsf{N_F}$, we obtain that $n \in D^\mathcal{J}$ for $n \in \mathbb{N}$. By abstract $\mathbb{C}_{\mathsf{fb}}$-equivalence of C and D, then, $n \in C^\mathcal{J}$ and thus $a \in C^\mathcal{I}$ must hold. Using abstract $\mathbb{C}_{\mathsf{fb}}$-equivalence again, we deduce that there exists an interpretation of feature names $f^\mathcal{I}(a)$ such that $a \in D^\mathcal{I}$. This leads to a contradiction, because $a \in D^\mathcal{I}$ can happen iff $f^\mathcal{I}(a) < f^\mathcal{I}(a)$. Therefore, we conclude that C and D cannot be abstractly $\mathbb{C}_{\mathsf{fb}}$-equivalent. □

4.2 $\mathcal{ALCSCC}$, $\mathcal{ALCSCC}^\infty$ and first-order logic

If a DL $\mathcal{L}$ is defined as a fragment of *first-order logic* (FOL), we can prove that the corresponding notion of L bisimulation is the most adequate for $\mathcal{L}$ by showing that, for a given class $\mathbb{C}$ of interpretations of interest, a FOL formula $\phi(x)$ is $\mathbb{C}$-invariant under L bisimulation iff it is $\mathbb{C}$-equivalent to some $\mathcal{L}$ concept. A formula $\phi(x)$ is $\mathbb{C}$*-invariant under* L *bisimulation* if for each L bisimulation ρ that relates $d \in \Delta^\mathcal{I}$ and $e \in \Delta^\mathcal{J}$ with $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ it holds that $\mathcal{I} \models \phi(d)$ iff $\mathcal{J} \models \phi(e)$.

We would like to prove that the above characterization holds for $\mathcal{ALCSCC}$ w.r.t. Pr bisimulation. However, as we will show later in this section, there are $\mathcal{ALCSCC}$ concepts that are not FOL-definable w.r.t. a class $\mathbb{C}$ of the desired form. In particular, we find sufficient conditions on a class $\mathbb{C}$ of interpretations such that every FOL formula $\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation iff there are $q, \ell \in \mathbb{N}$ such that $\phi(x)$ is $\mathbb{C}$-invariant under a parametrized version of Pr bisimulation, called *(q,ℓ)-bisimulation*. Here, q and ℓ parametrize the size of sets considered

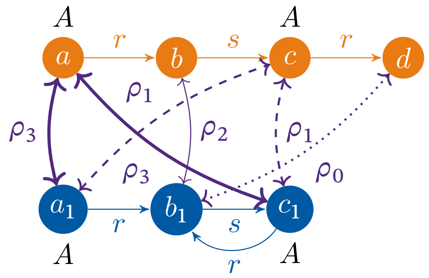


Figure 4.2: A Pr (1, 3)-bisimulation between two interpretations.

in the back-and-forth conditions and the depth at which these conditions are checked, respectively. We show that there are $\mathcal{ALCSCC}$ concepts that violate this property and are hence not FOL-definable w.r.t. $\mathbb{C}$.

For $\mathcal{ALCSCC}^\infty$, the existence of concepts that are not FOL-definable has already been showed in [16], and a similar proof was used to show that there are $\mathcal{ALCSCC}$ concept that cannot be expressed in the DL $\mathcal{ALCQ}$ [7]. Here, we generalize both results to several classes of interpretations. We adapt the bisimulation-based characterization of modal logic with graded modalities w.r.t. finite models in [84], which relies on the notion of (q,ℓ)-bisimulation.

Definition 4.7. *Let $\mathcal{I}$, $\mathcal{J}$ be interpretations of* $\mathsf{N_C}$, $\mathsf{N_R}$ *and* $q, \ell \in \mathbb{N}$. *The relation* $\rho \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{J}$ *is a* Pr $(q,0)$-bisimulation *between* $\mathcal{I}$, $\mathcal{J}$ *if it satisfies the (atomic) condition of Definition 4.2, and it is a* Pr $(q,\ell+1)$-bisimulation *if it is a Pr (q,ℓ)-bisimulation that additionally satisfies the following for all safe role types τ:*

(q,ℓ)-forth *if $(d, e) \in \rho$ and $D \subseteq \tau^\mathcal{I}(d)$ with $|D| \leqslant q$, then there are $E \subseteq \tau^\mathcal{J}(e)$ and a Pr (q,ℓ)-bisimulation ρ' that contains a bijection between D and E;*

(q,ℓ)-back *if $(d, e) \in \rho$ and $E \subseteq \tau^\mathcal{J}(e)$ with $|E| \leqslant q$, then there are $D \subseteq \tau^\mathcal{I}(d)$ and a Pr (q,ℓ)-bisimulation ρ' that contains a bijection between D and E.*

The notions of Pr (q,ℓ)-bisimilarity and $\mathbb{C}$-invariance w.r.t. Pr (q,ℓ)-bisimulation are defined similarly to how it was done in Definition 4.2.

For example, the relation ρ_3 depicted in Figure 4.2 is a Pr $(1,3)$-bisimulation relating a and a_1. We notice that these individuals cannot be Pr bisimilar, due to the presence of a loop between b_1 and c_1 that cannot be simulated in the other interpretation. Theorem 4.3 shows that all $\mathcal{ALCSCC}$ concept are invariant under Pr bisimulation. For Pr (q,ℓ)-bisimulation, this need not hold, as we can find an $\mathcal{ALCSCC}$ concept that is not invariant under Pr (q,ℓ)-bisimulation for all values of q and ℓ.

Theorem 4.8. *There is an $\mathcal{ALCSCC}$ concept C such that, for all values of q and ℓ, the concept C is not $\mathbb{C}_{\mathsf{fb}}$-invariant under Pr (q,ℓ)-bisimulation.*

Proof. Consider the $\mathcal{ALCSCC}$ concept $C := \mathsf{succ}(|r \cap A| = |r \cap \neg A|)$, which has been used in [16] to show that $\mathcal{ALCSCC}^\infty$ is not a fragment of FOL. For $n, m \in \mathbb{N}$, let $\mathcal{I}_{m,n}$ be the finitely branching interpretation containing individuals d and d_i for $i = 1, \dots, m+n$, where r is interpreted as the set of tuples (d, d_i) for $i = 1, \dots, m+n$, every d_i with $i = 1, \dots, m$ is in A and every other individual is not in A. Given $q \in \mathbb{N}$ we consider $\mathcal{I}_{q,q}$ and $\mathcal{I}_{q,q+1}$, and notice that

$d \in \Delta^{\mathcal{I}_{q,q}}$ and $d \in \Delta^{\mathcal{I}_{q,q+1}}$ are Pr (q,ℓ)-bisimilar: the relation mapping $d \in \Delta^{\mathcal{I}_{q,q}}$ to $d \in \Delta^{\mathcal{I}_{q,q+1}}$ and $d_i \in \Delta^{\mathcal{I}_{q,q}}$ to $d_i \in \Delta^{\mathcal{I}_{q,q+1}}$ is a Pr (q,ℓ)-bisimulation for all $\ell \in \mathbb{N}$. However, $d \in C^{\mathcal{I}_{q,q}}$ holds, whereas $d \notin C^{\mathcal{I}_{q,q+1}}$. □

Our goal is now to show that this cannot happen for $\mathcal{ALCSCC}$ concepts that are FOL-definable w.r.t. $\mathbb{C}_{\mathsf{fb}}$ or $\mathbb{C}_{\mathsf{fin}}$ (or more generally a class $\mathbb{C}$ of interpretations satisfying certain closure properties). The proof of this result uses certain locality properties of FOL formulae that are invariant under Pr bisimulation.

Locality of concepts and formulae.

We observe that the counterexample to invariance under (q,ℓ)-bisimulation used in the proof of Theorem 4.8 only depends on the choice of q and is independent of ℓ. In fact, if we were to only parameterize restricted Pr bisimulations by ℓ and ignore q, as we will do in Chapter 8 in the context of DLs with concrete domains, then for all $\mathcal{ALCSCC}$ concepts C we would be able to find a value ℓ such that C is invariant under Pr ℓ-bisimulation. This value ℓ is dependent on the structure of C, according to the following definition.

Definition 4.9. *The* depth *of an $\mathcal{ALCSCC}$ concept C is* 0 *if $C = A$, ℓ if $C = \neg D$ and D has depth ℓ,* $\max(\ell_1, \ell_2)$ *if $C = C_1 \sqcap C_2$ and C_i has depth ℓ_i, and $\ell + 1$ if $C = \mathsf{succ}(\mathtt{con})$ and ℓ is the maximum depth across all concepts occurring in* $\mathtt{con}$.

If C is an $\mathcal{ALCSCC}$ concept of depth ℓ, then for all interpretations $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$ we only need to consider elements reachable from d along chains of role successors of length at most ℓ to check if $d \in C^{\mathcal{I}}$ holds. The set of first-order formulae that satisfy a similar property is defined as follows.

Definition 4.10. *Let $\mathcal{I}$ be an interpretation. The* distance *of d and d' in $\mathcal{I}$ is the smallest value $\ell \in \mathbb{N}$ for which there is a sequence of elements $d_1, \dots, d_{\ell+1} \in \Delta^{\mathcal{I}}$ where $d_1 = d$, $d_{\ell+1} = d'$ and d_i is a role successor or predecessor of d_{i+1} for $i = 1, \dots, \ell$, or ∞ if such a number does not exist. The ℓ-*neighborhood *$\mathcal{N}_\ell^{\mathcal{I}}[\![d]\!]$ of d is derived from $\mathcal{I}$ by taking the substructure consisting of all individuals with distance at most ℓ from d.*

The class $\mathbb{C}$ of interpretations is closed under neighborhoods *if $\mathcal{N}_\ell^{\mathcal{I}}[\![d]\!] \in \mathbb{C}$ for all $\mathcal{I} \in \mathbb{C}$, $d \in \Delta^{\mathcal{I}}$ and $\ell \in \mathbb{N}$. The* FOL *formula $\phi(x)$ is* ℓ-local *w.r.t. $\mathbb{C}$ if for all $\mathcal{I} \in \mathbb{C}$ and all $d \in \Delta^{\mathcal{I}}$ we have that $\mathcal{I} \models \phi(d)$ iff $\mathcal{N}_\ell^{\mathcal{I}}[\![d]\!] \models \phi(d)$.*

We observed that every $\mathcal{ALCSCC}$ concept of depth ℓ is ℓ-local. Clearly, we cannot argue the same for first-order formulae $\phi(x)$ of quantifier depth ℓ. As an example, $\phi(x) = \exists y_1.\exists y_2.r(x, y_1) \wedge A(y_2)$ is not 2-local, as any potential individual replacing y_2 need not be in the 2-neighborhood of the individual replacing x. This formula is in particular not ℓ-local for all values of ℓ.

Fortunately, we can show that every formula $\phi(x)$ that is $\mathbb{C}$-invariant under Pr bisimulation is ℓ-local w.r.t. $\mathbb{C}$ for some value of ℓ dependent of the quantifier depth of $\phi(x)$, provided that $\mathbb{C}$ is closed under neighborhoods and under *finite disjoint unions* of elements of $\mathbb{C}$ as in Definition 2.31. Using the notation of Definition 2.31 we can prove, similarly to what done in Theorem 2.32, that $\rho := \{(d, (d, \nu)) \mid d \in \Delta^{\mathcal{I}_\nu}, \nu \in \mathbb{I}\}$ is a Pr bisimulation, and obtain the following property for formulae that are $\mathbb{C}$-invariant under Pr bisimulation.

Proposition 4.11. *Let $\mathbb{C}$ be closed under finite disjoint unions. If a* FOL *formula $\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation, then it is $\mathbb{C}$-invariant under finite disjoint unions.*

Crucially, $\mathbb{C}$-invariance under finite disjoint unions implies ℓ-locality w.r.t. $\mathbb{C}$ for some value of ℓ [51], provided that the class $\mathbb{C}$ is closed under both neighborhoods and finite disjoint unions[1]. A class $\mathbb{C}$ that satisfies these properties is called *localizable*; the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$ introduced in Chapter 2 are all localizable.

Lemma 4.12 ([86]). *If $\mathbb{C}$ is localizable, then any* FOL *formula $\phi(x)$ of quantifier depth q is $\mathbb{C}$-invariant under finite disjoint unions iff it is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$.*

Combining this lemma with Proposition 4.11, we can now link ℓ-locality with invariance under Pr bisimulation.

Corollary 4.13. *If $\mathbb{C}$ is localizable, then any* FOL *formula $\phi(x)$ of quantifier depth q that is $\mathbb{C}$-invariant under Pr bisimulation is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$.*

Our next goal is now to show that, for FOL formulae, invariance under Pr bisimulation is equivalent to invariance under Pr (q,ℓ) bisimulation for some $q, \ell \in \mathbb{N}$.

The case of tree-shaped neighborhoods.

In the proof of Theorem 4.8 we exhibit, for all $q \in \mathbb{N}$, two tree-shaped interpretations whose roots are Pr (q,ℓ)-bisimilar for all $\ell \in \mathbb{N}$, and that are distinguished by the fact that one root satisfies a certain $\mathcal{ALCSCC}$ concept C and the other does not. We show that this cannot happen for FOL formulae $\phi(x)$ that are ℓ-local and have quantifier depth q.

In particular, we consider roots of trees of depth ℓ as introduced in Definition 2.34. To prove that if two trees of depth ℓ that have Pr (q,ℓ)-bisimilar roots d, e are such that these roots satisfy the same formulae $\phi(x)$ of quantifier depth at most q, we make use of the Ehrenfeucht-Fraïsse method, which is based on the notion of *q-isomorphism* between d and e introduced in Definition 2.23. We show that a Pr (q,ℓ)-bisimulation between the roots of two trees of depth ℓ induces a q-isomorphism between these roots, leading to the following result.

Theorem 4.14. *If $\mathcal{I}$, $\mathcal{J}$ are trees of depth at most ℓ with roots d, e that are Pr (q,ℓ)-bisimilar, then these roots satisfy the same* FOL *formulae $\phi(x)$ of quantifier depth at most q.*

Proof. If d and e are Pr (q,ℓ)-bisimilar we can define a q-isomorphism $I_0, \dots, I_q$ between d and e such that for all $p \in I_{q-i}$ and $i = 0, \dots, q$ the following hold:

i-left if $\langle d_0 \cdots d_m \rangle$ with $d_0 = d$ is a path in $\mathcal{I}$ and $p(d_m)$ is defined, then for $j = 0, \dots, m$ there is $e_j \in \Delta^{\mathcal{J}}$ such that d_j, e_j are Pr $(q, \ell - j)$-bisimilar and $p(d_j) = e_j$, and $\langle e_0 \cdots e_m \rangle$ with $e_0 = e$ is a path in $\mathcal{J}$;

i-right if $\langle e_0 \cdots e_m \rangle$ with $e_0 = e$ is a path in $\mathcal{J}$ and $e_m = p(d_m)$, then for $j = 0, \dots, m$ there is $d_j \in \Delta^{\mathcal{I}}$ such that d_j, e_j are Pr $(q, \ell - j)$-bisimilar and $p(d_j) = e_j$, and $\langle d_0 \cdots d_m \rangle$ with $d_0 = d$ is a path in $\mathcal{I}$;

i-branches p is defined on individuals belonging at most i different branches of $\mathcal{I}$ and maps to individuals belonging to at most i diverging paths of $\mathcal{J}$.

[1]The conditions on $\mathbb{C}$ are not stated explicitly in [51] but are implicitly assumed.

Two paths are *diverging* if their length is greater than 1 and neither of the two is a prefix of the other. Clearly, $p := \{d \mapsto e\}$ satisfies all three properties and is a partial isomorphism: d and e satisfy the same concept names by (q,ℓ)-bisimilarity, and they vacuously agree on $\mathsf{N_R}$ since trees do not contain role loops. Let $I_q := \{\{d \mapsto e\}\}$. For $0 \leqslant i < q$, we assume that I_{q-i} is defined and show how to define $I_{q-(i+1)}$ so that i-forth and i-back in Definition 2.23 are satisfied.

Let $p \in I_{q-i}$ and $d' \in \Delta^{\mathcal{I}}$, consider the unique path $\langle d_0, \dots, d_{m'}\rangle$ with $m' \leqslant \ell$ between $d_0 := d$ and $d_{m'} := d'$ in $\mathcal{I}$ and let m with $0 \leqslant m < m'$ be the greatest value for which $p(d_{m'})$ is defined. If $m = m'$, we simply add p to $I_{q-(i+1)}$. Otherwise, for $m \leqslant j < m'$ we assume that the partial isomorphism p_j extending p with values for $d_m, \dots, d_j$ is defined and satisfies $(i+1)$-left, $(i+1)$-right and $(i+1)$-branches, and show how to extend p_j to p_{j+1} by adding a value e_{j+1} for d_{j+1} so that p_{j+1} also satisfies these conditions.

Let τ be the unique safe role type s.t. $(d_j, d_{j+1}) \in \tau^{\mathcal{I}}$. Then, the set D' of τ-successors of d_j for which p_j is defined must contain at most $i < q$ individuals: for $j = m$, this is a clear consequence of i-branches, and for $m < j < m'$ the set D' must be empty as otherwise $p_j(d_j)$ would already have been defined, and we would contradict our definition of m. Since d_j, e_j must be Pr $(q, \ell - j)$-bisimilar due to i-left, i-right and their $(i+1)$-versions, and $D := D' \cup \{d_{j+1}\} \subseteq \tau^{\mathcal{I}}(d_j)$ has size $i + 1 \leqslant q$, there exists a set $E \subseteq \tau^{\mathcal{J}}(e_j)$ of $i+1$ elements and a bijection $f\colon D \mapsto E$ such that d_x, $f(d_x)$ are Pr $(q, \ell - (j+1))$-bisimilar for $d_x \in D$. We notice that if $d_x \in D'$ then $p_j(d_x) \in \tau^{\mathcal{J}}(e_j)$ must hold because p_j is a partial isomorphism, and that d_x, $p_j(d_x)$ are $(q, \ell - (j+1))$-bisimilar by our assumptions on p_j, hence we can assume that $f(d_x) = p_j(d_x)$. Moreover, $e_{j+1} := f(d_{j+1}) \in E$ cannot be in the image of p_j: this is a direct consequence of i-branches for $j = m$ as p_j would otherwise map to values over $i+1$ different branches, and for $m < j < m'$ this would contradict the definition of m.

We define p_{j+1} by extending p_j with $p_{j+1}(d_{j+1}) := e_{j+1}$ and verify that it is a partial isomorphism. First, notice that p_j is injective by assumption and that $p_{j+1}(d_{j+1}) \neq p_{j+1}(d_x)$ if $d_x \neq d_{j+1}$ by definition, hence p_{j+1} is injective. Next, $d_x \in A^{\mathcal{I}}$ iff $p_{j+1}(d_x) \in A^{\mathcal{J}}$ holds if $p_j(d_x)$ is defined, so it is sufficient to notice that $d_{j+1} \in A^{\mathcal{I}}$ iff $p_{j+1}(d_{j+1}) \in A^{\mathcal{J}}$ follows from the fact that d_{j+1} and e_{j+1} are Pr $(q,\ell-(j+1))$-bisimilar thanks to the atomic condition to conclude that p_{j+1} is a partial isomorphism w.r.t. $\mathsf{N_C}$. To check that $(d_x, d_y) \in r^{\mathcal{I}}$ iff $(p_{j+1}(d_x), p_{j+1}(d_y)) \in r^{\mathcal{J}}$ for all d_x, d_y for which p_{j+1} is defined, we consider the cases not covered by p_j. In the first case, $d_y = d_{j+1}$ with $m \leqslant j < m'$ and so $(d_x, d_{j+1}) \in r^{\mathcal{I}}$ may occur iff $d_x = d_j$, and since we chose $p_{j+1}(d_{j+1})$ to be a τ-successor of $p_{j+1}(d_j)$ iff $d_{j+1} \in \tau^{\mathcal{I}}(d_j)$ we conclude that $(d_x, d_{j+1}) \in r^{\mathcal{I}}$ iff $(p_{j+1}(d_x), p_{j+1}(d_{j+1})) \in r^{\mathcal{I}}$. In the second case, $d_x = d_{j+1}$ and so $(d_{j+1}, d_y) \in r^{\mathcal{I}}$ may occur iff $d_y = d_{(j+1)+1}$ with $m < j' := j+1 < m'$, and so we fall in the first case applied to $d_y = d_{j'+1}$. We thus showed that p_{j+1} is a partial isomorphism w.r.t. $\mathsf{N_R}$ and we conclude that it is a partial isomorphism and add it to $I_{q-(i+1)}$.

The process above shows that $I_0, \dots, I_q$ satisfies the i-forth condition for $i = 0, \dots, q$. Using a similar strategy, we show, for a given $e' \in \Delta^{\mathcal{J}}$, how to add $p' \in I_{q-(i+1)}$ that extends $p \in I_{q-i}$ and such that $p(d') = e'$ for some $d' \in \Delta^{\mathcal{I}}$, thus showing that $I_0, \dots, I_q$ satisfies the i-back condition for $0 \leqslant i < q$. We obtain a q-isomorphism between d and e and conclude by Theorem 2.24 that they satisfy the same FOL formulae $\phi(x)$ of quantifier depth at most q. □

While not all interpretations in a class $\mathbb{C}$ of interest need to be tree-shaped, we show that, for every interpretation in $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ or $\mathbb{C}_{\mathsf{fin}}$, it is possible to find a Pr bisimilar interpretation in this class where the ℓ-neighborhood of a specific individual d is a tree with root d. Normally, this is achieved by unravelling [22], but this may yield an infinite interpretation, and is thus not suitable for our setting, where we are also interested in the class $\mathbb{C}_{\mathsf{fin}}$. We take instead the *partial* unravelling of $\mathcal{I}$, which preserves finiteness and finite branching. Intuitively, the *ℓ-unravelling* of an interpretation $\mathcal{I}$ at an element $d \in \Delta^\mathcal{I}$ applies unraveling up to length ℓ, and then adds a copy of $\mathcal{I}$ at the end. The exact definition of this operation, which is an adaptation of the unravelling operation described in [22], is as follows.

Definition 4.15. *Given an interpretation $\mathcal{I}$ and $\ell \in \mathbb{N}$, let $\mathcal{I}^u_\ell$ be the interpretation whose domain $\Delta^{\mathcal{I}^u_\ell}$ is the set of all paths of $\mathcal{I}$ of length at most ℓ, defined as follows for $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$:*

$$A^{\mathcal{I}^u_\ell} := \{p \in \Delta^{\mathcal{I}^u_\ell} \mid \mathsf{end}(p) \in A^\mathcal{I}\},$$
$$r^{\mathcal{I}^u_\ell} := \{(\langle d_0, \dots, d_k\rangle, \langle d_0, \dots, d_k, d_{k+1}\rangle) \in \Delta^{\mathcal{I}^u_\ell} \times \Delta^{\mathcal{I}^u_\ell} \mid (d_k, d_{k+1}) \in r^\mathcal{I}\}.$$

*The ℓ-*unravelling *$\mathcal{I}_\ell$ of $\mathcal{I}$ is obtained as the union of $\mathcal{I}$ and $\mathcal{I}^u_\ell$ where we additionally add to $r^{\mathcal{I}_\ell}$ all $(p, e) \in \Delta^{\mathcal{I}^u_\ell} \times \Delta^\mathcal{I}$ such that p has length ℓ and $(\mathsf{end}(p), e) \in r^\mathcal{I}$. Then, $\mathbb{C}$ is* closed under partial unravelling *if $\mathcal{I}_\ell \in \mathbb{C}$ for all $\mathcal{I} \in \mathbb{C}$ and $\ell \in \mathbb{N}$.*

As mentioned above, the ℓ-unravelling of $\mathcal{I}$ provides an element $\langle d\rangle$ that is Pr bisimilar to $d \in \mathcal{I}$ and whose ℓ-neighborhood is tree-shaped.

Proposition 4.16. *Let $\mathcal{I}^d_\ell$ be the ℓ-unravelling of the interpretation $\mathcal{I}$ at $d \in \Delta^\mathcal{I}$, and $\langle d\rangle$ the element corresponding to d in $\mathcal{I}_\ell$. Then,*

1. *The elements $d \in \Delta^\mathcal{I}$ and $\langle d\rangle \in \Delta^{\mathcal{I}^d_\ell}$ are Pr bisimilar.*
2. *The ℓ-neighborhood $\mathcal{N}^{\mathcal{I}^d_\ell}_\ell(\langle d\rangle)$ of $\langle d\rangle$ in $\mathcal{I}_\ell$ is a tree of depth at most ℓ with root $\langle d\rangle$.*

Proof. Using the notation of Definition 4.15, we prove that

$$\rho := \{(d, d) \mid d \in \Delta^\mathcal{I}\} \cup \{(d, p) \in \Delta^\mathcal{I} \times \Delta^{\mathcal{I}^u_\ell} \mid d = \mathsf{end}(p)\}$$

is the sought relation. Given that Pr bisimulations are closed under union, it is enough to show that all $(e, p) \in \rho$ satisfy the conditions of Definition 4.2 to conclude that ρ is a Pr bisimulation, as the first relation in the union trivially is an Pr bisimulation. For each $(e, p) \in \rho$, the (atomic) condition is implied by definition of $A^{\mathcal{I}_\ell}$ for $A \in \mathsf{N_C}$.

Next, we show that ρ satisfies (forth). Let $(e, p) \in \rho$ and $D \subseteq \tau^\mathcal{I}(e)$ a finite set for some safe role type τ over $\mathsf{N_R}$. If p is a directed path of length ℓ with starting point d, then every τ-successor of p is an element of $\Delta^\mathcal{I}$ and in particular it is a τ-successor of e. We define $D' := D$ and obtain a finite subset of $\tau^{\mathcal{I}_\ell}(p)$ such that ρ contains a bijection between D and D'. If, on the other hand, p has length less than ℓ, all of its τ-successors in $\mathcal{I}_\ell$ are directed paths of the form $p' = p\langle e'\rangle$ for which $(\mathsf{end}(p), e') \in \tau^\mathcal{I}$. Since $\mathsf{end}(p) = e$, we deduce that $p' := p\langle e'\rangle \in \tau^{\mathcal{I}_\ell}(p)$ holds for all $e' \in \tau^\mathcal{I}(e)$. Since ρ contains all tuples (e', p') of the form above, we conclude that it contains a bijection between D and the finite subset $D' := \{p\langle e'\rangle \mid e' \in D\}$ of $\tau^{\mathcal{I}_\ell}(p)$.

Finally, we show that the (back) direction holds. Let $D' \subseteq \tau^{\mathcal{I}_\ell}(p)$ be a finite set. If p has length ℓ, reusing our previous observations, we derive that $D' \subseteq \Delta^\mathcal{I}$. In particular, from $e = \mathsf{end}(p)$ and the definition of $r^{\mathcal{I}_\ell}$ we derive that $e' \in \tau^\mathcal{I}(e)$ for all $e' \in D'$. Thus, $D := D'$ is a finite subset of $\tau^\mathcal{I}(e)$ and, since $(e', e') \in \rho$ for all $e' \in \Delta^\mathcal{I}$, it follows that ρ contains a bijection between D and D'. If p has length less than ℓ, then each element of D' is a d-dipath p' such that $p' = p\langle e'\rangle$ and $(\mathsf{end}(p), e') \in \tau^\mathcal{I}$. Since $e = \mathsf{end}(p)$, it follows that $e' \in \tau^\mathcal{I}(e)$. Moreover, $(e', p') \in \rho$ by definition of ρ. We deduce that $D := \{\mathsf{end}(p') \mid p' \in D'\} \subseteq \tau^\mathcal{I}(e)$ and ρ contains a bijection between D and D'. The relation ρ satisfies all the conditions listed in Definition 4.2 and is thus a Pr bisimulation between $\mathcal{I}$ and $\mathcal{I}_\ell$. □

If $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$ are Pr (q,ℓ)-bisimilar, the result above implies that $\langle d \rangle$ and $\langle e \rangle$ are also Pr (q,ℓ)-bisimilar. This is obtained by showing that if $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a Pr (q,ℓ)-bisimulation between d and e and $\rho_d \subseteq \Delta^{\mathcal{I}_\ell} \times \Delta^{\mathcal{I}}$, $\rho_e \subseteq \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}_\ell}$ are Pr bisimulations between $\langle d \rangle$, d and e, $\langle e \rangle$ then the relation[2] $\rho' := \rho_d \circ \rho \circ \rho_e$ is a Pr (q,ℓ)-bisimulation between $\langle d \rangle$ and $\langle e \rangle$. Since $\langle d \rangle$ and $\langle e \rangle$ are the roots of two trees of depth ℓ, we can conclude the following using Theorem 4.14.

Corollary 4.17. *If $\mathbb{C}$ is closed under partial unravelling and $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ contain $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$ that are Pr (q,ℓ)-bisimilar, then $\langle d \rangle \in \Delta^{\mathcal{I}_\ell}$ and $\langle e \rangle \in \Delta^{\mathcal{J}_\ell}$ satisfy the same ℓ-local formulae $\phi(x)$ of quantifier depth at most q.*

$\mathcal{ALCSCC}$ goes beyond FOL.

Combining the results about locality and tree-shaped interpretations obtained so far, we show that there is no FOL formula $\phi(x)$ that behaves like the concept C in Theorem 4.8 w.r.t. Pr (q,ℓ)-bisimulation. Indeed, if $\phi(x)$ has quantifier depth q and is $\mathbb{C}$-invariant under Pr bisimulation, we are always able to find suitable values of q and ℓ and sufficient conditions on $\mathbb{C}$ such that $\phi(x)$ is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation.

The class $\mathbb{C}$ of interpretations is closed under partial unravelling if $\mathcal{I} \in \mathbb{C}$ implies $\mathcal{I}_\ell^d \in \mathbb{C}$. The following result links invariance under Pr bisimulation with invariance under Pr (q,ℓ)-bisimulation for FOL formulae.

Theorem 4.18. *Let $\mathbb{C}$ be localizable and closed under partial unravelling. For all* FOL *formulae $\phi(x)$, the following are equivalent:*

1. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation;*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$.*

Proof. The implication "2 ⇒ 1" is an immediate consequence of the fact that every Pr bisimulation is also a Pr (q,ℓ)-bisimulation for all $q, \ell \in \mathbb{N}$.

To prove the other direction, we assume 1. and that $\phi(x)$ has quantifier depth q. By Corollary 4.13 we deduce that $\phi(x)$ is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$. Given $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ and $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$, we know that the ℓ-unravellings $\mathcal{I}_\ell^d$ and $\mathcal{J}_\ell^e$ and the ℓ-neighborhoods $\mathcal{N}_d := \mathcal{N}_\ell^{\mathcal{I}_\ell^d}(\langle d \rangle)$ and $\mathcal{N}_e := \mathcal{N}_\ell^{\mathcal{I}_\ell^d}(\langle e \rangle)$ also belong to $\mathbb{C}$. Since $\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation and ℓ-local w.r.t. $\mathbb{C}$ we obtain

$$\begin{array}{ll} \mathcal{I} \models \phi(d) \text{ iff } \mathcal{I}_\ell^d \models \phi(\langle d \rangle) \text{ iff } \mathcal{N}_d \models \phi(\langle d \rangle) \text{ and} & \\ \mathcal{J} \models \phi(e) \text{ iff } \mathcal{J}_\ell^e \models \phi(\langle e \rangle) \text{ iff } \mathcal{N}_e \models \phi(\langle e \rangle). & \text{(by Proposition 4.16)} \end{array}$$

If ρ is a Pr (q,ℓ)-bisimulation with $(d, e) \in \rho$, then combining this relation with the Pr bisimulations linking d and $\langle d \rangle$ and d and $\langle e \rangle$ shows that there is a Pr (q,ℓ)-bisimulation ρ' between $\mathcal{I}_\ell^d$ and $\mathcal{I}_\ell^e$ with $(\langle d \rangle, \langle e \rangle) \in \rho'$. Since such a bisimulation looks only ℓ steps into the interpretation, the restriction of ρ' to the respective ℓ-neighborhoods $\mathcal{N}_d$ and $\mathcal{N}_e$ is also a Pr (q,ℓ)-bisimulation. Proposition 4.16 says that these neighborhoods are trees of depth at most ℓ, and thus we can apply Theorem 4.14 to obtain $\mathcal{N}_d \models \phi(\langle d \rangle)$ iff $\mathcal{N}_e \models \phi(\langle e \rangle)$, Therefore, (2) holds for $\phi(x)$. □

Together with Theorem 4.8, this yields the desired non-definability results since the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$, and $\mathbb{C}_{\mathsf{fin}}$ are localizable and closed under partial unravelling.

Corollary 4.19. *There are $\mathcal{ALCSCC}$ concepts that are not* FOL*-definable w.r.t.* $\mathbb{C}_{\mathsf{fb}}$.

[2] If $\rho_1 \subseteq A \times B$ and $\rho_2 \subseteq B \times C$ then $\rho_1 \circ \rho_2 := \{(a, c) \mid (a, b) \in \rho_1, (b, c) \in \rho_2\} \subseteq A \times C$.

The first-order fragment of $\mathcal{ALCSCC}$.

In [16], we have established that the FOL-definable subset of $\mathcal{ALCSCC}^\infty$ corresponds to the DL $\mathcal{ALCQ}t$. This DL can be seen both as the extension of $\mathcal{ALCQ}$ where safe roles types instead of just role names can be used in qualified number restrictions, and as the restriction of $\mathcal{ALCSCC}$ where only successor restrictions of the form $\mathsf{succ}(|\tau \cap C| \geqslant q)$ are available, where τ is a safe role type, $q \in \mathbb{N}$, and C is an $\mathcal{ALCQ}t$ concept. To make the relationship to qualified number restrictions clear, we write such successor restrictions as $(\geqslant q\, \tau.C)$, and call them qualified number restrictions. Saying that this result was proved in [16] for $\mathcal{ALCSCC}^\infty$ means that it was shown w.r.t. the class $\mathbb{C}_{\mathsf{all}}$. In the following we prove that it also holds for the classes $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$.

To conclude this section, we show how to leverage Theorem 4.18 to show that this characterization holds even w.r.t. finitely branching and finite interpretations, after proving that a first-order formula is invariant under Pr (q,ℓ)-bisimulation iff it is equivalent to some $\mathcal{ALCQ}t$ concept.

Clearly, every qualified number restriction $(\geqslant n\, r.C)$ or $(\leqslant n\, r.C)$ in $\mathcal{ALCQ}$ can be translated into $\mathcal{ALCQ}t$. For a given role name r, let T_r be the set of safe role types τ_1, ..., τ_k over N_R where r occurs positively. For a given natural number n, let P_n be the set of all tuple $(n_1, \dots, n_k)$ of natural numbers whose sum is exactly n. Then, $(\leqslant n\, r.C)$ is equivalent to the $\mathcal{ALCQ}t$ concept

$$\bigsqcup_{(n_1,\dots,n_k)\in P_n} \; \bigsqcap_{\tau_i \in T_r} (\leqslant n_i\, \tau_i.C^\star)$$

where $C^\star$ is recursively translated. Similarly, we define an $\mathcal{ALCQ}t$ concept that is equivalent ot $(\geqslant n\, r.C)$. On the other hand, the concept used to prove that $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ are more expressive than $\mathcal{ALCQ}$ in Corollary 4.4 is actually an $\mathcal{ALCQ}t$ concept, and therefore we deduce that $\mathcal{ALCQ}t$ is strictly more expressive than $\mathcal{ALCQ}$.

We notice that every $\mathcal{ALCQ}t$ concept C can be translated into a FOL formula $C^\sharp(x)$, and that unlike $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$, concepts in $\mathcal{ALCQ}t$ can only compare the cardinality of a set of role successors with a fixed natural number. In particular, we can define for every concept a coefficient q that is dependent on the numbers occurring in its qualified number restrictions, as follows.

Definition 4.20. *The* breadth *of an $\mathcal{ALCQ}t$ concept C is* 0 *if $C = A$, q if $C = \neg D$ and D has breadth q,* $\max(q_1, q_2)$ *if $C = C_1 \sqcap C_2$ and C_i has breadth q_i,* $\max(q, q')$ *if $C = (\geqslant q\, \tau.D)$ and D has breadth q'.*

The following example shows that the notion of breadth defined above is related to the number of quantified variables used in the first-order translation of $\mathcal{ALCQ}t$ concepts.

Example 4.21. *Let* $\mathsf{N_C} := \{A\}$, $\mathsf{N_R} := \{r\}$ *and consider the $\mathcal{ALCQ}t$ concept* $C := (\geqslant 3\, r.A)$. *According to Definition 4.20 this concept has breadth* 3*; its first-order translation*

$$\pi_x(C) = \exists x_1.\exists x_2.\exists x_3. \bigwedge_{i=1}^{3} (r(x,x_i) \wedge A(x_i) \wedge \bigwedge_{j=i+1}^{3} (x_i \neq x_j))$$

introduces three quantified variables, in addition to the free variable x. The concept $C' := (\leqslant 3\, r.A)$ *has first-order translation*

$$\pi_x(C') = \forall x_1.\forall x_2.\forall x_3.\forall x_4. \bigwedge_{i=1}^{4} (r(x,x_i) \wedge A(x_i)) \implies \bigvee_{i=1}^{4} \bigvee_{j=i+1}^{4} (x_i = x_j)$$

in which four quantified variables are used. There is thus a mismatch between the coefficient of the qualified number restriction in C' and the number of quantified variables in its first-order translation. This is not the case for the equivalent concept $C'' := \neg(\geqslant 4\, r.A)$, whose translation uses four quantified variables and which has breadth 4 according to the definition above.

We show that the breadth and the depth of an $\mathcal{ALCQ}t$ concept C are strictly connected to the values of q and ℓ for which C is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation. In particular, let us consider the subset $\mathcal{ALCQ}t_{q,\ell}$ of all $\mathcal{ALCQ}t$ concepts of depth at most ℓ and breadth at most q. We observe the following.

Theorem 4.22. *For all classes $\mathbb{C}$ of interpretations and all $q, \ell \in \mathbb{N}$, every $\mathcal{ALCQ}t_{q,\ell}$ concept is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation.*

Proof. Fixed $q \geqslant 0$, we prove that every $\mathcal{ALCQ}t_{q,\ell}$ concept is invariant under Pr (q,ℓ)-bisimulation by induction over ℓ. Let $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ be interpretations related by a Pr (q,ℓ)-bisimulation ρ with $(d, e) \in \rho$.

For $\ell = 0$, the fact that all $\mathcal{ALCQ}t_{q,0}$ concepts are Boolean combinations of concept names and that d and e satisfy the same concept names thanks to the atomic condition satisfied by ρ implies that they satisfy the same $\mathcal{ALCQ}t_{q,0}$ concepts, hence that $\mathcal{ALCQ}t_{q,0}$ concepts are $\mathbb{C}$-invariant under Pr $(q,0)$-bisimulation.

We assume inductively that every $\mathcal{ALCQ}t_{q,\ell}$ concept is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation and show that this implies that all $\mathcal{ALCQ}t_{q,\ell+1}$ concepts are $\mathbb{C}$-invariant under Pr $(q,\ell + 1)$-bisimulation. Let ρ be a Pr $(q,\ell + 1)$-bisimulation with $(d, e) \in \rho$. We show by structural induction over C an $\mathcal{ALCQ}t_{q,\ell+1}$ concept that d and e satisfy the same $\mathcal{ALCQ}t_{q,\ell+1}$ concepts. If $C = A$ is a concept name, this trivially follows from the fact that ρ satisfies the atomic condition. We inductively assume that if a $\mathcal{ALCQ}t_{q,\ell}$ concept D is a proper subconcept of C, then $d \in D^\mathcal{I}$ iff $e \in D^\mathcal{J}$. Let $C = (\geqslant q'\, \tau.D)$ with D an $\mathcal{ALCQ}t_{q,\ell}$ concept and $q' \leqslant q$. If $d \in C^\mathcal{I}$, then there is a set D_C of size $q' \leqslant q$ of τ-successors of d such that $d' \in D^\mathcal{I}$ for $d' \in D_C$. Thanks to the (q,ℓ)-forth condition, we find a set E_C of size $q' \leqslant q$ of τ-successors of e and a bijection h from D_C to E_C such that d' and $h(d')$ are Pr (q,ℓ)-bisimilar for $d' \in D_C$. Using our inductive hypothesis on ℓ, we deduce that $e' \in D^\mathcal{J}$ for $e' \in E_C$ and conclude that $e \in C^\mathcal{J}$. Similarly, we show that $e \in C^\mathcal{J}$ implies $d \in C^\mathcal{I}$, this time using the (q,ℓ)-back condition.

If $C = \neg D$, then the semantics of negation and our inductive hypothesis on D imply that $d \in (\neg D)^\mathcal{I}$ iff $d \notin D^\mathcal{I}$ iff $e \notin D^\mathcal{J}$ iff $e \in (\neg D)^\mathcal{J}$. The treatment is similar for $C = D_0 \sqcap D_1$. We conclude that d and e satisfy the same $\mathcal{ALCQ}t_{q,\ell+1}$ concepts, hence that $\mathcal{ALCQ}t_{q,\ell+1}$ concepts are $\mathbb{C}$-invariant under Pr $(q,\ell + 1)$-bisimulation. This concludes our proof by induction. □

We prove that $\mathcal{ALCQ}t_{q,\ell}$, unlike $\mathcal{ALCQ}t$, only contains finitely many concepts (up to $\mathbb{C}$-equivalence) if we assume that $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite. This is well-known for $\mathcal{ALC}$, i.e. the modal logic K [92] and for $\mathcal{ALCQ}$, i.e. modal logic with graded modalities [84] and the proof of these facts can be easily extended to $\mathcal{ALCQ}t$. We observe that if we only restricted w.r.t. q, then for $q \geqslant 1$ we could define concepts of arbitrary depth, and similarly if we only restricted w.r.t. ℓ we could write concepts of arbitrary breadth for $\ell \geqslant 1$, while restricting only w.r.t. ℓ is sufficient in logics such as $\mathcal{ALC}$ and $\mathcal{ALC}(\mathfrak{D})$ (as shown in Chapter 8).

Proposition 4.23. *If $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite sets, then for all values of q and ℓ and all classes of interpretations $\mathbb{C}$ the logic $\mathcal{ALCQ}t_{q,\ell}$, is finite (up to $\mathbb{C}$-equivalence).*

Proof. We fix $q \geqslant 0$ and proceed by induction over ℓ. For $\ell = 0$, we notice that every $\mathcal{ALCQ}t_{q,0}$ concept is a Boolean combination of concept names. Since $\mathsf{N_C}$ is assumed to be finite, we conclude that $\mathcal{ALCQ}t_{q,0}$ is finite up to $\mathbb{C}$-equivalence. Next, we inductively assume that $\mathcal{ALCQ}t_{q,\ell}$ is finite. Then, there exist finitely many qualified number restrictions $(\geqslant k\, \tau.C)$ with $k \leqslant q$, C an $\mathcal{ALCQ}t_{q,\ell}$ concept and τ a safe role type over $\mathsf{N_R}$ (up to $\mathbb{C}$-equivalence). This holds by finiteness of $\mathsf{N_C}$ and $\mathsf{N_R}$. Every $\mathcal{ALCQ}t_{q,\ell+1}$ concept is equivalent to a Boolean combination of $\mathcal{ALCQ}t_{q,\ell}$ concepts and qualified number restrictions of the form above. Since there are finitely many such combinations up to $\mathbb{C}$-equivalence, we conclude that $\mathcal{ALCQ}t_{q,\ell+1}$ is finite. □

Thanks to the above, we are able to define a concept $\mathsf{Bisim}^q_\ell[d]$ describing all individuals that are Pr (q,ℓ)-bisimilar with $d \in \Delta^{\mathcal{I}}$ on all classes of interpretations.

Definition 4.24. *Given an interpretation $\mathcal{I}$ with $d \in \Delta^{\mathcal{I}}$, a safe role type τ over $\mathsf{N_R}$ and $q, \ell \in \mathbb{N}$ we consider the mutually recursive $\mathcal{ALCQ}t_{q,\ell}$ concepts*

$$\mathsf{Atomic}[d] := \bigsqcap\{A \in \mathsf{N_C} \mid d \in A^{\mathcal{I}}\} \sqcap \bigsqcap\{\neg A \mid A \in \mathsf{N_C}, d \notin A^{\mathcal{I}}\} \qquad (\mathsf{atomic})$$

$$\mathsf{Forth}^{q,\ell}_{\tau}[d] := \bigsqcap\nolimits_{d' \in \tau^{\mathcal{I}}(d)} \mathsf{Forth}^{q,\ell}_{\tau,d'}[d] \qquad ((q,\ell)\text{-}\mathsf{forth})$$

$$\mathsf{Back}^{q,\ell}_{\tau}[d] := \begin{cases} \neg(\geqslant 1\, \tau.(\bigsqcap_{d' \in \tau^{\mathcal{I}}(d)} \neg\mathsf{Bisim}^q_\ell[d'])) & \textit{if } q \geqslant 1 \\ \top & \textit{otherwise} \end{cases} \qquad ((q,\ell)\text{-}\mathsf{back})$$

where, assuming that $k \geqslant 1$ is the number of τ-successors of d in $(\mathsf{Bisim}^q_\ell[d'])^{\mathcal{I}}$,

$$\mathsf{Forth}^{q,\ell}_{\tau,d'}[d] := \begin{cases} (\geqslant k\, \tau.\mathsf{Bisim}^q_\ell[d']) \sqcap \neg(\geqslant k+1\, \tau.\mathsf{Bisim}^q_\ell[d']) & \textit{if } k < q, \\ (\geqslant q\, \tau.\mathsf{Bisim}^q_\ell[d']) & \textit{otherwise;} \end{cases}$$

and finally

$$\mathsf{Bisim}^q_0[d] := \mathsf{Atomic}[d]$$

$$\mathsf{Bisim}^q_{\ell+1}[d] := \mathsf{Bisim}^q_\ell[d] \sqcap \bigsqcap\{\mathsf{Back}^{q,\ell}_\tau[d] \sqcap \mathsf{Forth}^{q,\ell}_\tau[d] \mid \tau \textit{ safe role type over } \mathsf{N_R}\}.$$

*We call $\mathsf{Bisim}^q_\ell[d]$ the (q,ℓ)-*characteristic $\mathcal{ALCQ}t$ concept *of d.*

If $\mathsf{N_C}$ and $\mathsf{N_R}$ are finite then Proposition 4.23 ensures that characteristic concepts are well-defined, even if $\mathcal{I}$ is not finitely branching, since the conjunctions in $\mathsf{Forth}^{q,\ell}_\tau[d]$ and $\mathsf{Back}^{q,\ell}_\tau[d]$ contain only finitely many non-equivalent conjuncts. Using the fact that (atomic), ((q,ℓ)-forth) and ((q,ℓ)-back) in Definition 4.24 encode the corresponding properties in Definition 4.7 we show that the relation $\rho_\ell := \{(d,e) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}} \mid e \in (\mathsf{Bisim}^q_i[d])^{\mathcal{J}}\}$ is a Pr (q,ℓ)-bisimulation and obtain the following correspondence.

Theorem 4.25. *Let $\mathsf{N_C}$, $\mathsf{N_R}$ be finite and $q, \ell \in \mathbb{N}$. Then, $d \in \Delta^{\mathcal{I}}$ and $e \in \Delta^{\mathcal{J}}$ are Pr(q,ℓ)-bisimilar iff they satisfy the same $\mathcal{ALCQ}t_{q,\ell}$ concepts.*

Proof. If d and e are Pr (q,ℓ)-bisimilar then they satisfy the same $\mathcal{ALCQ}t_{q,\ell}$ concepts by Theorem 4.22. We show by induction over ℓ that the relation ρ defined above satisfies all the conditions stated in Definition 4.7, which implies that ρ is a Pr (q,ℓ)-bisimulation. If d and e satisfy the same $\mathcal{ALCQ}t_{q,\ell}$ concepts, then $d \in (\mathsf{Bisim}^q_\ell[d])^{\mathcal{I}}$ implies that $e \in (\mathsf{Bisim}^q_\ell[d])^{\mathcal{J}}$ and so we conclude that $(d,e) \in \rho$.

For $\ell \in \mathbb{N}$, we observe that $e \in (\mathsf{Atomic}[d])^\mathcal{I}$ iff for all $A \in \mathsf{N_C}$ it holds that $d \in A^\mathcal{I}$ iff $e \in A^\mathcal{J}$; since this concept occurs as a conjunct in $\mathsf{Bisim}^q_\ell[d]$, we conclude that ρ_ℓ fulfills the atomic condition. In particular, this implies that ρ_0 is a Pr $(q,0)$-bisimulation.

Next, we inductively assume that ρ_ℓ is a Pr (q,ℓ)-bisimulation and show that $\rho_{\ell+1}$ is a Pr $(q,\ell+1)$-bisimulation. We start by showing that $\rho_{\ell+1}$ satisfies the (q,ℓ)-forth condition. Assume that $(d,e) \in \rho_{\ell+1}$ and let $D \subseteq \tau^\mathcal{I}(d)$ be a set of size $q' \leqslant q$ for some safe role type τ over $\mathsf{N_R}$. We partition D into sets $D_{d'}$ for $d' \in D$ with $D_{d'} := D \cap (\mathsf{Bisim}^q_\ell[d'])^\mathcal{I}$; then, it holds that $q_{d'} := |D_{d'}| \leqslant q$. In particular, $(\geqslant q'\ \tau.(\mathsf{Bisim}^q_\ell[d']))$ with $q' \geqslant q_{d'}$ is a conjunct of $\mathsf{Forth}^{q,\ell}_\tau[d]$, so we conclude that $e \in (\geqslant q'\ \tau.(\mathsf{Bisim}^q_\ell[d']))^\mathcal{J}$, hence $e \in (\geqslant q_{d'}\ \tau.(\mathsf{Bisim}^q_\ell[d']))^\mathcal{J}$. Thus, there exists a set $E_{d'} \subseteq \mathsf{Bisim}^q_\ell[d']^\mathcal{J}$ of τ-successors of e of size $q_{d'}$. Together with the definition of ρ_ℓ, we obtain that $D_{d'} \times E_{d'} \subseteq \rho_\ell$, and since the two sets are of the same size we can find a bijection $f_{d'} \subseteq \rho_\ell$ between them. By combining all mappings $f_{d'}$ with $d' \in D$ we are able to find a bijection $f \subseteq \rho_\ell$ between D and $E := \bigcup_{d' \in D} E_{d'}$. Since ρ_ℓ is inductively assumed to be a Pr (q,ℓ)-bisimulation, we conclude that $\rho_{\ell+1}$ satisfies (q,ℓ)-forth.

Finally, we show that $\rho_{\ell+1}$ satisfies the (q,ℓ)-back condition. Assume that $(d,e) \in \rho_{\ell+1}$ and let E be a subset of $\tau^\mathcal{J}(e)$ of cardinality $q' \leqslant q$. Since $e \in (\mathsf{Bisim}^q_{\ell+1}[d])^\mathcal{J}$ and $\mathsf{Back}^{q,\ell}_\tau[d]$ is a conjunct of $\mathsf{Bisim}^q_{\ell+1}[d]$, we deduce that for every $e' \in E$ there is some τ-successor d' of d such that $e' \in \mathsf{Bisim}^q_\ell[d']^\mathcal{J}$. As done in the previous paragraph, then, we define sets $E_{d'} := E \cap (\mathsf{Bisim}^q_\ell[d'])^\mathcal{J}$ with $q_{d'} := |E_{d'}| \leqslant q$ and use them to find a set $D \subseteq \tau^\mathcal{I}(d)$ of size q' and a bijection $f\colon E \to D$ included in ρ_ℓ, concluding that $\rho_{\ell+1}$ satisfies (q,ℓ)-back.

Since $\rho_{\ell+1}$ satisfies all the conditions of Definition 4.7, we conclude that it is a Pr $(q,\ell+1)$-bisimulation with $(d,e) \in \rho_{\ell+1}$. □

Summing up all these results, we obtain the characterization of $\mathcal{ALCQ}t_{q,\ell}$ over a class of interpretations $\mathbb{C}$ that is localizable and closed under partial unravelling as the fragment of FOL that is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation.

Theorem 4.26. *Let $\mathbb{C}$ be localizable and closed under partial unravelling and $\mathsf{N_C}$, $\mathsf{N_R}$ be finite. For all* FOL *formulae $\phi(x)$, the following are equivalent:*

1. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr bisimulation;*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation for some $q, \ell \in \mathbb{N}$;*
3. *$\phi(x)$ is $\mathbb{C}$-equivalent to some $\mathcal{ALCQ}t$ concept.*

Proof. The equivalence between (1) and (2) is showed in Theorem 4.18. We focus on proving that (2) and (3) are equivalent.

Let $C_\phi := \bigsqcup\{\mathsf{Bisim}^q_\ell[d] \mid \mathcal{I} \in \mathbb{C}, d \in \Delta^\mathcal{I} \text{ and } \mathcal{I} \models \phi(d)\}$. This is a well-formed concept by Proposition 4.23, and we use it to show that (2) implies (3). First, assume that $\mathcal{I} \models \phi(d)$ with $\mathcal{I} \in \mathbb{C}$ and $d \in \Delta^\mathcal{I}$. Then, $d \in C^\mathcal{I}_\phi$ trivially follows from the fact that $\mathsf{Bisim}^q_\ell[d]$ occurs as a disjunct in C_ϕ. Vice versa, if $d \in C^\mathcal{I}_\phi$ then $d \in (\mathsf{Bisim}^q_\ell[e])^\mathcal{I}$ for some $\mathcal{J} \in \mathbb{C}$ and $e \in \Delta^\mathcal{J}$, and in particular $\mathcal{J} \models \phi(e)$. We saw in the proof of Theorem 4.25 that this implies that d and e are Pr (q,ℓ)-bisimilar, hence that $\mathcal{I} \models \phi(d)$ by (2). Thus, $\phi(x)$ and C_ϕ are $\mathbb{C}$-equivalent.

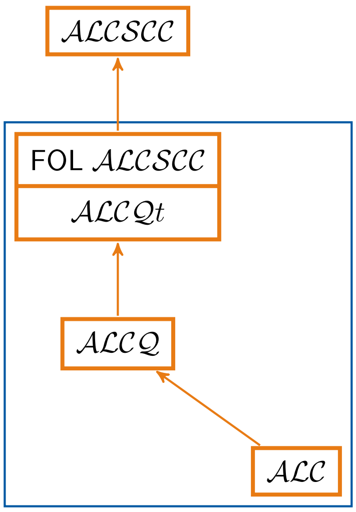
Next, let $\phi(x)$ be $\mathbb{C}$-equivalent to the $\mathcal{ALCQ}t$ concept C. If $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ contain $d \in \Delta^\mathcal{I}$ and $e \in \Delta^\mathcal{J}$ that are Pr (q,ℓ)-bisimilar, the $\mathbb{C}$-invariance of C under Pr (q,ℓ)-bisimulation (cf. Theorem 4.8) and our assumption yield $d \in \phi^\mathcal{I}$ iff $d \in C^\mathcal{I}_\phi$ iff $e \in C^\mathcal{J}_\phi$ iff $e \in \phi^\mathcal{J}$. Thus, $\phi(x)$ is $\mathbb{C}$-invariant under Pr (q,ℓ)-bisimulation. □

As a corollary of this theorem and Theorem 4.18 we obtain the characterization of $\mathcal{ALCQ}t$ as the first-order fragment of $\mathcal{ALCSCC}^\infty$ and $\mathcal{ALCSCC}$ w.r.t. the classes of all, of all finitely branching and of all finite interpretations.

Corollary 4.27. *Let $\mathsf{N_C}$, $\mathsf{N_R}$ be finite and $\mathbb{C}$ be $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ or $\mathbb{C}_{\mathsf{fin}}$. Then, an $\mathcal{ALCSCC}^\infty$ concept is* FOL*-definable w.r.t. $\mathbb{C}$ iff it is $\mathbb{C}$-equivalent to an $\mathcal{ALCQ}t$ concept.*

Summary

We analyzed the expressive power of $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ as concept languages using the notion of *local Presburger (Pr) bisimulation*, which strengthens the known notion of *counting bisimulation* [80] by applying the back-and-forth conditions to *safe role types* rather than role names. We showed that both $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ concepts are invariant under Pr bisimulation and used this property to show non-definability results w.r.t. these DLs. In [16] we showed that the first-order definable fragment of $\mathcal{ALCSCC}^\infty$ corresponds to the DL $\mathcal{ALCQ}t$, by showing that this DL is exactly the fragment of first-order logic that is invariant under Pr bisimulation. Here, we generalized this result to both $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ w.r.t. arbitrary, finitely branching and finite interpretations, using an approach based on strong locality properties of FOL that follows the treatment of Otto for graded modal logic (i.e. $\mathcal{ALCQ}$) in terms of counting bisimulation [84]. Using (q,ℓ)-bisimulations, we showed that $\mathcal{ALCQ}t$ and $\mathcal{ALCSCC}$ can be separated, thus showing that the latter DL contains concepts that are not first-order definable and is more expressive than the former DL. Our results on the relative expressive power of $\mathcal{ALC}$, $\mathcal{ALCQ}$, $\mathcal{ALCQ}t$, $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ are summarized in the following diagram.



FOL-definable

In this diagram, an arrow from a node N to a node N′ means that the concept language N can be expressed in N′ and that N is strictly less expressive than N′.

5 Knowledge Bases that Count, and what They Can and Cannot Count

In Chapter 4 we analyzed the relationship between FOL formulae $\phi(x)$ and $\mathcal{ALCSCC}$ (and $\mathcal{ALCSCC}^{\infty}$) concepts over the classes $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$ of arbitrary, finitely branching and finite interpretations of $\mathsf{N_C}$ and $\mathsf{N_R}$. For each of these classes of interpretations $\mathbb{C}$, we determined that $\mathcal{ALCQ}t$ is the fragment of FOL formulae $\phi(x)$ that are $\mathbb{C}$-invariant under Pr bisimulation.

The goal of this chapter is to study the relationship between FOL sentences and TBoxes, CBoxes and ECBoxes written using $\mathcal{ALCSCC}^{\infty}$ concepts. Due to the increased expressive power of these formalisms, we cannot apply the methods used in the previous chapter, as many of the employed transformations do not preserve the cardinality of the interpretations of the concepts at hand. Therefore, we cannot readily obtain a characterization that works w.r.t. finite and finitely branching interpretations. Instead, we focus on $\mathcal{ALCSCC}^{\infty}$ concepts over $\mathbb{C}_{\mathsf{all}}$, and lift the relationship between this DL and $\mathcal{ALCQ}t$ to (Boolean) TBoxes, to (Boolean) CBoxes and to ECBoxes written in these DLs. We show that a (Boolean) $\mathcal{ALCSCC}^{\infty}$ TBox is FOL-definable w.r.t. $\mathbb{C}_{\mathsf{all}}$ iff it is $\mathbb{C}_{\mathsf{all}}$-equivalent to a (Boolean) $\mathcal{ALCQ}t$ TBox, and that an $\mathcal{ALCSCC}^{\infty}$ ECBox is FOL-definable w.r.t. $\mathbb{C}_{\mathsf{all}}$ iff it is $\mathbb{C}_{\mathsf{all}}$-equivalent to a Boolean $\mathcal{ALCQ}t$ CBox. Finally, we provide inexpressivity results that separate the expressive power of CBoxes, RCBoxes and (Boolean) TBoxes w.r.t. $\mathbb{C}_{\mathsf{all}}$. The work contained in this chapter is based on the paper:

[14] Baader, F., De Bortoli, F.: Description Logics That Count, and What They Can and Cannot Count. In: Kovacs, L., Korovin, K., Reger, G. (eds.) ANDREI-60. Automated New-era Deductive Reasoning Event in Iberia. EPiC Series in Computing, pp. 1–25. EasyChair (2020). `https://doi.org/10.29007/ltzn`

Non-expressivity using disjoint unions. It is straightforward to deduce from Definition 3.6 that the CI $C \sqsubseteq D$, the CR $|C \sqcap \neg D| \leqslant 0$ and the inequality $|C \sqcap \neg D| \leqslant |\bot|$ are $\mathbb{C}_{\mathsf{all}}$-equivalent. This allows us to conclude that TBoxes are expressible w.r.t. $\mathbb{C}_{\mathsf{all}}$ using CBoxes, ECBoxes or RCBoxes. It is also clear that CBoxes and RCBoxes are special instances of an ECBox. The expressive power of CBoxes and RCBoxes, on the other hand, appears to be orthogonal. Indeed, CBoxes only allow us to compare concept cardinalities with a fixed number, and this is explicitly disallowed in RCBoxes (we can only simulate 0 using $|\bot|$). On the other hand, RCBoxes enable us to compare the cardinalities of different concepts whereas this is not possible in CBoxes.

In Definition 2.31 we defined the notion of $\mathbb{C}$-*invariance under disjoint unions* for TBoxes. Using the notation of Definition 2.31, we recall that a TBox is invariant under disjoint unions if for every family of interpretations $(\mathcal{I}_\nu)_{\nu\in\mathbb{I}}$, the disjoint union $\mathcal{I}$ of this family is a model of $\mathcal{T}$ iff every $\mathcal{I}_\nu$ with $\nu \in \mathbb{I}$ is a model of $\mathcal{T}$.

While TBoxes written in the DLs that we investigated in Chapter 4 are $\mathbb{C}_{\mathsf{all}}$-invariant under disjoint unions, the other formalisms analyzed in this chapter may be not. Some, such as RCBoxes and ERCBoxes, are only *closed under disjoint unions*, i.e. if every interpretation $\mathcal{I}_\nu$ with $\nu \in \mathbb{I}$ is a model of a (E)RCBox $\mathcal{R}$, then the disjoint union of all $\mathcal{I}_\nu$ with $\nu \in \mathbb{I}$ is also a model of $\mathcal{R}$. Differently from ERCBoxes, RCBoxes are also *invariant under disjoint copies*, i.e. invariant under disjoint unions of families where every interpretation is the same. Using this property, we deduce that ERCBoxes are more expressive than RCBoxes and less expressive than ECBoxes.

Proposition 5.1. *If $\mathcal{L} \in \{\mathcal{ALCQ}, \mathcal{ALCQ}t, \mathcal{ALCSCC}, \mathcal{ALCSCC}^\infty\}$, then*

1. *the models of $\mathcal{L}$ TBoxes are $\mathbb{C}_{\mathsf{all}}$-invariant under disjoint union;*
2. *the models of $\mathcal{L}$ RCBoxes are $\mathbb{C}_{\mathsf{all}}$-closed under disjoint union, $\mathbb{C}_{\mathsf{all}}$-invariant under disjoint copies but in general not $\mathbb{C}_{\mathsf{all}}$-invariant under disjoint union;*
3. *the models of $\mathcal{L}$ ERCBoxes are $\mathbb{C}_{\mathsf{all}}$-closed under disjoint union, but in general not $\mathbb{C}_{\mathsf{all}}$-invariant under disjoint copies nor $\mathbb{C}_{\mathsf{all}}$-invariant under disjoint union;*
4. *the models of $\mathcal{L}$ ECBoxes or CBoxes are in general not closed under disjoint union;*
5. *the models of Boolean $\mathcal{L}$ TBoxes are not closed under disjoint union.*

The first point is proved by adapting Corollary 2.33 to each $\mathcal{L}$ by using the notion of counting or Presburger bisimulation. Regarding the positive statement in (2) we notice that, for all families of interpretations $(\mathcal{I}_\nu)_{\nu\in\mathbb{I}}$ whose disjoint union is $\mathcal{I}$, the identity $|C^\mathcal{I}| = \sum_{\nu\in\mathbb{I}} |C^{\mathcal{I}_\nu}|$ holds for all $\mathcal{L}$ concepts C. Since the space of solutions of a homogeneous system of inequalities is closed under addition and scaling, we deduce in particular that closure under disjoint unions and invariance under disjoint copies hold for RCBoxes. As for the negative statement in (2), consider the RCBox $|A|+|B| \leqslant |C|$ for concept names A, B, C. If we consider interpretations $\mathcal{I}$ and $\mathcal{J}$ in which $A^\mathcal{I}$ contains one element, $B^\mathcal{I}$ one element, $C^\mathcal{I}$ one element, $A^\mathcal{J}$ one element, $B^\mathcal{J}$ one element, and $C^\mathcal{J}$ three elements, then the disjoint union of $\mathcal{I}$ and $\mathcal{J}$ is a model of the RCBox, but $\mathcal{I}$ is not. For (3) an argument similar to the one for (2) can be made to show closure under disjoint unions. However, it is easy to see that the ERCBox $|A| \geqslant 2$ is not invariant under disjoint copies, using a proof similar to the one employed to show in Chapter 2 that the models of $|A| \leqslant 1$ are not closed under disjoint union. This, in turn, shows that (4), holds. Finally, we notice that the models of the Boolean TBox $(A \sqsubseteq \bot) \vee (B \sqsubseteq \bot)$ are not closed under disjoint union. As an immediate consequence of Proposition 5.1, we obtain the following inexpressibility results.

Proposition 5.2. *If $\mathcal{L} \in \{\mathcal{ALCQ}, \mathcal{ALCQ}t, \mathcal{ALCSCC}, \mathcal{ALCSCC}^\infty\}$, then*

- *$\mathcal{L}$ TBoxes in general cannot express $\mathcal{L}$ RCBoxes, ECBoxes, CBoxes, and Boolean TBoxes;*
- *$\mathcal{L}$ RCBoxes in general cannot express $\mathcal{L}$ ECBoxes, CBoxes, and Boolean TBoxes;*
- *$\mathcal{L}$ ERCBoxes cannot express $\mathcal{L}$ ECBoxes, CBoxes, and Boolean TBoxes.*

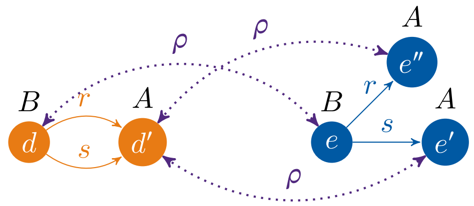


Figure 5.1: Two interpretations $\mathcal{I}$ and $\mathcal{J}$ and a global counting bisimulation ρ, which is not a global Pr bisimulation.

5.1 Expressive Power of (Boolean) TBoxes

In this setting, we pose finer conditions on the notion of Pr bisimulation ρ introduced in Definition 4.2 and require that every element in the two interpretations related by ρ is covered.

Definition 5.3. *A Pr bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$ is* global *if for every $d \in \Delta^{\mathcal{I}}$ there exists $e \in \Delta^{\mathcal{J}}$ such that $(d, e) \in \rho$ (and vice versa). We say that $\mathcal{I}$ and $\mathcal{J}$ are* globally Pr bisimilar *if they are related by a global Pr bisimulation. A* FOL *sentence ϕ is $\mathbb{C}_{\mathsf{all}}$-*invariant under global Pr bisimulation *if for all globally Pr bisimilar interpretations $\mathcal{I}$, $\mathcal{J}$ it holds that $\mathcal{I} \models \phi$ iff $\mathcal{J} \models \phi$.*

The following proposition is easily derived by combining Definition 5.3 and the fact that $\mathcal{ALCSCC}^{\infty}$ concepts are $\mathbb{C}_{\mathsf{all}}$-invariant under Pr bisimulation [16], combined with the analogous result for $\mathcal{ALCQ}$ w.r.t. counting bisimulation of [80].

Proposition 5.4. *Every (Boolean) $\mathcal{ALCSCC}^{\infty}$ (resp. $\mathcal{ALCQ}$) TBox is $\mathbb{C}_{\mathsf{all}}$-invariant under global Pr (resp. counting) bisimulation.*

Proof. The proof for $\mathcal{ALCQ}$ is detailed in [80]. For $\mathcal{ALCSCC}^{\infty}$, we only need to show that two interpretations $\mathcal{I}$ and $\mathcal{J}$ that are related by a global Pr bisimulation ρ satisfy the same $\mathcal{ALCSCC}^{\infty}$ CIs $C \sqsubseteq D$. Given a CI of this kind, assume that $\mathcal{I}$ is not a model of $C \sqsubseteq D$. Then, there exists $d \in (C \sqcap \neg D)^{\mathcal{I}}$. By assumption, there exists $e \in \Delta^{\mathcal{I}}$ such that $(d, e) \in \rho$, and since $C \sqcap \neg D$ is an $\mathcal{ALCSCC}^{\infty}$ concept, Theorem 4.3 implies that $e \in (C \sqcap \neg D)^{\mathcal{J}}$. Hence, $\mathcal{J}$ is not a model of $C \sqsubseteq D$. Similarly, we prove that if $\mathcal{J}$ is not a model of $C \sqsubseteq D$, then neither is $\mathcal{I}$. Since every Boolean TBox is a Boolean combination of CIs, we conclude that $\mathcal{I}$ and $\mathcal{J}$ satisfy the same Boolean TBoxes. □

We proved earlier that Boolean $\mathcal{ALCSCC}^{\infty}$ TBoxes are more expressive than $\mathcal{ALCSCC}^{\infty}$ TBoxes, as they are not closed under disjoint unions. Here, we apply the notion of global counting bisimulation to show that Boolean $\mathcal{ALCQ}$ TBoxes cannot express $\mathcal{ALCSCC}^{\infty}$ TBoxes.

Corollary 5.5. *There is no Boolean $\mathcal{ALCQ}$ TBox that is $\mathbb{C}_{\mathsf{all}}$-equivalent to the $\mathcal{ALCSCC}^{\infty}$ TBox $\mathcal{T} = \{B \sqsubseteq \mathsf{succ}(|r \cap s \cap A| \geqslant 1)$.*

Proof. To prove this corollary, we use the two interpretations depicted in Figure 5.1 which are related by a global counting bisimulation ρ. However, the interpretation on the left is a model of $\mathcal{T}$, whereas the one on the right is not, which shows that $\mathcal{T}$ cannot be equivalent to a Boolean $\mathcal{ALCQ}$ TBox by Proposition 5.4. □

$\mathcal{ALCSCC}^{\infty}$ TBoxes are not FOL-definable. In Theorem 4.18 we showed that for each FOL formula $\phi(x)$ $\mathbb{C}_{\text{all}}$-invariance under Pr bisimulation holds iff $\phi(x)$ is $\mathbb{C}_{\text{all}}$-invariant under Pr (q,ℓ)-bisimulation, where q is the quantifier depth of $\phi(x)$ and $\ell = 2^q - 1$, and used this fact to prove that there are $\mathcal{ALCSCC}^{\infty}$ concepts that are not FOL-definable w.r.t. $\mathbb{C}_{\text{all}}$. We can employ the same strategy to show that there are $\mathcal{ALCSCC}^{\infty}$ TBoxes that are not FOL-definable w.r.t. $\mathbb{C}_{\text{all}}$.

We will prove in Theorem 5.13 that a FOL sentence ϕ is $\mathbb{C}_{\text{all}}$-invariant under global Pr bisimulation iff it is $\mathbb{C}_{\text{all}}$-equivalent to a Boolean $\mathcal{ALCQ}t$ TBox $\mathcal{T}_{\phi}$. In particular, $\mathcal{T}_{\phi}$ must be a Boolean $\mathcal{ALCQ}t_{q,\ell}$ TBox (see Chapter 4) for some values of q and ℓ, which are not explicitly related to the quantifier depth of ϕ. Introducing a notion of *global Pr (q,ℓ)-bisimulation* that combines Definitions 4.7 and 5.3 and by adapting the proof of Proposition 5.4, we deduce that every Boolean $\mathcal{ALCQ}t_{q,\ell}$ TBox and thus $\mathcal{T}_{\phi}$ is $\mathbb{C}_{\text{all}}$-invariant under global Pr (q,ℓ)-bisimulation.

To show that $\mathbb{C}_{\text{all}}$-invariance under global Pr (q,ℓ)-bisimulation implies $\mathbb{C}_{\text{all}}$-invariance under global Pr bisimulation, it is sufficient to show that every global Pr bisimulation is a global Pr (q,ℓ)-bisimulation for all values $q, \ell \in \mathbb{N}$. Finally, we show an analogous of Theorem 4.8 for an $\mathcal{ALCSCC}^{\infty}$ TBox.

Theorem 5.6. *There is an $\mathcal{ALCSCC}^{\infty}$ TBox $\mathcal{T}$ such that for all $q, \ell \in \mathbb{N}$, $\mathcal{T}$ is not $\mathbb{C}_{\text{all}}$-invariant under global Pr (q,ℓ)-bisimulation.*

Proof. We consider the $\mathcal{ALCSCC}^{\infty}$ TBox $\mathcal{T}$ consisting of the CI $\top \sqsubseteq \mathsf{succ}(|r \cap A| = |r \cap \neg A|)$. By looking at the proof of Theorem 4.8 we notice that the Pr (q,ℓ)-bisimulation ρ between the interpretations $\mathcal{I}_{q,q}$ and $\mathcal{I}_{q,q+1}$ is global. However, $\mathcal{I}_{q,q}$ is a model of $\mathcal{T}$, while $\mathcal{I}_{q,q+1}$ is not, obtaining a contradiction. We conclude that $\mathcal{T}$ is not $\mathbb{C}_{\text{all}}$-invariant under global Pr (q,ℓ)-bisimulation. □

We thus showed that there exists $\mathcal{ALCSCC}^{\infty}$ TBoxes that are not FOL-definable.

Corollary 5.7. *There exists an $\mathcal{ALCSCC}^{\infty}$ TBox that is not FOL-definable.*

First-order definable (Boolean) TBoxes. As anticipated in the previous paragraph, global bisimulations can be used to characterize the set of FOL sentences that are $\mathbb{C}_{\text{all}}$-equivalent to Boolean TBoxes written in a certain DL. This is the case for global counting bisimulation in the context of $\mathcal{ALCQ}$ [80]. Here, we show that global Pr bisimulation characterizes the set of FOL sentences that are $\mathbb{C}_{\text{all}}$-equivalent to a Boolean $\mathcal{ALCQ}t$ TBox, and thus obtain that every Boolean $\mathcal{ALCSCC}^{\infty}$ TBox that is FOL-definable w.r.t. $\mathbb{C}_{\text{all}}$ is $\mathbb{C}_{\text{all}}$-equivalent to a Boolean $\mathcal{ALCQ}t$ TBox.

Previously, the semantic restriction of $\mathcal{ALCSCC}$ to finitely branching interpretations disallowed us from using strong model-theoretic properties of first-order logic over arbitrary interpretations that concern infinite sets of formulae. This is different in $\mathcal{ALCSCC}^{\infty}$, whose semantics is defined w.r.t. $\mathbb{C}_{\text{all}}$. First, we can use the *compactness property* which we mentioned in Chapter 2 to extract a finite unsatisfiable set of FOL sentences from an infinite unsatisfiable set of sentences. Second, we can use a restricted form of compactness for countable sets of FOL formulae $\phi(x_1, \dots, x_n)$ w.r.t. an interpretation $\mathcal{I}$, provided that this is *ω-saturated* (see e.g. [38]).

Definition 5.8. *Given an interpretation $\mathcal{I}$ of $\mathsf{N_C}$ and $\mathsf{N_R}$, let $\Phi(x_1, \dots, x_n)$ be a countable (possibly infinite) set of FOL formulae with free variables in $\{x_1, \dots, x_n\}$ that use names from $\mathsf{N_C}$ and $\mathsf{N_R}$ as predicates symbols and that are additionally allowed to use individuals d from a finite subset of $\Delta^{\mathcal{I}}$ as constant symbols, so that $d^{\mathcal{I}} := d$ for all individuals in this set.*

We say that $\Phi(x_1, \ldots, x_n)$ is realizable *in $\mathcal{I}$ if there is an assignment w of $x_1, \ldots, x_n$ in $\mathcal{I}$ such that $\mathcal{I}, w \models \phi(y_1, \ldots, y_k)$ for all $\phi(y_1, \ldots, y_k) \in \Phi(x_1, \ldots, x_n)$, and that $\Phi(x_1, \ldots, x_n)$ is* finitely realizable *if all its finite subsets are realizable in $\mathcal{I}$.*

The interpretation $\mathcal{I}$ is ω-saturated if every set $\Phi(x_1, \ldots, x_n)$ of the form above is realizable in $\mathcal{I}$ if and only if it is finitely realizable in $\mathcal{I}$.

A fundamental result for FOL is that, though not every interpretation $\mathcal{I}$ is ω-saturated, we may assume without loss of generality that $\mathcal{I}$ is as such, as long as we are only interested in the first-order theory of $\mathcal{I}$, i.e. the set of sentences that are satisfied in this interpretation (cf. [38]).

Theorem 5.9. *For every interpretation $\mathcal{I}$ there exists an ω-saturated interpretation $\mathcal{I}^\star$ that satisfies the same first-order sentences as $\mathcal{I}$.*

This property of FOL does not hold when restricting to finitely branching interpretations.

Proposition 5.10. *There is a finitely branching interpretation $\mathcal{I}$ for which there is no finitely branching interpretation $\mathcal{I}^\star$ that satisfies the same first-order sentences and is ω-saturated.*

Proof. Let $\mathcal{I}$ be the finitely branching interpretation whose domain contains all tuples (m, n) of natural numbers such that $n \leqslant m$ and where

$$r^{\mathcal{I}} := \{((m, 0), (m + 1, 0)) \mid m \in \mathbb{N}\} \cup \{((m, 0), (m, i)) \mid m \in \mathbb{N} \text{ and } i = 1, \ldots, m\}.$$

Intuitively, $\mathcal{I}$ contains an element with m r-successors for all values of m. This means that $\mathcal{I}$ satisfies the sentence $\exists x.\exists_{\geqslant m} y.r(x, y)$ for all values of m.

Assume by contradiction that a finitely branching interpretation $\mathcal{I}^\star$ of the form above exists. Then, the fact that $\mathcal{I}^\star$ and $\mathcal{I}$ satisfy the same first-order sentences implies that the set of first-order formulae $\Gamma(x)$ containing $\exists_{\geqslant m} y.r(x, y)$ for $m \in \mathbb{N}$ is finitely realizable in $\mathcal{I}^\star$. Since this interpretation is ω-saturated, we deduce that $\Gamma(x)$ is realizable in $\mathcal{I}^\star$ by some variable assignment w. This leads to a contradiction: indeed, the fact that $\mathcal{I}^\star$ is finitely branching implies that $w(x)$ has at most m r-successors for some value of m, and therefore that $\mathcal{I}^\star, w \not\models \exists_{\geqslant m+1} y.r(x, y)$. We conclude that no such interpretation $\mathcal{I}^\star$ exists. □

In Chapter 4 we showed that it is possible to describe the class of individuals that are Pr (q,ℓ)-bisimilar to $d \in \Delta^{\mathcal{I}}$ using an $\mathcal{ALCQ}t$ concept, called the *characteristic concept* of d. For Pr bisimulation, this is not the case: if d belongs to an interpretation that is not finitely branching, we may not find an $\mathcal{ALCQ}t$ concept that describes all the individuals that are Pr bisimilar to d. However, if $\mathcal{I}$ is ω-saturated we can show that the set of $\mathcal{ALCQ}t$ concepts satisfied by d effectively describes the class of individuals that belong to ω-saturated interpretations and that are Pr bisimilar to d. We can further lift this result to show that the set of $\mathcal{ALCQ}t$ CIs that are satisfied by $\mathcal{I}$ describe all ω-saturated interpretations that are globally Pr bisimilar to $\mathcal{I}$. For $\mathcal{ALCQ}$, a similar property w.r.t. counting bisimulation has been showed in [80].

A further result that we will need in the upcoming proof is Hall's theorem [58], which provides a way to reduce the problem of choosing distinct representatives $s_1, \ldots, s_n$ with $s_i \in S_i$ for $i = 1, \ldots, n$ from a family of sets $F = (S_1, \ldots, S_n)$ to checking the satisfiability of a finite set of cardinality constraints. If those representatives exist, we say that F has a *system of distinct representatives (SDR).*

Hall's Theorem. *The family $F = (S_1, \ldots, S_n)$ has a system of distinct representatives iff for all index sets $I \subseteq \{1, \ldots, n\}$ we have $\left|\bigcup_{i \in I} S_i\right| \geqslant |I|$.*

The following lemma is an immediate consequence of Hall's theorem and shows that we can describe the existence of a SDR for a finite family of sets of τ-successors in $\mathcal{ALCQ}t$, if these sets correspond to extensions of $\mathcal{ALCQ}t$ concepts.

Lemma 5.11. *Given $\mathcal{ALCQ}t$ concepts $C_1, \dots, C_n$, a safe role type τ and an interpretation $\mathcal{I}$ with $d \in \Delta^{\mathcal{I}}$, the family of sets $\tau^{\mathcal{I}}(d) \cap C_i^{\mathcal{I}}$ has a SDR iff $d \in C_{\mathsf{sdr}}^{\mathcal{I}}$ where*

$$C_{\mathsf{sdr}} := \bigsqcap \{ (\geqslant k\, \tau.C_{i_1} \sqcup \cdots \sqcup C_{i_k}) \mid \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\} \textit{ contains } k \textit{ elements} \}.$$

We leverage Lemma 5.11 to prove the following property of ω-saturated interpretations.

Theorem 5.12. *If $\mathcal{I}$, $\mathcal{J}$ are ω-saturated interpretations that satisfy the same $\mathcal{ALCQ}t$ CIs, then they are globally Pr bisimilar.*

Proof. Given the assumptions stated above, we show that the relation

$$\rho := \{(d, e) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}} \mid d \text{ and } e \text{ satisfy the same } \mathcal{ALCQ}t \text{ concepts}\}$$

is a global Pr bisimulation between $\mathcal{I}$ and $\mathcal{J}$. First, we show that this relation is global in the sense of Definition 5.3.

For $d \in \Delta^{\mathcal{I}}$, let Γ be the set of all FOL formulae $C^{\sharp}(x)$ that are the translation of an $\mathcal{ALCQ}t$ concept C such that $d \in C^{\mathcal{I}}$. Clearly, Γ is realizable in $\mathcal{I}$ using the variable assignment $\{x \mapsto d\}$. We show that Γ is finitely realizable in $\mathcal{J}$; since $\mathcal{J}$ is ω-saturated, this means that Γ is realizable in $\mathcal{J}$ using a variable assignment $\{x \mapsto e\}$, which implies that $e \in \Delta^{\mathcal{J}}$ satisfies the same $\mathcal{ALCQ}t$ concepts as d, hence $(d, e) \in \rho$. Every finite subset Γ' of Γ induces an $\mathcal{ALCQ}t$ concept G that is the conjunction of all $\mathcal{ALCQ}t$ concepts whose translation is in Γ'. Then, the fact that $d \in G^{\mathcal{I}}$ implies that $\mathcal{I}$ does not satisfy the CI $G \sqsubseteq \bot$. By assumption, we deduce that $\mathcal{J}$ also does not satisfy this CI, and thus that there exists $e \in G^{\mathcal{J}}$. Then, $\{x \mapsto e\}$ realizes Γ' in $\mathcal{J}$. We conclude that Γ is finitely realizable in $\mathcal{J}$. The other direction, i.e. showing that for $e \in \Delta^{\mathcal{J}}$ there is $d \in \Delta^{\mathcal{I}}$ such that $(d, e) \in \rho$, can be proved analogously.

Next, we show that ρ is a Pr bisimulation. Since $(d, e) \in \rho$ satisfy in particular the same concept names, we deduce that ρ trivially satisfies the atomic condition. To show that forth holds for ρ, let $(d, e) \in \rho$ and $D \subseteq \tau^{\mathcal{I}}(d)$ be a finite set. We introduce a fresh variable $x_{d'}$ for every $d' \in D$ and consider the set of FOL formulae Γ_d obtained as the union of

$$\Gamma^{\neq} := \{x_{d'} \neq x_{d''} \mid d', d'' \in D \text{ and } d' \neq d''\} \qquad \text{(variables are all distinct)}$$
$$\Gamma_d^{\tau} := \{\tau(d, x_{d'}) \mid d' \in D\} \qquad (\omega\text{-successors of } e_2)$$
$$T_{d'} := \{\pi_{x_{d'}}(C) \mid C \text{ is a } \mathcal{ALCQ}t \text{ concept and } d' \in C^{\mathcal{I}}\} \text{ for } d' \in D \qquad (\mathcal{ALCQ}t\text{-type of } d)$$

Then, Γ_d is realizable in $\mathcal{I}$ by the assignment w mapping $x_{d'}$ to d' for $d' \in D$.

We show that the set Γ_e obtained by replacing d with e in Γ is finitely realizable and thus realizable (by ω-saturation) in $\mathcal{J}$. Let $\Gamma_d' \subseteq \Gamma_d$ be a finite set containing w.l.o.g. both $\Gamma^{\neq}$ and Γ_d^{τ} (as these sets are both finite) and let Γ_e' be the corresponding finite subset of Γ_e. Further, let $t_{d'}$ be the intersection of $T_{d'}$ and Γ_d' (thus Γ_e') for $d \in D'$; this set induces a concept $C_{d'}$ that is the conjunction of all concepts C for which $\pi_{x_{d'}}(C) \in t_{d'}$. Since Γ_d' is realized in $\mathcal{I}$ by the assignment w above, we deduce that D' constitues a SDR for the family of sets $\tau^{\mathcal{I}}(d) \cap C_{d'}^{\mathcal{I}}$ with $d' \in D$, thus that $d \in C_{\mathsf{sdr}}^{\mathcal{I}}$ by Lemma 5.11. Lemma 5.11 together with the fact that d and e satisfy the same $\mathcal{ALCQ}t$ concepts allow us to conclude that the family of sets $\tau^{\mathcal{J}}(e) \cap C_{d'}^{\mathcal{J}}$ with

$d' \in D$ has a SDR, represented by the set E that contains an element e' for each $d' \in D$. We conclude that the assignment w' mapping x to e and $x_{d'}$ to the corresponding $e' \in E$ realizes Γ'_e in $\mathcal{J}$, therefore that Γ_e is realizable in $\mathcal{J}$.

Let w' be an assignment that realizes Γ_e in $\mathcal{J}$ and let $E := \{w'(d') \mid d' \in D\}$. We deduce that d' and $w'(d')$ satisfy the same $\mathcal{ALCQ}t$ concepts, hence $(d', w'(d')) \in \rho$, from the fact that $\mathcal{J}, w' \models T_{d'}$ for $d' \in D$. The mapping $d' \mapsto w'(d')$ constitutes a bijection from D to E thanks to $\mathcal{J}, w' \models \Gamma^{\neq}$. Finally, $\mathcal{J}, w' \models \Gamma^{\tau}_e$ guarantees that $E \subseteq \tau^{\mathcal{J}}(e)$.

We conclude that ρ satifies the forth condition. Similarly, we prove that ρ satisfies the back condition, hence that it is a Pr bisimulation. □

Thanks to this property of ω-saturated interpretations, we can characterize the set of Boolean $\mathcal{ALCQ}t$ TBoxes as the set of FOL sentences that are $\mathbb{C}_{\mathsf{all}}$-invariant under global Pr bisimulation. The analogous result for $\mathcal{ALCQ}$ w.r.t counting bisimulation has been proved in [80].

Theorem 5.13. *Let ϕ be a first-order sentence. Then the following are equivalent:*

1. *There exists a Boolean $\mathcal{ALCQ}t$ TBox that is $\mathbb{C}_{\mathsf{all}}$-equivalent to ϕ;*

2. *The sentence ϕ is $\mathbb{C}_{\mathsf{all}}$-invariant under global Pr bisimulation.*

Proof. Direction (1 $\implies$ 2) is a direct consequence of Proposition 5.4, since $\mathcal{ALCQ}t$ is a sublogic of $\mathcal{ALCSCC}^{\infty}$. Let Cons($\phi$) be the set of Boolean $\mathcal{ALCQ}$ TBoxes entailed by a first-order sentence ϕ. We prove that (2 $\implies$ 1), showing that if we assume (2) and Cons(ϕ) $\not\models \phi$ we are able to derive a contradiction. By first-order compactness, if Cons(ϕ) $\models \phi$ there is a finite set of Boolean TBoxes Γ in Cons(ϕ) entailing ϕ. Since Cons(ϕ) is closed under conjunction, the Boolean $\mathcal{ALCQ}$ TBox $\mathcal{T} := \bigwedge \Gamma$ belongs to Cons(ϕ) and $\mathcal{T} \models \phi$, hence the conclusion $\mathcal{T} \equiv \phi$.

If we assume that Cons(ϕ) $\not\models \phi$, then Cons(ϕ) $\cup \{\neg\phi\}$ is satisfied by an interpretation $\mathcal{I}^-$ that is w.l.o.g. ω-saturated (thanks to Theorem 5.9). Moreover, we are able to show that if $\mathcal{G}$ is the set of $\mathcal{ALCQ}$ CIs $C \sqsubseteq D$ or their negation $\neg(C \sqsubseteq D)$ satisfied by $\mathcal{I}^-$, then $\mathcal{G} \cup \{\phi\}$ has a model $\mathcal{I}^+$ that is w.l.o.g. ω-saturated. Otherwise, first-order compactness would yield a finite subset $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}' \cup \{\phi\}$ was also unsatisfiable. This would imply that $\phi \to \neg \bigwedge \mathcal{G}'$ is a tautology and $\neg \bigwedge \mathcal{G}' \in$ Cons(ϕ) would follow accordingly. However, this would lead to a contradiction, since both $\bigwedge \mathcal{G}'$ and $\neg \bigwedge \mathcal{G}'$ are now satisfied by $\mathcal{I}^-$.

We observe that $\mathcal{I}^-$ and $\mathcal{I}^+$ satisfy the same $\mathcal{ALCQ}t$ CIs, and they are consequently globally Pr bisimilar by Theorem 5.12. Finally, we contradict (2.) since $\mathcal{I}^+ \models \phi$ but $\mathcal{I}^- \not\models \phi$. Therefore, we conclude that Cons(ϕ) $\models \phi$. □

We can refine this characterization to $\mathcal{ALCQ}t$ TBoxes by considering first-order sentences that are additionally $\mathbb{C}_{\mathsf{all}}$-invariant under (arbitrary) disjoint unions.

Theorem 5.14. *Let ϕ be a first-order sentence. Then the following are equivalent:*

1. *There exists an $\mathcal{ALCQ}t$ (resp. $\mathcal{ALCQ}$) TBox $\mathcal{T}$ that is $\mathbb{C}_{\mathsf{all}}$-equivalent to ϕ.*

2. *The sentence ϕ is $\mathbb{C}_{\mathsf{all}}$-invariant under global Pr (resp. counting) bisimulation and disjoint unions.*

Proof. For $\mathcal{ALCQ}$, the proof is detailed in [80, Theorem 7]. For $\mathcal{ALCSCC}^{\infty}$, this is proved analogously. In particular, the proof is similar to that of Theorem 5.13, with the following modifications.

First, we take $\mathrm{Cons}(\phi)$ as the set of CIs over $\mathcal{ALCQ}t$ concepts entailed by ϕ. The model $\mathcal{I}^-$ of $\mathrm{Cons}(\phi) \cup \{\neg\phi\}$ is considered as above. For every CI $\mathfrak{c} := (C \sqsubseteq D)$ that is not entailed by ϕ, we are able to find a model $\mathcal{I}_\mathfrak{c}$ of ϕ that is not a model of $\mathfrak{c}$. We then define $\mathcal{I}^+$ as the disjoint union of all interpretations $\mathcal{I}_\mathfrak{c}$ obtained this way, and assume w.l.o.g. that $\mathcal{I}^+$ is ω-saturated. Clearly, both $\mathcal{I}^-$ and $\mathcal{I}^+$ satisfy the same $\mathcal{ALCQ}t$ CIs and thus are globally Pr bisimilar by Theorem 5.12. However, $\mathcal{I}^+ \models \phi$ but $\mathcal{I}^- \not\models \phi$ and therefore we must conclude that $\mathrm{Cons}(\phi) \models \phi$. □

Combining Proposition 5.4 with Theorems 5.13 and 5.14, we thus obtain the following characterizations of the first order fragments of (Boolean) $\mathcal{ALCSCC}^\infty$ TBoxes.

Theorem 5.15. *Let $\mathcal{T}$ be a (Boolean) $\mathcal{ALCSCC}^\infty$ TBox. Then the following are equivalent:*

1. *$\mathcal{T}$ is* FOL*-definable w.r.t. $\mathbb{C}_{\mathsf{all}}$;*
2. *$\mathcal{T}$ is $\mathbb{C}_{\mathsf{all}}$-equivalent to a (Boolean) $\mathcal{ALCQ}t$ TBox.*

5.2 Expressive Power of (Boolean) CBoxes and ECBoxes

We turn our attention to (Boolean) CBoxes and ECBoxes. Analogously to our treatment of (Boolean) TBoxes, we relate the expressive power of Boolean CBoxes to an appropriate notion of bisimulation. In order to deal with CRs rather than CIs, we need to extend our notion of a global bisimulation to one that can also compare cardinalities of sets on the global level. The following definition is inspired by the *first-order counting games* used in [56] to analyze extensions of first-order logic by certain counting quantifiers.

Definition 5.16. *A Pr (resp. counting) bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$ is* comparative *if for each finite subset $D \subseteq \Delta^\mathcal{I}$ there is a set $E \subseteq \Delta^\mathcal{J}$ such that ρ contains a bijection between D and E (and vice versa). We say that $\mathcal{I}$ and $\mathcal{J}$ are* comparatively Pr (resp. counting) bisimilar *if they are related by a comparative Pr (resp. counting) bisimulation. A first-order sentence ϕ is* $\mathbb{C}_{\mathsf{all}}$-invariant under comparative Pr (resp. counting) bisimulation *if for all comparately Pr (resp. counting) bisimilar interpretations $\mathcal{I}$ and $\mathcal{J}$ it holds that $\mathcal{I} \models \phi$ iff $\mathcal{J} \models \phi$.*

We show that comparatively bisimilar interpretations $\mathcal{I}$ and $\mathcal{J}$ assign to every concept C sets $C^\mathcal{I}$ and $C^\mathcal{J}$ that are either of the same size or both infinite, leading to the following.

Theorem 5.17. *Every $\mathcal{ALCQ}$ ECBox is $\mathbb{C}_{\mathsf{all}}$-invariant under comparative counting bisimulation, and every $\mathcal{ALCSCC}^\infty$ ECBox is $\mathbb{C}_{\mathsf{all}}$-invariant under comparative Pr bisimulation.*

Proof. We illustrate the proof for Pr bisimulations; the case for counting bisimulations is proved analogously. Let ρ be a comparative Pr bisimulation between the interpretations $\mathcal{I}$ and $\mathcal{J}$. We show that, under this assumption, $|C^\mathcal{I}| = |C^\mathcal{J}|$ holds for all $\mathcal{ALCSCC}^\infty$ concepts C. Since every extended CR occurring in an ECBox $\mathcal{E}$ is of the form

$$N_0 + N_1|C_1| + \cdots + N_k|C_k| \leqslant M_0 + M_1|D_1| + \cdots + M_\ell|D_\ell|$$

with N_i, M_j natural numbers and C_i, D_j $\mathcal{ALCSCC}^\infty$ concepts, showing the above implies that $\mathcal{E}$ is evaluated in the same way in $\mathcal{I}$ and $\mathcal{J}$, and thus that $\mathcal{E}$ is invariant under comparative Pr bisimulation.

Let n be a natural number. If $|C^\mathcal{I}| \geqslant n$ and $D \subseteq C^\mathcal{I}$ is a set of n distinct elements, the fact that ρ is comparative implies that there is a set $E \subseteq \Delta^\mathcal{J}$ of n elements and a bijection $\rho' \subseteq \rho$

between D and E. By Theorem 4.3 this implies that $E \subseteq C^{\mathcal{J}}$ and thus that $|C^{\mathcal{J}}| \geqslant n$. Similarly, we show that $|C^{\mathcal{J}}| \geqslant n$ implies $|C^{\mathcal{I}}| \geqslant n$.

If $|C^{\mathcal{I}}| = n$ for some natural number n, then the above implies that $|C^{\mathcal{J}}| = n$, since $|C^{\mathcal{I}}| \not\geqslant n+1$ iff $|C^{\mathcal{J}}| \not\geqslant n+1$. If $C^{\mathcal{I}}$ is infinite, then the above implies that $C^{\mathcal{J}}$ must be infinite too, since $|C^{\mathcal{J}}| \geqslant n$ iff $|C^{\mathcal{I}}| \geqslant n$ holds for all values of n. We conclude that $|C^{\mathcal{I}}| = |C^{\mathcal{J}}|$ for all $\mathcal{ALCSCC}^{\infty}$ concepts C, thus that ECBoxes are invariant under comparative Pr bisimulation. □

Next, we want to show that Boolean $\mathcal{L}$ CBoxes are exactly the first-order sentences that are $\mathbb{C}_{\mathsf{all}}$-invariant under the corresponding notion of comparative bisimulation. We show that an analogous of Theorem 5.12 holds for comparative bisimulations w.r.t. CRs. In this case, we apply Hall's Theorem to show that the existence of a SDR for the family of sets $C_1^{\mathcal{I}}, \dots, C_n^{\mathcal{I}}$ where $C_1, \dots, C_n$ are $\mathcal{L}$ concepts can be described using a Boolean $\mathcal{L}$ CBox, similarly to what done in Lemma 5.11 using qualified number restrictions.

Lemma 5.18. *Let $\mathcal{L} \in \{\mathcal{ALCQ}, \mathcal{ALCQ}t\}$, $\mathcal{I} \in \mathbb{C}_{\mathsf{all}}$, and $C_1, \dots, C_n$ $\mathcal{L}$ concepts. Then the family $(C_1^{\mathcal{I}}, \dots, C_n^{\mathcal{I}})$ has an SDR iff $\mathcal{I} \models \mathcal{C}_{\mathsf{sdr}}$ where*

$$\mathcal{C}_{\mathsf{sdr}} := \big\{|C_{i_1} \sqcup \cdots \sqcup C_{i_k}| \geqslant k \mid \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\} \text{ contains } k \text{ elements}\big\}.$$

Proposition 5.19. *If $\mathcal{I}$, $\mathcal{J}$ are ω-saturated interpretations that satisfy the same $\mathcal{ALCQ}t$ (resp. $\mathcal{ALCQ}$) CRs, then they are comparatively Pr (resp. counting) bisimilar.*

Proof. We illustrate the proof for Pr bisimulations; the case for counting bisimulations is proved analogously. Given the assumptions stated above, we show that the relation

$$\rho := \{(d, e) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}} \mid d \text{ and } e \text{ satisfy the same } \mathcal{ALCQ}t \text{ concepts}\}$$

is a comparative Pr bisimulation between $\mathcal{I}$ and $\mathcal{J}$. It is already known that ρ is an Pr bisimulation: this has been shown in Theorem 5.12 for $\mathcal{ALCQ}t$. Thus, we focus on showing that ρ is comparative in the sense of Definition 5.16.

Given a finite set $D \subseteq \Delta^{\mathcal{I}}$, we introduce a fresh variable $x_{d'}$ for each $d' \in D$ and consider the set of FOL formulae Γ obtained as the union of

$$\Gamma^{\neq} := \{x_{d'} \neq x_{d''} \mid d', d'' \in D \text{ and } d' \neq d''\},$$
$$\Theta_{d'} := \{C^{\sharp}(x_{d'}) \mid C \text{ is an } \mathcal{L} \text{ concept and } d' \in C^{\mathcal{I}}\} \text{ for } d' \in D$$

and show that Γ is finitely realizable and thus realizable (by ω-saturation) in $\mathcal{J}$.

Let $\Gamma' \subseteq \Gamma$ be a finite set that contains w.l.o.g. $\Gamma^{\neq}$ (as this set is finite). For $d' \in D$ we consider the $\mathcal{L}$ concept description

$$C_{d'} := \bigsqcap \{C \mid C^{\sharp}(x_{d'}) \in \Gamma' \cap \Theta_{d'}\},$$

which is well-defined since Γ' is finite. Since the variable assignment w that maps $x_{d'}$ to d' for $d' \in D$ trivially realizes Γ in $\mathcal{I}$, we obtain that

$$\mathcal{I}, w \models \textstyle\bigwedge_{i=1}^{n} C_i^{\sharp}(x_i) \wedge \bigwedge_{1 \leqslant i < j \leqslant n} (x_i \neq x_j)$$

which implies that D induces a SDR for the family of sets $C_{d'}^{\mathcal{I}}$ with $d' \in D$. Then, by Lemma 5.18, we obtain that $\mathcal{I}$ is a model of the CBox $\mathcal{C}_{\mathsf{sdr}}$ defined in this lemma. Since $\mathcal{I}$ and $\mathcal{J}$ satisfy the same CRs, we deduce that $\mathcal{J}$ is also a model of $\mathcal{C}_{\mathsf{sdr}}$. By Lemma 5.18, this implies that the family of sets $C^{\mathcal{J}}_{d'}$ with $d' \in D$ also has an SDR E. Let $e' \in E$ the element associated to $d' \in D$, i.e. $e' \in C^{\mathcal{J}}_{d'}$. Then, the variable assignment w' that maps $x_{d'}$ to e' for $d' \in D$ realizes Γ' in $\mathcal{J}$.

We obtain that Γ is realizable in $\mathcal{J}$ by a variable assignment w'. This implies that $d' \in D$ and $w'(d')$ satisfy the same $\mathcal{ALCQ}t$ concepts for $d' \in D$ and thus that $(d', w'(d)) \in \rho$. Furthermore, the mapping $d' \mapsto w'$ acts as a bijection between D and $E := \{w'(d') \mid d' \in D\}$. We conclude that ρ is comparative, as the other direction can be similarly proved. □

We are now ready to prove the main theorem of this section.

Theorem 5.20. *Let ϕ be a first-order sentence. Then the following are equivalent:*

1. *There exists a Boolean $\mathcal{ALCQ}t$ (resp. $\mathcal{ALCQ}$) CBox $\mathcal{C}$ that is $\mathbb{C}_{\mathsf{all}}$-equivalent to ϕ.*
2. *The sentence ϕ is $\mathbb{C}_{\mathsf{all}}$-invariant under comparative Pr (resp. counting) bisimulation.*

Proof. We illustrate the proof for Pr bisimulations; the case for counting bisimulations is proved analogously. The direction (1 ⇒ 2) is a direct consequence of Theorem 5.17 since Boolean $\mathcal{ALCQ}t$ CBoxes are a special case of $\mathcal{ALCQ}t$ ECBoxes.

Let Cons(ϕ) denote the set of Boolean $\mathcal{ALCQ}t$ CBoxes entailed by the first-order sentence ϕ. We prove (2 ⇒ 1) by showing that (2) implies Cons(ϕ) $\models \phi$. In fact, if this is the case, then compactness of first-order logic yields a finite set of Boolean $\mathcal{ALCQ}t$ CBoxes $\Gamma \subseteq$ Cons(ϕ) entailing ϕ. But then the conjunction $\mathcal{C} := \bigwedge \Gamma$ of the elements of Γ also belongs to Cons(ϕ), and thus we have that $\mathcal{C}$ is a Boolean $\mathcal{ALCQ}t$ CBox that is equivalent to ϕ.

We prove Cons(ϕ) $\models \phi$ by contradiction. Thus, assume that Cons(ϕ) $\not\models \phi$. Then Cons(ϕ)$\cup\{\neg\phi\}$ has a model $\mathcal{I}^-$, of which we can assume without loss of generality that it is ω-saturated (thanks to Theorem 5.9).

Now, let $\mathcal{G}$ denote the set of $\mathcal{ALCQ}t$ CRs that are satisfied by $\mathcal{I}^-$. We claim that $\mathcal{G} \cup \{\phi\}$ has a model. In fact, otherwise first-order compactness would yield a finite subset $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}' \cup \{\phi\}$ also does not have a model. However, this would imply that $\phi \to \neg \bigwedge \mathcal{G}'$ is a tautology, which would yield $\neg \bigwedge \mathcal{G}' \in$ Cons(ϕ). This lead to a contradiction since now both $\bigwedge \mathcal{G}'$ and $\neg \bigwedge \mathcal{G}'$ would need to be satisfied by $\mathcal{I}^-$. Thus, we have shown that $\mathcal{G} \cup \{\phi\}$ has a model $\mathcal{I}^+$, of which can again assume that it is ω-saturated.

We observe that $\mathcal{I}^-$ and $\mathcal{I}^+$ both satisfy exactly the CRs occurring in $\mathcal{G}$. Since these two interpretations are also ω-saturated, Proposition 5.19 implies that they are also comparatively Pr bisimilar. This contradicts our assumption that (2) holds since we have $\mathcal{I}^+ \models \phi$, but $\mathcal{I}^- \not\models \phi$. Thus, we have shown that (2) implies Cons(ϕ) $\models \phi$, which concludes our proof. □

Since ECBoxes are $\mathbb{C}_{\mathsf{all}}$-invariant under comparative bisimulation by Theorem 5.17, Theorem 5.20 yields the following characterization of the FOL-definable fragment of ECBoxes w.r.t. $\mathbb{C}_{\mathsf{all}}$ for the DLs $\mathcal{ALCQ}$ and $\mathcal{ALCQ}t$.

Theorem 5.21. *Let $\mathcal{L} \in \{\mathcal{ALCQ}, \mathcal{ALCQ}t\}$ and $\mathcal{E}$ be an $\mathcal{L}$ ECBox. Then the following are equivalent:*

1. *There exists a first-order sentence ϕ that is $\mathbb{C}_{\mathsf{all}}$-equivalent to $\mathcal{E}$.*
2. *$\mathcal{E}$ is $\mathbb{C}_{\mathsf{all}}$-equivalent to a Boolean $\mathcal{L}$ CBox $\mathcal{C}$.*

It remains to show that there are $\mathcal{ALCQ}$ ECBoxes that are not equivalent to a first-order sentence. Since it uses a technique different from the ones employed until now, we defer the proof of this result to the next section.

We close the analysis of comparative Pr bisimulation by giving a characterization of the first-order fragment of $\mathcal{ALCSCC}^\infty$ ECBoxes.

Theorem 5.22. *Let $\mathcal{E}$ be an $\mathcal{ALCSCC}^\infty$ ECBox. Then the following are equivalent:*

1. *There exists a first-order sentence ϕ that is $\mathbb{C}_{\mathsf{all}}$-equivalent to $\mathcal{E}$.*
2. *$\mathcal{E}$ is $\mathbb{C}_{\mathsf{all}}$-equivalent to a Boolean $\mathcal{ALCQ}t$ CBox $\mathcal{C}$.*

Proof. To prove ($1 \Rightarrow 2$), assume that ϕ is a first-order sentence equivalent to $\mathcal{E}$. It is easy to show that $\mathcal{ALCSCC}^\infty$ ECBoxes are $\mathbb{C}_{\mathsf{all}}$-invariant under comparative $\mathcal{ALCQ}t$ bisimulation. Therefore, ϕ is also $\mathbb{C}_{\mathsf{all}}$-invariant under comparative $\mathcal{ALCQ}t$ bisimulation. By Theorem 5.20, this implies that ϕ, and hence $\mathcal{E}$, is $\mathbb{C}_{\mathsf{all}}$-equivalent to a Boolean $\mathcal{ALCQ}t$ CBox $\mathcal{C}$.

($2 \Rightarrow 1$) is an immediate consequence of the fact that Boolean $\mathcal{ALCQ}t$ CBoxes have a first-order translation (obtained by combining Chapter 2 with the first-order translation of $\mathcal{ALCQ}t$). □

5.3 ECBoxes and the 0-1 law for FOL

Let ϕ be a first-order sentence over a finite relational signature δ. We denote by $L_n(\delta)$ the set of interpretations over the signature δ with domain $\{1, \dots, n\}$, and with $L_n(\phi)$ the number of these interpretations that are models of ϕ. We then set

$$\ell(\phi) := \lim_{n \to \infty} \frac{L_n(\phi)}{L_n(\delta)}. \tag{5.1}$$

Theorem 5.23 (0-1 law of FOL [46]). *For every first-order sentence ϕ, the limit $\ell(\phi)$ always exists and is equal to* 0 *or* 1.

One can use this theorem to prove that a sentence of a certain logic cannot be equivalent to a first-order sentence by showing that the corresponding limit either does not exist or is a number different from 0 or 1. An example for the former case would be a formula whose models are exactly the interpretations whose domain has even cardinality. We show now that ECBoxes can yield examples for the latter case.

Proposition 5.24. *The ECBox $\mathcal{E} := |A| \leqslant |\neg A|$ is not expressible as a first-order sentence.*

Proof. By contradiction, assume that $\mathcal{E}$ is equivalent to some first-order sentence ϕ. We restrict our attention to the relational signature $\delta := \{A\}$ since the only relation symbol contained in $\mathcal{E}$ is the concept name A. If we consider interpretations $\mathcal{I}$ with domain $\Delta^\mathcal{I} = \{1, \dots, n\}$, then there are 2^n possible ways of interpreting $A^\mathcal{I}$, which shows that $L_n(\delta) = 2^n$. Among these interpretations, the ones where $|A^\mathcal{I}| = j$ for $0 \leqslant j \leqslant n$ are exactly $\binom{n}{j}$. Therefore, the number of interpretations with domain $\{1, \dots, n\}$ over δ satisfying $\mathcal{E}$, and hence ϕ, is

$$L_n(\phi) = \textstyle\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{j}. \tag{5.2}$$

Let $\ell_n(\phi) := L_n(\phi)/L_n(\delta)$. We show that the sequence $L := (\ell_n(\phi))_{n\geqslant 1}$ is convergent and $\ell(\phi) := \lim_{n\to\infty} \ell_n(\phi) = 1/2$.[1] This yields a contradiction: by Theorem 5.23, it should hold that $\ell(\phi) = 0$ or $\ell(\phi) = 1$.

We split the sequence L into two subsequences $L_1 := (\ell_{2n}(\phi))_{n\geqslant 1}$ and $L_2 := (\ell_{2n+1}(\phi))_{n\geqslant 1}$. To show that L converges to 1/2, it is sufficient to prove that both L_1 and L_2 have this limit. First, note that for $n \geqslant 1$ the following identities hold (which can, e.g., be shown by an application of Newton's binomial theorem):

$$2^{2n+1} = 2 \cdot \textstyle\sum_{j=0}^{n}\binom{2n+1}{j} \qquad (5.3) \qquad\qquad \textstyle\sum_{j=0}^{n}\binom{2n}{j} = \frac{1}{2}\cdot\left(2^{2n} + \binom{2n}{n}\right) \qquad (5.4)$$

By (5.3), our claim clearly holds for L_2. Indeed, for $n \geqslant 1$ we have

$$\ell_{2n+1}(\phi) = \frac{\sum_{j=0}^{n}\binom{2n+1}{j}}{2^{2n+1}} \overset{(5.3)}{=} \frac{\sum_{j=0}^{n}\binom{2n+1}{j}}{2\cdot\sum_{j=0}^{n}\binom{2n+1}{j}} = \frac{1}{2}.$$

Regarding the other subsequence, note that the n-th term of L_1 corresponds to

$$\ell_{2n}(\phi) = \frac{\sum_{j=0}^{n}\binom{2n}{j}}{2^{2n}} \overset{(5.4)}{=} \frac{1}{2} + \frac{1}{2}\cdot\frac{\binom{2n}{n}}{4^n}.$$

We know that the following asymptotic equivalence holds [76]:

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}. \qquad (5.5)$$

Hence, we deduce that

$$\lim_{n\to\infty} \ell_{2n}(\phi) = \frac{1}{2} + \frac{1}{2}\lim_{n\to\infty}\frac{\binom{2n}{n}}{4^n} = \frac{1}{2} + \frac{1}{2}\lim_{n\to\infty}\frac{1}{\sqrt{\pi n}} = \frac{1}{2}.$$

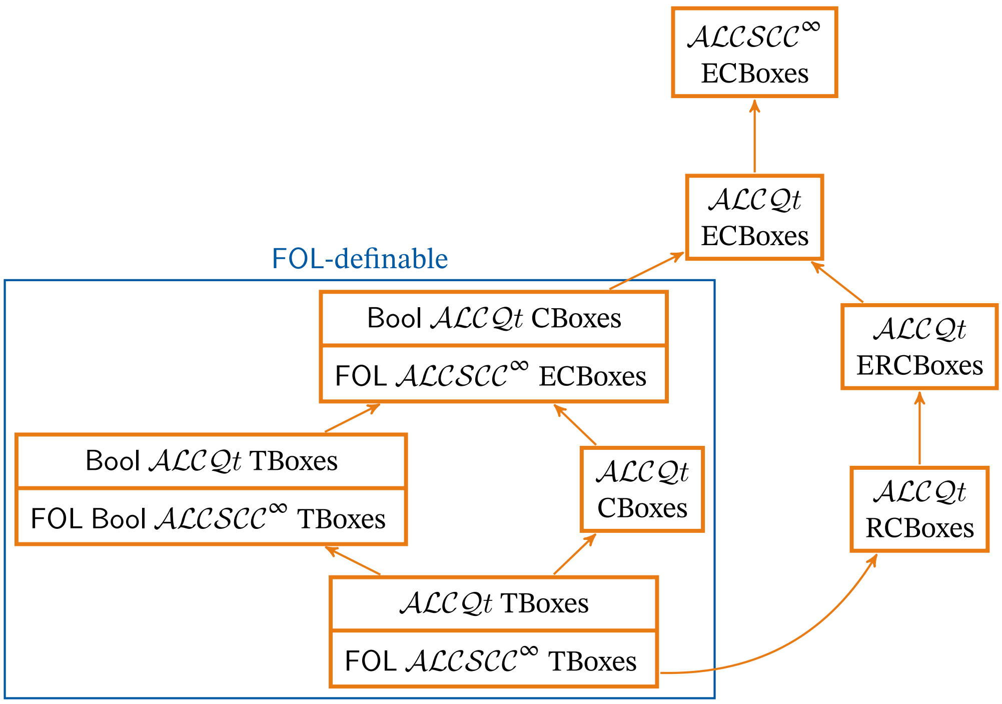
This yields the convergence of L_1 to 1/2 as desired. □

The interpretations used to compute $\ell(\phi)$ are always finite, and therefore Proposition 5.24 holds also w.r.t. the classes $\mathbb{C}_{\mathsf{fb}}$ and $\mathbb{C}_{\mathsf{fin}}$ of finitely branching and finite models.

Summary

We analyzed the expressive power of knowledge bases written using local and global cardinality constraints using the notions of *global* and *comparative Pr bisimulations*. We showed that the set of Boolean $\mathcal{ALCQ}t$ TBoxes is the fragment of first-order logic that is invariant under global Pr bisimulation, and that Boolean $\mathcal{ALCQ}t$ CBoxes play a similar role w.r.t. comparative Pr bisimulation. Using the 0-1 law of first-order logic, we showed that even simple RCBoxes cannot be defined in first-order logic, both w.r.t. arbitrary and finite interpretations. A classification of the expressive power results obtained in this chapter is depicted in the following diagram.

[1] This was already stated in [56], but without proof.



A visual representation of the expressivity hierarchy for TBoxes and their extensions. An arrow from a node N to a node N′ means that all the languages in N, which are equivalent, are strictly less expressive than those in N′.

6 The Precise Complexity of Reasoning with ω-admissible Concrete Domains

In this chapter we focus on extensions of $\mathcal{ALC}$ by ω-admissible concrete domains $\mathfrak{D}$ whose CSP is decidable in exponential time. We show that, under these assumptions, the consistency problem for $\mathcal{ALC}(\mathfrak{D})$ ontologies is ExpTime-complete. *Ontologies*, in this setting, consist of a TBox together with a set of *assertions*, called ABox, that state conditions on named individuals.

The work contained in this chapter is based on the paper:

[34] Borgwardt, S., De Bortoli, F., Koopmann, P.: The Precise Complexity of Reasoning in $\mathcal{ALC}$ with ω-Admissible Concrete Domains. In: Giordano, L., Jung, J.C., Ozaki, A. (eds.) Proceedings of the 37th International Workshop on Description Logics (DL'24). CEUR Workshop Proceedings. CEUR-WS, Bergen, Norway (2024)

ExpTime-ω-admissible Concrete Domains

In the following, we consider *ontologies* $\mathcal{O}$ that are the union of a TBox $\mathcal{T}$ and an *ABox* $\mathcal{A}$, which is a finite set of *concept assertions* $C(a)$ and *role assertions* $r(a, b)$ where C is a concept description, $r \in \mathsf{N_R}$ a role name and a, b are *named individuals* taken from a countable set $\mathsf{N_I}$ of *individual names* that is disjoint from $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$. An interpretation $\mathcal{I}$ of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ is defined over individual names by mapping $a \in \mathsf{N_I}$ to an individual $a^\mathcal{I} \in \Delta^\mathcal{I}$. We say that $\mathcal{I}$ is a *model* of $C(a)$ if $a^\mathcal{I} \in C^\mathcal{I}$ and of $r(a, b)$ if $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$. The ontology $\mathcal{O} := \mathcal{T} \cup \mathcal{A}$ is *satisfiable* if there is an interpretation $\mathcal{I}$ that is a model of $\mathcal{T}$ and that is a model of $\mathcal{A}$, i.e. it satisfies all assertions in $\mathcal{A}$.

We focus on ExpTime-ω-admissible concrete domains, which differ from ω-admissible domains as introduced in Chapter 2 in that we require their CSP to be decidable in exponential time instead of simply decidable.

Definition 6.1. *A concrete domain $\mathfrak{D}$ is* ExpTime-ω-admissible *if it is ω-admissible and* CSP($\mathfrak{D}$) *is decidable in ExpTime.*

Checking Consistency

Let now $\mathfrak{D}$ be an ExpTime-ω-admissible concrete domain, $\mathcal{O} = \mathcal{A} \cup \mathcal{T}$ be an $\mathcal{ALC}(\mathfrak{D})$ ontology, and $\mathcal{M}$ be the set of all subconcepts appearing in $\mathcal{O}$ and their negations. For the type elimination algorithm, we start by defining the central notion of *types*, which is standard.

Definition 6.2. *A set $t \subseteq \mathcal{M}$ is a* type *w.r.t. $\mathcal{O}$ if it satisfies the following properties:*

- *if $C \sqsubseteq D \in \mathcal{T}$ and $C \in t$, then $D \in t$;*
- *if $\top \in \mathcal{M}$, then $\top \in t$;*
- *if $\neg D \in \mathcal{M}$, then $D \in t$ iff $\neg D \notin t$;*
- *if $D \sqcap D' \in \mathcal{M}$, then $D \sqcap D' \in t$ iff $D \in t$ and $D' \in t$.*

Given a model $\mathcal{I}$ of $\mathcal{O}$ and an individual $d \in \Delta^{\mathcal{I}}$, the type of d *w.r.t. $\mathcal{O}$ is the set*

$$t_{\mathcal{I}}(d) := \{C \in \mathcal{M} \mid d \in C^{\mathcal{I}}\}.$$

Clearly, $t_{\mathcal{I}}(d)$ satisfies the four conditions required to be a type w.r.t. $\mathcal{O}$. We use this connection between individuals and types to define *augmented types* that represent the relationship between an individual, its role successors, and the CD-restrictions that ought to be satisfied. Hereafter, let n_{ex} be the number of existential restrictions $\exists r.C$ in $\mathcal{M}$, and n_{cd} the number of CD-restrictions $\exists p_1, \dots, p_k.P$ in $\mathcal{M}$. The maximal arity of predicates P occurring in $\mathcal{M}$ is denoted by n_{ar}, and we define $n_{\mathcal{O}} := n_{\mathsf{ex}} + n_{\mathsf{cd}} \cdot n_{\mathsf{ar}}$. Intuitively, each non-negated existential restriction in a type needs a successor (and associated type) to be realized, while CD-restrictions may require n_{ar} role successors to fulfill a certain constraint. Therefore, $n_{\mathcal{O}}$ is an upper bound on the number of successors needed to satisfy all the non-negated restrictions occurring in a type t w.r.t. $\mathcal{O}$.

Given a type t_0, we define a constraint system associated with a sequence of types $t_1, \dots, t_{n_{\mathcal{O}}}$ representing the role successors of a domain element with type t_0. This system contains a variable f^i for each feature f of an individual with type t_i in order to express the relevant CD-restrictions. Concrete features that are not represented in this system can remain undefined since their values are irrelevant for satisfying the CD-restrictions.

Definition 6.3. *A* local system *for a type t_0 w.r.t. a sequence of types $t_1, \dots, t_{n_{\mathcal{O}}}$ is a complete constraint system $\mathfrak{C}$ for which there exists a* successor function $\mathsf{succ}\colon \mathsf{N_R}(\mathcal{O}) \to \mathcal{P}(\{1, \dots, n_{\mathcal{O}}\})$*, such that, for all $\exists p_1, \dots, p_k.P \in \mathcal{M}$, the following condition holds:*

$\exists p_1, \dots, p_k.P \in t_0$ *iff there is $P(v_1, \dots, v_k) \in \mathfrak{C}$ for some variables $v_1, \dots, v_k$ such that*

$$v_i = \begin{cases} f^0 & \text{if } p_i = f \text{, or} \\ f^j & \text{if } p_i = rf \text{ and } j \in \mathsf{succ}(r). \end{cases}$$

We use a *sequence* instead of a *set* of types for the role successors, since there can be TBoxes that require the existence of successors with the same type that only differ in their feature values.

Example 6.4. *For the consistent ontology $\mathcal{O} := \{\top \sqsubseteq \exists rf, rf.{<}\}$ over $\mathfrak{Q} = (\mathbb{Q}, <, =, >)$, we have $\mathcal{M} = \{\top, \neg\top, \exists rf, rf.{<}, \neg\exists rf, rf.{<}\}$, and the only type is $t = \{\top, \exists rf, rf.{<}\}$. Any r-successors witnessing $\exists rf, rf.{<}$ for an element in a model of $\mathcal{O}$ have the same type t. However, we cannot express the restriction on their f-values by the (unsatisfiable) constraint ${<}(f^t, f^t)$, but need to consider two copies t_1, t_2 of t to get the (satisfiable) constraint ${<}(f^1, f^2)$.*

To merge the local systems associated to types of adjacent elements in a model, we introduce the following operation. For two local systems $\mathfrak{C}, \mathfrak{C}'$, the *merged system* $\mathfrak{C} \lhd_i \mathfrak{C}'$ is obtained as the union of $\mathfrak{C}$ and $\mathfrak{C}'$ after replacing all variables f^j in $\mathfrak{C}'$ by fresh variables $f^{j'}$, and subsequently replacing the variables $f^{0'}$ in $\mathfrak{C}'$ by f^i. This operation identifies all features with index i in $\mathfrak{C}$ with those of index 0 in $\mathfrak{C}'$, while keeping the remaining variables separate.

Definition 6.5. *An* augmented type *for $\mathcal{O}$ is a tuple* $\mathfrak{t} := (t_0, \dots, t_{n_\mathcal{O}}, \mathfrak{C}_\mathfrak{t})$ *where $t_0, \dots, t_{n_\mathcal{O}}$ are types for $\mathcal{O}$ and $\mathfrak{C}_\mathfrak{t}$ is a local system for t_0 w.r.t. $t_1, \dots, t_{n_\mathcal{O}}$ with a successor function* $\mathsf{succ}_\mathfrak{t}$. *The* root *of* $\mathfrak{t}$ *is* $\mathsf{root}(\mathfrak{t}) := t_0$. *The augmented type* $\mathfrak{t}$ *is* locally realizable *if $\mathfrak{C}_\mathfrak{t}$ has a solution and, for all concepts $\exists r.C \in \mathcal{M}$, it holds that*

$$\exists r.C \in \mathsf{root}(\mathfrak{t}) \text{ iff there is } i \in \mathsf{succ}_\mathfrak{t}(r) \text{ such that } C \in t_i.$$

An augmented type $\mathfrak{t}'$ patches $\mathfrak{t}$ *at $i \in \mathsf{succ}_\mathfrak{t}(r)$ if $\mathsf{root}(\mathfrak{t}') = t_i$ and the system $\mathfrak{C}_\mathfrak{t} \lhd_i \mathfrak{C}_{\mathfrak{t}'}$ has a solution. A set of augmented types $\mathbb{T}$* patches $\mathfrak{t}$ *if, for every role name r and every $i \in \mathsf{succ}_\mathfrak{t}(r)$, there is a $\mathfrak{t}' \in \mathbb{T}$ that patches $\mathfrak{t}$ at i.*

For the ontology $\mathcal{O}$ introduced in Example 6.4 we have $n_\mathcal{O} = 2$, since $\mathcal{M}$ only contains one CD-restriction over a binary predicate. Using infix notation, all augmented types $\mathfrak{t} = (t, t, t, \mathfrak{C}_\mathfrak{t})$ for $\mathcal{O}$ are such that $\mathfrak{C}_\mathfrak{t}$ contains the constraints $f^i = f^i$ for $i = 0, 1, 2$, and either $f^1 < f^2$ or $f^2 < f^1$. There are augmented types $\mathfrak{t}$ that are not locally realizable, for instance if $\mathfrak{C}_\mathfrak{t}$ contains $f^0 = f^1$, $f^0 = f^2$, and $f^1 < f^2$. On the other hand, there is a locally realizable augmented type using the constraints $f^0 < f^1$, $f^1 < f^2$, and $f^0 < f^2$, which can patch itself both at $i \in \{1, 2\}$.

To additionally handle named individuals and concept and role assertions, we introduce a structure $\mathfrak{t}_\mathcal{A}$ that describes all ABox individuals and their connections simultaneously, similar to the common notion of *precompletion*. The associated constraint system $\mathfrak{C}_\mathcal{A}$ now uses variables $f^{a,i}$ indexed with individual names a in addition to natural numbers i.

Definition 6.6. *An* ABox type *for $\mathcal{O}$ is a tuple* $\mathfrak{t}_\mathcal{A} := \big((\mathfrak{t}_a)_{a \in \mathsf{N_I}(\mathcal{A})}, \mathcal{A}_R, \mathfrak{C}_\mathcal{A}\big)$, *where $\mathfrak{t}_a$ are augmented types, $\mathcal{A}_R$ is a set of role assertions over $\mathsf{N_I}(\mathcal{A})$ and $\mathsf{N_R}(\mathcal{O})$, and $\mathfrak{C}_\mathcal{A}$ is a complete constraint system, such that, for every $a \in \mathsf{N_I}(\mathcal{A})$,*

- *for every concept assertion $C(a) \in \mathcal{A}$, we have $C \in \mathsf{root}(\mathfrak{t}_a)$;*
- *for every role assertion $r(a, b) \in \mathcal{A}$, we have $r(a, b) \in \mathcal{A}_R$;*
- *for every $\neg\exists r.C \in \mathsf{root}(\mathfrak{t}_a)$ and $r(a, b) \in \mathcal{A}_R$, we have $C \notin \mathsf{root}(\mathfrak{t}_b)$;*
- *for every $P(f_1^{j_1}, \dots, f_k^{j_k}) \in \mathfrak{C}_{\mathfrak{t}_a}$, we have $P(f_1^{a,j_1}, \dots, f_k^{a,j_k}) \in \mathfrak{C}_\mathcal{A}$;*
- *for every $\neg\exists p_1, \dots, p_k.P \in \mathsf{root}(\mathfrak{t}_a)$, there can be no $P(v_1, \dots, v_k) \in \mathfrak{C}_\mathcal{A}$ with*

$$v_i = \begin{cases} f^{a,0} & \text{if } p_i = f, \\ f^{b,0} & \text{if } p_i = rf \text{ and } r(a,b) \in \mathcal{A}_R, \text{ or} \\ f^{a,j} & \text{if } p_i = rf \text{ and } j \in \mathsf{succ}_{\mathfrak{t}_a}(r); \end{cases}$$

- *the system $\mathfrak{C}_\mathcal{A}$ has a solution.*

Positive occurrences of existential role or CD-restrictions in the ABox type do not need to be handled, as these are satisfied by anonymous successors described in the augmented types $\mathfrak{t}_a$.

Using the notions of ABox type and augmented types, we define Algorithm 1 which is based on type elimination and constitutes a decision procedure for the consistency of $\mathcal{O}$.

Algorithm 1 Elimination algorithm for consistency of $\mathcal{ALC}(\mathfrak{D})$ ontologies

Input: An $\mathcal{ALC}(\mathfrak{D})$ ontology $\mathcal{O} = \mathcal{A} \cup \mathcal{T}$
Output: CONSISTENT if $\mathcal{O}$ is consistent, and INCONSISTENT otherwise

```
M ← all subconcepts occurring in O and their negations
T ← all augmented types for O
while there is t ∈ T that is not locally realizable or not patched by T do
    T ← T \ {t}
if there is an ABox type t_A for O with t_a ∈ T for all a ∈ N_I(A) then
    return CONSISTENT
else
    return INCONSISTENT
```

Soundness

To prove that Algorithm 1 is sound, we show how to use the set $\mathbb{T}$ and the ABox type $\mathfrak{t}_{\mathcal{A}} = ((\mathfrak{t}_a)_{a \in \mathsf{N_I}(\mathcal{A})}, \mathcal{A}_R, \mathfrak{C}_{\mathcal{A}})$ obtained after a successful run of the elimination algorithm to define a forest-shaped interpretation $\mathcal{I}$ that is a model of $\mathcal{O}$. The domain of this model consists of pairs (a, w), where $a \in \mathsf{N_I}$ designates a tree-shaped part of $\mathcal{I}$ whose structure is given by the words w over the alphabet $\Sigma := \mathbb{T} \times \{0, \dots, n_{\mathcal{O}}\}$. A pair $(\mathfrak{t}, i) \in \Sigma$ describes an augmented type and the position relative to the restriction that this augmented type fulfills w.r.t. its parent in the tree. For a word $w \in \Sigma^+$, we define $\mathsf{end}(w) := \mathfrak{t}$ if $(\mathfrak{t}, j)$ occurs at the last position of w for some $j \in \{0, \dots, n_{\mathcal{O}}\}$.

We start defining the domain of $\mathcal{I}$ by $\Delta^0 := \{(a, w_a) \mid a \in \mathsf{N_I}(\mathcal{A}),\ w_a := (\mathfrak{t}_a, 0)\}$. Observe that $w_a \in \Sigma$, since $\mathfrak{t}_a \in \mathbb{T}$. Assuming that Δ^m is defined, we define Δ^{m+1} based on Δ^m, and subsequently construct the domain of $\mathcal{I}$ as the union of all sets Δ^m. Given $(a, w) \in \Delta^m$ with $\mathsf{end}(w) = \mathfrak{t}$, we observe that, for every $i \in \mathsf{succ}_{\mathfrak{t}}(r)$, there is an augmented type $\mathfrak{u}^i \in \mathbb{T}$ patching $\mathfrak{t}$ at i, as otherwise $\mathfrak{t}$ would have been eliminated from $\mathbb{T}$. We use these augmented types to define $\Delta_r^{m+1}[a, w] := \{(a, w \cdot (\mathfrak{u}^i, i)) \mid i \in \mathsf{succ}_{\mathfrak{t}}(r)\}$ to then obtain

$$\Delta^{m+1} := \Delta^m \cup \bigcup \left\{\Delta_r^{m+1}[a, w] \mid (a, w) \in \Delta^m \text{ and } r \in \mathsf{N_R}\right\}$$

and set $\Delta^{\mathcal{I}} := \bigcup_{m \in \mathbb{N}} \Delta^m$. The interpretation of individual, concept, and role names over $\mathcal{I}$ is given by

$$\begin{aligned}
a^{\mathcal{I}} &:= (a, w_a),\\
A^{\mathcal{I}} &:= \left\{(a, w) \in \Delta^{\mathcal{I}} \mid \mathsf{end}(w) = \mathfrak{t} \text{ and } A \in \mathsf{root}(\mathfrak{t})\right\},\\
r^{\mathcal{I}} &:= \left\{((a, w_a), (b, w_b)) \mid r(a, b) \in \mathcal{A}_R\right\} \cup\\
&\quad\ \left\{((a, w), (a, w')) \mid (a, w) \in \Delta^m \text{ and } (a, w') \in \Delta_r^{m+1}[a, w] \text{ with } m \in \mathbb{N}\right\}.
\end{aligned}$$

Defining the interpretation of feature names in $\mathcal{I}$ requires more work. Given $(a, w) \in \Delta^{\mathcal{I}}$ with $\mathsf{end}(w) = \mathfrak{t}$, let $\mathfrak{C}_{a,w}$ be the constraint system obtained by replacing every variable f^0 in $\mathfrak{C}_{\mathfrak{t}}$ with $f^{a,w}$ and every other variable f^i in $\mathfrak{C}_{\mathfrak{t}}$ with $f^{a,u}$, where $u \in \Sigma^+$ is the unique word of the form $w \cdot (\mathfrak{t}', i)$ for which $(a, u) \in \Delta^{\mathcal{I}}$. Correspondingly, let $\mathfrak{C}^0_{\mathcal{A}}$ be the result of replacing all variables $f^{a,0}$ in $\mathfrak{C}_{\mathcal{A}}$ by f^{a,w_a} and $f^{a,i}$ by $f^{a,u}$, where u is the unique word of the form $w_a \cdot (\mathfrak{t}', i)$ for which $(a, u) \in \Delta^{\mathcal{I}}$. For $m \in \mathbb{N}$, let $\mathfrak{C}^m$ be the union of $\mathfrak{C}^0_{\mathcal{A}}$ and all constraint systems $\mathfrak{C}_{a,w}$ for which $(a, w) \in \Delta^m$.

Lemma 6.7. *For every $m \in \mathbb{N}$, the constraint system $\mathfrak{C}^m$ has a solution.*

Proof. We prove the claim by induction over $m \in \mathbb{N}$. For the base case $m = 0$, observe that $\mathfrak{C}^0$ is equal to $\mathfrak{C}^0_{\mathcal{A}}$ since $\mathfrak{C}^0_{\mathcal{A}}$ already contains all local systems of the form $\mathfrak{C}_{a,w_a}$ (see Definition 6.6). Since $\mathfrak{C}^0_{\mathcal{A}}$ is equal to $\mathfrak{C}_{\mathcal{A}}$ up to renaming of variables, the fact that $\mathfrak{C}_{\mathcal{A}}$ has a solution implies that $\mathfrak{C}^0 = \mathfrak{C}^0_{\mathcal{A}}$ has a solution as well.

For the inductive step, we assume that $\mathfrak{C}^m$ has a solution and show how to extend it to a solution of $\mathfrak{C}^{m+1}$. We begin by observing that any constraint system $\mathfrak{C}$ that has a solution h can be extended to a complete constraint system by using h to add any missing constraints; i.e. if there is no constraint $P(v_1, \dots, v_k)$ for $v_1, \dots, v_k \in V(\mathfrak{C})$, but $\mathfrak{D}$ has k-ary predicates, then we can complete $\mathfrak{C}$ by adding the unique $P(v_1, \dots, v_k)$ for which $(h(v_1), \dots, h(v_k)) \in P^D$ (cf. JEPD). Moreover, this complete constraint system also has h as a solution. Since $\mathfrak{C}^m$ has a solution, let now $\mathfrak{B}$ be the satisfiable, complete system obtained by extending $\mathfrak{C}^m$ in this way.

Let $(a, w) \in \Delta^{m+1} \setminus \Delta^m$. By construction, there is a unique non-empty word $w' \in \Delta^m$ and a symbol $(\mathfrak{t}, i) \in \Sigma$ such that $w = w' \cdot (\mathfrak{t}, i)$. We notice that $\mathfrak{B}$ and $\mathfrak{C}_{a,w}$ are complete systems that agree on the constraints over their shared variables $V(\mathfrak{B}) \cap V(\mathfrak{C}_{a,w})$. This holds since all shared variables are of the form $f^{a,w}$, which can occur in $\mathfrak{B}$ only inside $\mathfrak{C}_{a,w'}$. Both $\mathsf{end}(w')$ and $\mathfrak{t}$ belong to $\mathbb{T}$, and $\mathfrak{t}$ patches $\mathsf{end}(w')$ at i, thus $\mathfrak{C}_{a,w'} \cup \mathfrak{C}_{a,w}$ has a solution (cf. Definition 6.5). In particular, the relations over the concrete domain $\mathfrak{D}$ satisfy JEPD, hence there cannot be a tuple of variables $v_1, \dots, v_k$ such that $P(v_1, \dots, v_k) \in \mathfrak{C}_{a,w}$ and $P'(v_1, \dots, v_k) \in \mathfrak{C}_{a,w'} \subseteq \mathfrak{B}$ with $P \neq P'$. Since $\mathfrak{B}$ and $\mathfrak{C}_{a,w}$ are complete, agree on the constraints over their shared variables, and both have a solution ($\mathfrak{B}$ by inductive hypothesis, and $\mathfrak{C}_{a,w}$ because $\mathfrak{t} \in \mathbb{T}$), property AP implies that $\mathfrak{B} \cup \mathfrak{C}_{a,w}$ has a solution, which we can use to extend $\mathfrak{B}$ to a complete constraint that includes $\mathfrak{C}_{a,w}$.

We can repeat this process for every $(a, w) \in \Delta^{m+1} \setminus \Delta^m$, because the different constraint systems $\mathfrak{C}_{a,w}$ do not share variables, and thus we obtain a constraint system $\mathfrak{B}'$ that is complete, has a solution, and includes $\mathfrak{C}^{m+1}$. Therefore, we conclude that $\mathfrak{C}^{m+1}$ has a solution. □

Thanks to Lemma 6.7 we can define a suitable interpretation of feature names for $\mathcal{I}$. Let $\mathfrak{C}^{\mathcal{I}}$ be the union of all systems $\mathfrak{C}^m$ for $m \in \mathbb{N}$. Every finite system $\mathfrak{B} \subseteq \mathfrak{C}^{\mathcal{I}}$ is also a subsystem of $\mathfrak{C}^m$ for some $m \in \mathbb{N}$. Since $\mathfrak{C}^m$ has a solution, it follows that $\mathfrak{B}$ has a solution. Every finite subsystem of $\mathfrak{C}^{\mathcal{I}}$ has a solution; since $\mathfrak{D}$ has the homomorphism ω-compactness property, we infer that $\mathfrak{C}^{\mathcal{I}}$ has a solution $h^{\mathcal{I}}$. Using this solution, we define for every feature name f the interpretation $f^{\mathcal{I}}(a, w) := h^{\mathcal{I}}(f^{a,w})$ if $f^{a,w}$ occurs in $\mathfrak{C}^{\mathcal{I}}$, and leave it undefined otherwise.

Lemma 6.8. *If $C \in \mathcal{M}$ and $(a, w) \in \Delta^{\mathcal{I}}$ with $\mathsf{end}(w) = \mathfrak{t}$, then $C \in \mathsf{root}(\mathfrak{t})$ iff $(a, w) \in C^{\mathcal{I}}$.*

Proof. We prove this claim by structural induction over $C \in \mathcal{M}$. We first prove the two base cases where C is either a concept name or an existential CD-restriction.

- The case $C = A$ is trivially covered by the definition of $A^{\mathcal{I}}$.
- Let $C = \exists p_1, \dots, p_k.P \in \mathcal{M}$. If $C \in \mathsf{root}(\mathfrak{t})$, by Definition 6.3 and because of $\mathfrak{t} \in \mathbb{T}$ there exists a constraint $P(f_1^{j_1}, \dots, f_k^{j_k}) \in \mathfrak{C}_{\mathfrak{t}}$ such that for $i = 1, \dots, k$
  - if $p_i = f_i$, then $j_i = 0$;
  - if $p_i = r_i f_i$, then $j_i \in \mathsf{succ}_{\mathfrak{t}}(r_i)$ and there exists $\mathfrak{u}^i \in \mathbb{T}$ that patches $\mathfrak{t}$ at j_i.

Using these indices and augmented types, we define for $i = 1, \dots, k$ the word

$$w^i := \begin{cases} w & \text{if } p_i = f_i, \\ w \cdot (\mathfrak{u}^i, j_i) & \text{if } p_i = r_i f_i. \end{cases}$$

It follows that $P(f_1^{a,w^1}, \dots, f_k^{a,w^k}) \in \mathfrak{C}^{\mathcal{I}}$, thus that $\big(f_1^{\mathcal{I}}(a, w^1), \dots, f_k^{\mathcal{I}}(a, w^k)\big) \in P^D$ by definition of $\mathcal{I}$. Due to the construction of w^i, we know that $f_i^{\mathcal{I}}(a, w^i) \in p_i^{\mathcal{I}}(a, w)$ holds for $i = 1, \dots, k$, which allows us to conclude that $(a, w) \in C^{\mathcal{I}}$.

Vice versa, assume that $C \notin \mathsf{root}(\mathfrak{t})$. By Definition 6.2, this means that $\neg C \in \mathsf{root}(\mathfrak{t})$. We show that $(c_1, \dots, c_k) \notin P^D$ for all $c_1 \in p_1^{\mathcal{I}}(w)$, ..., $c_k \in p_k^{\mathcal{I}}(w)$. First, we consider the case that $w = w_a$ for some $a \in \mathsf{N_I}(\mathcal{A})$, and find individual names a^i and words w^i such that the variable f^{a^i,w^i} reflects the origin of the value c_i, as follows. By construction of $\mathcal{I}$, one of the following three cases must hold for each $i \in \{1, \dots, k\}$.

- If $p_i = f_i$ and $c_i = f_i^{\mathcal{I}}(a, w_a) = h^{\mathcal{I}}(f_i^{a,w_a})$, then we set $a^i := a$ and $w^i := w_a$.
- If $p_i = r_i f_i$ and $c_i = f_i^{\mathcal{I}}(b, w_b) = h^{\mathcal{I}}(f_i^{b,w_b})$ for some individual name b with $r_i(a, b) \in \mathcal{A}_R$, then we set $a^i := b$ and $w^i := w_b$.
- If $p_i = r_i f_i$ and $c_i = f_i^{\mathcal{I}}(a, w') = h^{\mathcal{I}}(f_i^{a,w'})$ for some r_i-successor (a, w') of (a, w) with $w' = w \cdot (\mathfrak{t}', j)$ and $j \in \mathsf{succ}_{\mathfrak{t}}(r_i)$, then we set $a^i := a$ and $w^i := w'$.

By Definition 6.6 and the fact that $\neg C \in \mathsf{root}(\mathfrak{t})$, we know that $P\big(f_1^{a^1,w^1}, \dots, f_k^{a^k,w^k}\big)$ cannot be contained in $\mathfrak{C}^0_{\mathcal{A}}$. However, by construction of $\mathcal{I}$, the fact that the corresponding feature values are defined means that the variables $f_1^{a^1,w^1}, \dots, f_k^{a^k,w^k}$ must all occur in $\mathfrak{C}^0_{\mathcal{A}}$. Hence, $P'\big(f_1^{a^1,w^1}, \dots, f_k^{a^k,w^k}\big) \in \mathfrak{C}^0_{\mathcal{A}} \subseteq \mathfrak{C}^{\mathcal{I}}$ for some k-ary predicate P' disjoint with P by completeness of $\mathfrak{C}^0_{\mathcal{A}}$. This means that $(c_1, \dots, c_k) \notin P^D$, since $(c_1, \dots, c_k) = \big(h^{\mathcal{I}}(f_1^{a^1,w^1}), \dots, h^{\mathcal{I}}(f_k^{a^k,w^k})\big) \in (P')^D$. This allows us to conclude that $(a, w_a) \notin C^{\mathcal{I}}$.

For the case that $w \neq w_a$ for all $a \in \mathsf{N_I}(\mathcal{A})$, we can use similar, but simpler, arguments, based on $\mathfrak{C}_w$ (Definition 6.3) instead of $\mathfrak{C}^0_{\mathcal{A}}$ (Definition 6.6).

Let us now assume that the claim holds for $D, C_1, C_2 \in \mathcal{M}$ and prove the inductive cases.

- If $C = \neg D \in \mathcal{M}$, then $C \in \mathsf{root}(\mathfrak{t})$ iff $D \notin \mathsf{root}(\mathfrak{t})$ iff $(a, w) \notin D^{\mathcal{I}}$ iff $(a, w) \in C^{\mathcal{I}}$, where the equivalences hold due to Definition 6.2, the inductive hypothesis, and the semantics of negation, respectively.

- If $C = C_1 \sqcap C_2 \in \mathcal{M}$, then $C \in \mathsf{root}(\mathfrak{t})$ iff $C_1 \in \mathsf{root}(\mathfrak{t})$ and $C_2 \in \mathsf{root}(\mathfrak{t})$ iff $(a, w) \in C_1^{\mathcal{I}}$ and $(a, w) \in C_2^{\mathcal{I}}$ iff $(a, w) \in C^{\mathcal{I}}$, where the equivalences hold similarly to the case of concept negation.

- If $C = \exists r.D \in \mathcal{M}$, then $C \in \mathsf{root}(\mathfrak{t})$ and $\mathfrak{t} \in \mathbb{T}$ imply that there are $i \in \mathsf{succ}_{\mathfrak{t}}(r)$ and an augmented type $\mathfrak{t}' \in \mathbb{T}$ that patches $\mathfrak{t}$ at i such that $((a, w), (a, w')) \in r^{\mathcal{I}}$ with $w' = w \cdot (\mathfrak{t}', i)$ and $D \in \mathsf{root}(\mathfrak{t}')$. By definition of $\mathcal{I}$ and inductive hypothesis, we deduce that $(a, w') \in D^{\mathcal{I}}$, which in turn implies $(a, w) \in C^{\mathcal{I}}$.

  Vice versa, assume that $C \notin \mathsf{root}(\mathfrak{t})$, and thus $\neg C \in \mathsf{root}(\mathfrak{t})$. Any r-successor e of (a, w) must be of one of the following forms.

- If $w = w_a = (\mathsf{t}_a, 0)$, $e = (b, w_b)$, and $r(a,b) \in \mathcal{A}_R$, then $\neg C \in \mathsf{root}(\mathsf{t}_a)$ implies that $D \notin \mathsf{root}(\mathsf{t}_b)$ by Definition 6.6. Since $\mathsf{end}(w_b) = \mathsf{t}_b$, we thus obtain $e \notin D^{\mathcal{I}}$ by inductive hypothesis.
- If $e = (a, w')$ and $\mathsf{end}(w') = \mathsf{t}'$ patches some $i \in \mathsf{succ}_{\mathsf{t}}(r)$, then $\neg C \in \mathsf{root}(\mathsf{t})$ implies that $D \notin \mathsf{root}(\mathsf{t}')$ (see Definition 6.5). By inductive hypothesis, we conclude that $e \notin D^{\mathcal{I}}$.

This shows that no r-successor of (a, w) can satisfy D in $\mathcal{I}$, and thus $(a, w) \notin C^{\mathcal{I}}$.

□

By Lemma 6.8 and Definition 6.2, we know that $\mathcal{I}$ is a model of $\mathcal{T}$. By Definition 6.6 we obtain that for each $C(a) \in \mathcal{A}$, we have $C \in \mathsf{root}(\mathsf{t}_a)$, and thus $a^{\mathcal{I}} = (a, w_a) = (a, (\mathsf{t}_a, 0)) \in C^{\mathcal{I}}$. Similarly, whenever $r(a,b) \in \mathcal{A}$, then $r(a,b) \in \mathcal{A}_R$, and thus $(a^{\mathcal{I}}, b^{\mathcal{I}}) = ((a, w_a), (b, w_b)) \in r^{\mathcal{I}}$ by the construction of $\mathcal{I}$. Therefore, we conclude that $\mathcal{I}$ is also a model of $\mathcal{A}$, and thus of $\mathcal{O}$. We successfully proved the soundness of Algorithm 1.

Theorem 6.9 (Soundness). *If Algorithm 1 returns consistent, then $\mathcal{O}$ is consistent.*

Completeness

To show that Algorithm 1 is complete, let $\mathcal{I}$ be a model of $\mathcal{O}$ and define the set $T_{\mathcal{I}}$ of all types that are realized in $\mathcal{I}$, that is,

$$T_{\mathcal{I}} := \{t_{\mathcal{I}}(d) \mid d \in \Delta^{\mathcal{I}}\}.$$

Given that $\mathcal{I}$ is a model of $\mathcal{O}$, every element of $T_{\mathcal{I}}$ is a type according to Definition 6.2. Using the elements of $T_{\mathcal{I}}$, we construct a set $\mathbb{T}_{\mathcal{I}} := \{\mathsf{t}_{\mathcal{I}}(d) \mid d \in \Delta^{\mathcal{I}}\}$ of augmented types. For any domain element $d \in \Delta^{\mathcal{I}}$, we build the augmented type $\mathsf{t}_{\mathcal{I}}(d) = (t_0, \dots, t_{n_{\mathcal{O}}}, \mathfrak{C}_d)$ as follows.

First, we set $t_0 := t_{\mathcal{I}}(d) \in T_{\mathcal{I}}$. Assuming that $\exists r_i.C_i \in \mathcal{M}$ for $i = 1, \dots, n_{\mathsf{ex}}$, we select types $t_1, \dots, t_{n_{\mathsf{ex}}}$ to add to t that realize the (possibly negated) existential role restrictions occurring in t_0. If $\exists r_i.C_i \in t_0$, then we can select t_i as the type of an r-successor d' of d such that $d' \in C_i^{\mathcal{I}}$; otherwise, we pick t_i as the type of an arbitrary element in $\Delta^{\mathcal{I}}$.

Next, we assume that $\exists p_1^i, \dots, p_k^i.P_i \in \mathcal{M}$ for $i = 1, \dots, n_{\mathsf{cd}}$ and define the function $\mathsf{off}(i,j) := n_{\mathsf{ex}} + (i-1) \cdot n_{\mathsf{ar}} + j$ to be able to refer to the j-th path in the i-th CD-restriction above. We select types $t_{\mathsf{off}(i,1)}, \dots, t_{\mathsf{off}(i,n_{\mathsf{ar}})}$ that realize the (possibly negated) existential CD-restrictions occurring in t_0 for $i = 1, \dots, n_{\mathsf{cd}}$. If $\exists p_1^i, \dots, p_k^i.P_i \in t_0$, then there exist values $v_j^i \in (p_j^i)^{\mathcal{I}}(d)$ for $j = 1, \dots, k$ such that $(v_1^i, \dots, v_k^i) \in P^D$. If $p_j = rf$ holds for some feature name f and some role name r, let $t_{\mathsf{off}(i,j)}$ be the type of an r-successor d' of d such that $v_j^i = f^{\mathcal{I}}(d)$. For every $j = 1, \dots, n_{\mathsf{ar}}$ for which $t_{\mathsf{off}(i,j)}$ has not been selected this way, let $t_{\mathsf{off}(i,j)}$ be the type of an arbitrary individual in $\Delta^{\mathcal{I}}$. Similarly, if $\exists p_1^i, \dots, p_k^i.P_i \notin t_0$ then we set $t_{\mathsf{off}(i,j)}$ be the type of an arbitrary individual in $\Delta^{\mathcal{I}}$ for $j = 1, \dots, n_{\mathsf{ar}}$.

The two processes described in the two previous paragraphs yield a sequence of types $t_1, \dots, t_{n_{\mathcal{O}}}$ that occur in $T_{\mathcal{I}}$ and can thus be associated to individuals $d_1, \dots, d_{n_{\mathcal{O}}} \in \Delta^{\mathcal{I}}$. Using these individuals, we define the local system associated to our augmented type $\mathsf{t}_{\mathcal{I}}(d)$. First, we define the constraint system $\mathfrak{C}_d$ that contains the constraint $P\big(f_1^{i_1}, \dots, f_k^{i_k}\big)$ iff $\big(f_1^{\mathcal{I}}(d_{i_1}), \dots, f_k^{\mathcal{I}}(d_{i_k})\big) \in P^D$ for all $i_1, \dots, i_k \in \{0, \dots, n_{\mathcal{O}}\}$. The successor function succ_d associated to this constraint system assigns to $r \in \mathsf{N_R}$ all $i \in \{1, \dots, n_{\mathcal{O}}\}$ for which d_i is an r-successor of d. This concludes our definition of $\mathsf{t}_{\mathcal{I}}(d)$ for $d \in \Delta^{\mathcal{I}}$, and thus of $\mathbb{T}_{\mathcal{I}}$.

Claim 6.10. *Every augmented type in $\mathbb{T}_\mathcal{I}$ is locally realizable and patched in $\mathbb{T}_\mathcal{I}$.*

Proof. We notice that $\mathfrak{C}_d$ is complete and that $\mathfrak{C}_d$ and succ_d satisfy all the conditions stated in Definition 6.3 and thus constitute a local system. In addition, the mapping $v_d(f^i) := f^\mathcal{I}(d_i)$ is a solution of $\mathfrak{C}_d$, and thanks to our choice of types $t_0, \dots, t_{n_\mathcal{O}}$ we also obtain that the augmented type $\mathfrak{t}_\mathcal{I}(d)$ constructed using this process is locally realizable.

Moreover, for a given augmented type $\mathfrak{t} = \mathfrak{t}_\mathcal{I}(d)$ in $\mathbb{T}_\mathcal{I}$ and $i \in \{0, \dots, n_\mathcal{O}\}$, our construction yields that $\mathfrak{t}' = \mathfrak{t}_\mathcal{I}(d_i)$ patches $\mathfrak{t}$ at i; here, d_i is the domain element chosen for t_i in the construction of $\mathfrak{t}_\mathcal{I}(d)$. The first condition, namely that $\mathsf{root}(\mathfrak{t}') = t_i$, is fulfilled by construction. The second condition, i.e. that $\mathfrak{C}_\mathfrak{t} \lhd_i \mathfrak{C}_{\mathfrak{t}'}$ has a solution, follows from the fact that the individual solutions v_d, v_{d_i} constructed above agree on the values of the shared variables $v_d(f^i) = f^\mathcal{I}(d_i) = v_{d_i}(f^0)$. Together with the conditions proved during the construction of the augmented types, this ensures that $\mathfrak{t}$ is patched by $\mathbb{T}_\mathcal{I}$. □

Therefore, no augmented type $\mathfrak{t} \in \mathbb{T}_\mathcal{I}$ is eliminated during a run of Algorithm 1, and so $\mathbb{T}_\mathcal{I} \subseteq \mathbb{T}$. We further deduce that $\mathbb{T}$ cannot become empty, since $T_\mathcal{I}$ is non-empty. Using $\mathbb{T}_\mathcal{I}$ together with our model $\mathcal{I}$ of $\mathcal{O}$ we derive an ABox type $\mathfrak{t}_\mathcal{A}^\mathcal{I} = \big((\mathfrak{t}_a)_{a \in \mathsf{N_I}(\mathcal{A})}, \mathcal{A}_R, \mathfrak{C}_\mathcal{A}\big)$ for $\mathcal{O}$. We define each $\mathfrak{t}_a$, $a \in \mathsf{N_I}(\mathcal{A})$, as $\mathfrak{t}_a := \mathfrak{t}_\mathcal{I}(a^\mathcal{I}) \in \mathbb{T}_\mathcal{I}$. Assuming that $d_{a,i}$ is the domain element used to establish the i-th type in $\mathfrak{t}_a$ for $i \in \{0, \dots, n_\mathcal{O}\}$, we define the constraint system $\mathfrak{C}_\mathcal{A}$ s.t.

$$P(f_1^{a_1,i_1}, \dots, f_k^{a_k,i_k}) \in \mathfrak{C}_\mathcal{A} \text{ iff } (f_1^\mathcal{I}(d_{a_1,i_1}), \dots, f_k^\mathcal{I}(d_{a_k,i_k})) \in P^D.$$

Finally, we define the set $\mathcal{A}_R$ to consist of all role assertions $r(a,b)$ for which $a, b \in \mathsf{N_I}(\mathcal{A})$, $r \in \mathsf{N_R}(\mathcal{O})$, and $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$.

Lemma 6.11. *The object $\mathfrak{t}_\mathcal{A}^\mathcal{I}$ is an ABox type for $\mathcal{O}$ with $\mathfrak{t}_a \in \mathbb{T}_\mathcal{I}$ for all $a \in \mathsf{N_I}(\mathcal{A})$.*

Proof. We show that $\mathfrak{t}_\mathcal{A}^\mathcal{I}$ satisfies all conditions stated in Definition 6.6. If $C(a) \in \mathcal{A}$ then $a^\mathcal{I} \in C^\mathcal{I}$ and since $\mathsf{root}(\mathfrak{t}_a) = t_\mathcal{I}(a^\mathcal{I})$, we deduce that $C \in \mathsf{root}(\mathfrak{t}_a)$. Similarly, if $r(a,b) \in \mathcal{A}$ then $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$ holds, which by definition of $\mathcal{A}_R$ implies that $r(a,b) \in \mathcal{A}_R$. If $\neg\exists r.C \in \mathsf{root}(\mathfrak{t}_a)$ and $r(a,b) \in \mathcal{A}_R$, the fact that b is an r-successor of a in $\mathcal{I}$ clearly implies that $C \notin \mathsf{root}(\mathfrak{t}_b) = t_\mathcal{I}(b^\mathcal{I})$ must hold. We turn our attention to $\mathfrak{C}_\mathcal{A}$. If $P(f_1^{j_1}, \dots, f_k^{j_k}) \in \mathfrak{C}_{\mathfrak{t}_a}$, then $(f_1^\mathcal{I}(d_{a,j_1}), \dots, f_k^\mathcal{I}(d_{a,j_k})) \in P^D$, and by definition of $\mathfrak{C}_\mathcal{A}$ we obtain $P(f_1^{a_1,i_1}, \dots, f_k^{a_k,i_k}) \in \mathfrak{C}_\mathcal{A}$. Next, assume that $a \in \mathsf{N_I}(\mathcal{A})$ and that $P(v_1, \dots, v_k) \in \mathfrak{C}_\mathcal{A}$ holds, where each variable v_i is of one of the forms described in Definition 6.6 for a. Then, by definition of $\mathfrak{C}_\mathcal{A}$ we can find values $c_i \in p_i^\mathcal{I}(a^\mathcal{I})$ for $i = 1, \dots, k$ such that $(c_1, \dots, c_k) \in P^D$. Therefore, it follows that $\exists p_1, \dots, p_k.P \in \mathsf{root}(\mathfrak{t}_a)$. At last, we observe that $\mathfrak{C}_\mathcal{A}$ has a solution, given by the interpretation of feature names over $\mathcal{I}$. We conclude that $\mathfrak{t}_\mathcal{A}^\mathcal{I}$ is an ABox type for $\mathcal{O}$. □

Thus, there is a suitable ABox type for $\mathcal{O}$, and Algorithm 1 returns CONSISTENT. We successfully showed that Algorithm 1 is complete.

Theorem 6.12 (Completeness). *If $\mathcal{O}$ is consistent, then Algorithm 1 returns consistent.*

Termination and Complexity

By showing that Algorithm 1 runs in exponential time w.r.t. the size of the input ontology $\mathcal{O}$, we obtain the following complexity result, where the hardness is a trivial consequence of the fact that $\mathcal{ALC}(\mathfrak{D})$ is an extension of $\mathcal{ALC}$.

Theorem 6.13. *Let $\mathfrak{D}$ be an ExpTime-ω-admissible concrete domain. Then, the consistency problem for $\mathcal{ALC}(\mathfrak{D})$ ontologies is ExpTime-complete.*

Proof. It only remains to show that Algorithm 1 runs in exponential time. Each augmented type $\mathfrak{t}$ contains $n_{\mathcal{O}} + 1$ types, each of size polynomial in the input. Moreover, fixing the types in $\mathfrak{t}$, there are at most $n_{\mathfrak{D}} \cdot |V_\ell|^{n_{\mathsf{ar}}}$ distinct local systems, where $n_{\mathfrak{D}}$ is the number of predicates in $\mathfrak{D}$ and $V_\ell \leq n_{\mathcal{O}} \cdot |\mathsf{N_F}|$ the number of variables occurring in the system. It follows that the number of augmented types is at most exponential, and thus the loop in Line 3 takes at most exponentially many iterations. In each iteration, we need to test whether some $\mathfrak{t} \in \mathbb{T}$ is not locally realizable or not patched, which amounts to a number of tests polynomial in $|\mathbb{T}|$, and thus exponential in the size of $\mathcal{O}$. Each such test involves checking the satisfiability of complete constraint systems $\mathfrak{C}_\mathfrak{t}$ and merged systems $\mathfrak{C}_\mathfrak{t} \lhd_i \mathfrak{C}_{\mathfrak{t}'}$. Since each (merged) system is of polynomial size, and CSP($\mathfrak{D}$) is in ExpTime, these tests each take exponential time. It follows that the loop in Line 3 takes at most exponential time in total. It remains to show that also the search for ABox types in Line 5 takes at most exponential time. For this, it suffices to observe that there can be at most exponentially many ABox types for $\mathcal{O}$, which each have a polynomial-size constraint system attached. This is not hard to see from Definition 6.6, since there is exactly one augmented type for every individual in $\mathcal{A}$, the set of possible role assertions in $\mathcal{A}_R$ is polynomial in $\mathcal{O}$, and the set of variables in $\mathfrak{C}_\mathcal{A}$ is polynomial in the size of the involved augmented types. □

By looking at the model built in the proof of Theorem 6.9, we notice that each element has a finite number of role successors. Therefore, we can establish the following *finitely branching model property* of $\mathcal{ALC}(\mathfrak{D})$ ontologies.

Corollary 6.14. *An $\mathcal{ALC}(\mathfrak{D})$ ontology is consistent iff it has a finitely branching model.*

This property justifies the usage of a tableaux algorithm with appropriate blocking conditions to check consistency of an $\mathcal{ALC}(\mathfrak{D})$ ontology (as done in [79]). It further allows to adapt the results obtained in Chapter 9, which are stated w.r.t. finitely branching interpretations, since Corollary 6.14 implies that checking consistency of $\mathcal{ALC}(\mathfrak{D})$ ontologies w.r.t. arbitrary or finitely branching interpretations yields the same result, at the same complexity.

Summary

In this chapter, we turned our attention to DLs that integrate concrete domain reasoning. In particular, we introduced the notion of ExpTime-ω-admissible concrete domain, and proved that the consistency problem for $\mathcal{ALC}(\mathfrak{D})$ ontologies is ExpTime-complete if $\mathfrak{D}$ is ExpTime-ω-admissible, thus proving the conjecture posed in [79] regarding ω-admissible concrete domains, and further showing that ABoxes containing concept and role assertions can be added without an increase in complexity.

7 The Abstract Expressive Power of Logics with Concrete Domains

In this chapter, we introduce FOL with concrete domains and present two variants of the notion of abstract expressive power, one where one can use auxiliary predicates on the first-order side to express sentences of the logic with concrete domains, and one where this is not allowed.

We prove that FOL($\mathfrak{D}$) and DLs extended with concrete domain restrictions over the concrete domain $\mathfrak{D}$ share a number of interesting formal properties with FOL, provided that $\mathfrak{D}$ is homomorphism ω-compact and strongly positive. Then, we show, on the one hand, that FOL with a unary concrete domain can be expressed in FOL if we are allowed to use auxiliary predicates. In addition, if we restrict the logic with unary concrete domain to a decidable fragment like the guarded or the two-variable fragment with counting, then decidability on the concrete domain side yields decidability of the whole logic. On the other hand, we show that $\mathcal{ALC}$ extended with a JD concrete domain cannot be expressed in FOL. We also show that adding such a concrete domain to the decidable two-variable fragment of first-order logic causes undecidability.

The work contained in this chapter is based on the papers:

[17] Baader, F., De Bortoli, F.: The Abstract Expressive Power of First-Order and Description Logics with Concrete Domains. In: Proceedings of the 39th ACM/SIGAPP Symposium on Applied Computing. SAC ’24, pp. 754–761. ACM, New York, NY, USA (2024). `https://doi.org/10.1145/3605098.3635984`

[15] Baader, F., De Bortoli, F.: Logics with Concrete Domains: First-Order Properties, Abstract Expressive Power, and (Un)Decidability. SIGAPP Applied Computing Review **24**(3), 5–17 (2024). `https://doi.org/10.1145/3699839.3699840`

7.1 First-Order Logic with Concrete Domains and Abstract Expressive Power

We introduce first-order logic with concrete domains, from which we can obtain DLs with concrete domains as fragments. Then, we define the notion of abstract expressive power of a logic with concrete domains.

FOL with concrete domains. In Chapter 2 we defined a *concrete domain* $\mathfrak{D}$ to be a fixed relational structure. We assume that the concrete domain $\mathfrak{D}$ has signature τ, and consider a first-order signature σ (which may also contain function symbols), and a countable set $\mathcal{F}$ of *feature symbols*. The formulae of *first-order logic with the concrete domain* $\mathfrak{D}$, $\mathsf{FOL}_{\mathfrak{D}}[\sigma, \mathcal{F}]$ (or simply $\mathsf{FOL}(\mathfrak{D})$ if σ and $\mathcal{F}$ are irrelevant or clear from the context), are obtained by extending the inductive definition for FOL seen in Chapter 2 with the following base cases:

- *definedness predicates* $\mathrm{Def}(f)(t)$ with $f \in \mathcal{F}$ and t a σ-term,
- *concrete domain predicates* $P(f_1, \dots, f_n)(t_1, \dots, t_n)$ with $P \in \tau$ of arity n, $f_i \in \mathcal{F}$, and t_i σ-terms.

The semantics of $\mathsf{FOL}(\mathfrak{D})$ formulae is defined inductively, using a first-order interpretation $\mathcal{I}$ for σ extended with a set $\mathfrak{F}$ of partial functions $f^{\mathfrak{F}} \colon I \rightharpoonup D$ for $f \in \mathcal{F}$, and an assignment w mapping variables to individuals in $\mathcal{I}$. The semantics of terms, Boolean connectives and first-order quantifiers is defined as in Chapter 2, where we denote the interpretation of a term t by $\mathcal{I}$ and w as $t^{\mathcal{I},w}$. The new predicates are interpreted as follows, where $\mathbf{f} := f_1, \dots, f_n$ and $\mathbf{t} := t_1, \dots, t_n$:

- $(\mathcal{I}, \mathfrak{F}), w \models \mathrm{Def}(f)(t)$ if $f^{\mathfrak{F}}(t^{\mathcal{I},w})$ is defined, and
- $(\mathcal{I}, \mathfrak{F}), w \models P(\mathbf{f})(\mathbf{t})$ if $(f_1^{\mathfrak{F}}(t_1^{\mathcal{I},w}), \dots, f_n^{\mathfrak{F}}(t_n^{\mathcal{I},w})) \in P^D$.

Note that $(f_1^{\mathfrak{F}}(t_1^{\mathcal{I},w}), \dots, f_n^{\mathfrak{F}}(t_n^{\mathcal{I},w})) \in P^D$ entails that $f_i^{\mathfrak{F}}(t_i^{\mathcal{I},w})$ must be defined for $i = 1, \dots, n$. The tuple $(\mathcal{I}, \mathfrak{F})$ is a *model* of the $\mathsf{FOL}(\mathfrak{D})$ sentence ϕ (i.e., formula without free variables), in symbols $(\mathcal{I}, \mathfrak{F}) \models \phi$, if $(\mathcal{I}, \mathfrak{F}), w \models \phi$ for some (and thus all) assignments w. The notion of *entailment* between $\mathsf{FOL}(\mathfrak{D})$ sentences is similar to that of first-order logic, i.e. ϕ *entails* ψ, in symbols $\phi \models \psi$, if every model of ϕ is a model of ψ.

If $\mathcal{DL}$ has a first-order translation π_x mapping concepts C to FOL formulae with one free variable x (cf. Chapter 2) we can extend this translation function to map concepts of $\mathcal{DL}(\mathfrak{D})$ to formulae of $\mathsf{FOL}(\mathfrak{D})$, by providing the translation of CD-restrictions. We consider the variables $\mathbf{x} := x_1, \dots, x_k$ and the feature paths $\mathbf{p} := p_1, \dots, p_k$ and define $I \subseteq \{1, \dots, k\}$ such that $p_i = r_i f_i$ if $i \in I$ and $p_i = f_i$ otherwise. We then define the sequence of variables $\mathbf{y} := y_1, \dots, y_k$ by setting $y_i = x_i$ if $i \in I$ and $y_i = x$ otherwise, and $\mathbf{z}$ as the sequence of variables y_i with $i \in I$. The translation of CD-restrictions is then defined as follows, where $\gamma(x, \mathbf{y}) := \bigwedge_{i \in I} r_i(x, y_i) \wedge \bigwedge_{i=1}^{k} \mathrm{Def}(f_i)(y_i)$:

$$
\begin{aligned}
\pi_x(\exists \mathbf{p}.P(\mathbf{x})) &:= \exists \mathbf{z}. \left(\textstyle\bigwedge_{i \in I} r_i(x, y_i) \wedge P(f_1, \dots, f_k)(\mathbf{y}) \right), \\
\pi_x(\forall \mathbf{p}.P(\mathbf{x})) &:= \forall \mathbf{z}. \left(\gamma(x, \mathbf{y}) \rightarrow P(f_1, \dots, f_k)(\mathbf{y}) \right).
\end{aligned}
\tag{7.1}
$$

The semantics of TBoxes (i.e., finite sets of CIs $C \sqsubseteq D$) of the DL $\mathcal{DL}(\mathfrak{D})$ is then defined in the usual way by translation into $\mathsf{FOL}(\mathfrak{D})$ sentences: $C \sqsubseteq D$ is translated into $\forall x. \pi_x(C) \rightarrow \pi_x(D)$. It is easy to see that the semantics of CD-restrictions given by the translation in (7.1) coincides with the direct model-theoretic semantics in Chapter 2. In [79], extensions of the predicates of a concrete domain $\mathfrak{D}$ by disjunctions of its base predicates are allowed to be used in CD-restrictions, whereas in [24] even predicates first-order definable from the base predicates are considered. These extensions can clearly also be translated into $\mathsf{FOL}(\mathfrak{D})$. We denote them as $\mathcal{DL}_{\vee+}(\mathfrak{D})$ and $\mathcal{DL}_{\mathsf{fo}}(\mathfrak{D})$, respectively.

Abstract expressive power. If we want to compare the expressive power of (a fragment of) FOL with that of (a fragment of) FOL($\mathfrak{D}$), we have the problem that the semantic structures they are based on differ in that, for the latter, one additionally has a collection of partial functions into the concrete domain. To overcome this difference, we say that the first-order interpretation $\mathcal{I}$ is an *abstract model* of the FOL($\mathfrak{D}$) sentence ϕ, in symbols $\mathcal{I} \models_{\mathfrak{D}} \phi$, if there is an interpretation of the feature symbols $\mathfrak{F}$ such that $(\mathcal{I}, \mathfrak{F}) \models \phi$. The FOL sentence ψ is an *abstract definition* of the FOL($\mathfrak{D}$) sentence ϕ if the abstract models of ϕ are exactly the models of ψ. In this case we also say that ϕ and ψ are *abstractly equivalent*.

Example 7.1. *Consider the unary concrete domain $\mathfrak{N}_{\mathsf{par}} := (\mathbb{N}, \mathtt{even}, \mathtt{odd})$ where $\mathtt{even}, \mathtt{odd}$ are unary relations with the standard meaning. The $\mathcal{ALC}(\mathfrak{N}_{\mathsf{par}})$ TBox*

$$\mathcal{T} := \{A \sqsubseteq \exists f.\mathtt{even}(x), B \sqsubseteq \exists f.\mathtt{odd}(x)\}$$

is abstractly equivalent to the $\mathcal{ALC}$ TBox $\mathcal{T}' := \{A \sqsubseteq \neg B\}$. In fact, A and B must be interpreted as disjoint sets in any model of $\mathcal{T}$. Conversely, any model of $\mathcal{T}'$ can be extended to a model of $\mathcal{T}$ by defining f to yield 0 for the elements of A, 1 for the elements of B, and no value for all other elements.

In Section 7.3 we show that such a definability result always holds for unary concrete domains. However, to obtain this result one may need to introduce auxiliary predicates to express the CD-restrictions. The following definition allows for such additional predicates. Let ϕ be an FOL($\mathfrak{D}$) sentence and ψ an FOL sentence that may contain auxiliary predicates not occurring in ϕ. Then ψ is an *abstract projective definition* of ϕ if the abstract models of ϕ are exactly the *reducts* of the models of ψ, where in a reduct we just forget about the interpretation of the auxiliary predicates. In this case we say that ϕ and ψ are *abstractly projectively equivalent*. The abstract expressive power of (a fragment of) FOL($\mathfrak{D}$) is determined by which classes of abstract models can be defined by its sentences.

Definition 7.2. *The* abstract expressive power *of a fragment F of* FOL($\mathfrak{D}$) *is said to be* (projectively) contained *in a fragment G of* FOL *if every sentence of F has an abstract (projective) definition in G.*

Example 7.3. *In Chapter 1 we have given an example showing that, for a concrete domain $\mathfrak{D}$ over the integers with predicates $x > y$ and $x > 0$, the abstract expressive power of $\mathcal{ALC}(\mathfrak{D})$ is not contained in* FOL*. The argument we have used there (which is based on the fact that* FOL *is compact, but $\mathcal{ALC}(\mathfrak{D})$ is not) also works in the projective setting. In fact, the CI (1.1) enforces that, for any element of PO, there is a positive integer such that the length of all hpp-chains issuing from it are bounded by this number. Assume that ψ is an* FOL *sentence that is an abstract projective definition of this CI. Clearly we can write, for all $n \geq 1$, an* FOL *sentence ψ_n that says that the constant a is an element of PO and the starting point of an hpp-chain of length n. Then any finite subset of $\{\psi\} \cup \{\psi_n \mid n \geq 1\}$ is satisfiable, but the whole set cannot be satisfiable since the CI (1.1) enforces a finite bound on the length of chains issuing from a. Since* FOL *is compact, this shows that ψ cannot be a first-order sentence.*

However, compactness of $\mathcal{ALC}(\mathfrak{D})$ for a given concrete domain $\mathfrak{D}$ does not guarantee that its abstract expressive power is projectively contained in FOL.

Example 7.4. *Consider the concrete domain $\mathfrak{Q}_{>} := (\mathbb{Q}, >)$. The results shown later in this section imply that the logic* FOL($\mathfrak{Q}_{>}$) *is compact, and thus also its fragment $\mathcal{ALC}(\mathfrak{Q}_{>})$. Nevertheless, the* *abstract expressive power of $\mathcal{ALC}(\mathfrak{Q}_>)$ is not projectively contained in* FOL. *To see this, consider the TBox*

$$\mathcal{T} := \{\top \sqsubseteq \neg(\forall f, f.{>}(x_1, x_2)) \sqcap \forall f, rf.{>}(x_1, x_2)\}. \tag{7.2}$$

First note that, having the conjunct $\neg(\forall f, f.{>}(x_1, x_2))$ on the right-hand side of the CI in $\mathcal{T}$, is equivalent to requesting that $f^{\mathfrak{F}}$ is a total function in every model $(\mathcal{I}, \mathfrak{F})$ of $\mathcal{T}$. In fact, if $d \in I$ is not an element of $\forall f, f.{>}(x_1, x_2)$, then $f^{\mathfrak{F}}(d)$ must be defined since otherwise (according to (7.1)) the universal CD-restriction would be trivially satisfied. Conversely, if $f^{\mathfrak{F}}(d)$ is defined, then d not belonging to $\forall f, f.{>}(x_1, x_2)$ boils down to requiring $f^{\mathfrak{F}}(d) \leqslant f^{\mathfrak{F}}(d)$, which is trivially true.

Now, assume that there is a FOL *formula ψ that is an abstract projective definition of $\mathcal{T}$. Then $(\mathbb{Q}, >)$, where $>$ is the interpretation of r, is an abstract model of $\mathcal{T}$. In fact, one can use the identity function to interpret the feature f. Thus, $(\mathbb{Q}, >)$ can be extended to a model of ψ (by appropriate interpretations of the auxiliary predicates contained in ψ, if any). Since $(\mathbb{Q}, >)$ satisfies the formula $\tau := \forall x, y.(r(x, y) \vee x = y \vee r(y, x))$, we conclude that $\psi \wedge \tau$ is satisfiable. The upward Löwenheim-Skolem property of* FOL *yields an uncountable model of $\psi \wedge \tau$. Since ψ is an abstract projective definition of $\mathcal{T}$, the reduct $\mathfrak{R}$ of this uncountable model to the signature consisting of r must be extendable to a model of $\mathcal{T}$. This means that there is an interpretation $f^{\mathfrak{F}}$ of f such that $(\mathfrak{R}, \mathfrak{F})$ is a model of $\mathcal{T}$. As shown above, $f^{\mathfrak{F}}$ must be defined on every element of $\mathfrak{R}$. Let ν, μ be distinct elements of $\mathfrak{R}$. Since $\mathfrak{R}$ satisfies τ, we know that either $r(\nu, \mu)$ or $r(\mu, \nu)$ holds in $\mathfrak{R}$. Then the restriction $\forall f, rf.{>}(x_1, x_2)$ yields $f^{\mathfrak{F}}(\nu) \neq f^{\mathfrak{F}}(\mu)$, and thus $f^{\mathfrak{F}}$ is injective. However, since $\mathfrak{R}$ is uncountable and $\mathbb{Q}$ is countable, there cannot be an injective function from the domain of $\mathfrak{R}$ to $\mathbb{Q}$.*

7.2 First-order Properties of Logics with Concrete Domains

We mentioned in Chapter 2 that, as a consequence of its semidecidability, FOL satisfies the *compactness property* and is *recursively enumerable*. There are other interesting formal characteristics satisfied by FOL, usually shown in any textbook in logic and model theory [45, 48, 59]. Assuming that Φ is an at most countable set of sentences in our target language and ϕ, ψ are sentences, these properties can overall be specified as follows:

(Downward) Löwenheim-Skolem If Φ is satisfiable, then it has a model whose domain is at most countable;

(Upward) Löwenheim-Skolem If Φ has a model with an infinite domain, then it has a model with an uncountable domain;

(Countable) Compactness If every finite subset of Φ is satisfiable, then Φ is satisfiable;

Recursive enumerability The set of unsatisfiable sentences is recursively enumerable (r.e.).

Craig interpolation if ϕ entails ψ, then there is a sentence χ whose signature is a subset of the intersection of the signatures of ϕ and ψ, called *Craig interpolant*, such that ϕ entails χ and χ entails ψ.

We show that, if we assume $\mathfrak{D}$ to be homomorphism ω-compact and strongly positive, then FOL($\mathfrak{D}$) and $\mathcal{ALC}(\mathfrak{D})$ share most of these properties with FOL. The main tool for showing our results is a satisfiability-preserving translation of sets of FOL($\mathfrak{D}$) sentences into sets of FOL sentences.

Rewriting to first-order logic. If $\phi(\mathbf{x})$ is a $\mathsf{FOL}(\mathfrak{D})$ formula we derive a FOL formula $\phi^{\mathsf{FOL}}(\mathbf{x})$ by replacing every atom of the form $P(f_1, \dots, f_n)(t_1, \dots, t_n)$ with $P^{f_1,\dots,f_n}(t_1, \dots, t_n)$, where for every n-ary concrete domain predicate P and features $f_1, \dots, f_n$ we assume that $P^{f_1,\dots,f_n}$ is a new n-ary predicate symbol in the first-order signature. Similarly, every atom $\mathrm{Def}(f)(t)$ is replaced with $\mathrm{Def}_f(t)$ where Def_f is a new unary predicate symbol for every feature f.

In addition to making these replacements in the given $\mathsf{FOL}(\mathfrak{D})$ formulae, our rewriting into FOL also needs to specify the semantics of the new predicates $P^{f_1,\dots,f_n}$ and Def_f. Since predicates of the form $P^{f_1,\dots,f_n}$ may also occur negated, we must also specify how the negation of a concrete domain relation interacts with other relations. Given a concrete domain relation P, the concrete domain $\mathfrak{D}$ need not have a relation symbol for the negation of P. However, since $\mathfrak{D}$ is strongly positive, the negation of the n-ary relation P is defined by a quantifier-free, positive formula $\phi_{\neg P}(x_1, \dots, x_n)$ over the relations of $\mathfrak{D}$. Using this formula, we introduce a first-order formula $\phi_{\neg P}^{f_1,\dots,f_n}(x_1, \dots, x_n)$ to represent the negation of $P^{f_1,\dots,f_n}$, where each atom $R(x_{i_1}, \dots, x_{i_k})$ with $1 \leqslant i_1, \dots, i_k \leqslant n$ in $\phi_{\neg P}(x_1, \dots, x_n)$ is replaced by $R^{f_{i_1},\dots,f_{i_k}}(x_{i_1}, \dots, x_{i_k})$.

The interaction between the concrete domain relations is captured by considering constraints systems built from them. Every set Γ of atoms of the form $P^{f_1,\dots,f_n}(x_1, \dots, x_n)$ induces the constraint system

$$\hat{\Gamma} := \{P(f^1_{x_1}, \dots, f^n_{x_n}) \mid P^{f^1,\dots,f^n}(x_1, \dots, x_n) \in \Gamma\},$$

where f_x is a new variable for each feature symbol f and variable x.

Let Φ be an at most countable set of $\mathsf{FOL}(\mathfrak{D})$ sentences. We translate Φ into a set of FOL sentences Φ^{FOL} using the procedure described above. To capture the semantics of the concrete domain predicates and the definedness predicate, we additionally consider the set of FOL sentences $\Psi^{\mathfrak{D}} := \Psi_1 \cup \Psi_2$ where:

- the set Ψ_1 contains for each predicate symbol $P^{f_1,\dots,f_n}$ that occurs in a sentence of Φ^{FOL} the sentences

$$\forall \mathbf{x}. P^{f_1,\dots,f_n}(\mathbf{x}) \to \mathrm{Def}_{f_1}(x_1) \wedge \cdots \wedge \mathrm{Def}_{f_n}(x_n),$$
$$\forall \mathbf{x}. \neg P^{f_1,\dots,f_n}(\mathbf{x}) \to \phi_{\neg P}^{f_1,\dots,f_n}(\mathbf{x}) \vee \textstyle\bigvee_{i=1}^{n}(\neg\, \mathrm{Def}_{f_i}(x_i)),$$

  with $\mathbf{x} := x_1, \dots, x_n$, and

- the set Ψ_2 contains the sentence $\forall \mathbf{x}. \bigwedge \Gamma \ \to\ \bot$ if Γ is a finite set of atomic formulae $P^{f_1,\dots,f_n}(x_1, \dots, x_n)$ for which $P^{f_1,\dots,f_n}$ occurs in Ψ_1, and the constraint system $\hat{\Gamma}$ is unsatisfiable in $\mathfrak{D}$, where $\mathbf{x}$ collects all the variables occurring in Γ.

Theorem 7.5. *Let $\mathfrak{D}$ be a strongly positive, homomorphism ω-compact concrete domain. The set Φ of* $\mathsf{FOL}(\mathfrak{D})$ *formulae is satisfiable in* $\mathsf{FOL}(\mathfrak{D})$ *iff* $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$ *is satisfiable in* FOL.

Proof. "$\Leftarrow$" Assume that $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$ is satisfiable. Since this is a countable set of first-order formulae, we apply the downward Löwenheim-Skolem property of FOL to get an at most countable model $\mathcal{I}$ of $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$. We show that we can extend $\mathcal{I}$ with an interpretation $\mathfrak{F}$ of the features such that $(\mathcal{I}, \mathfrak{F})$ is a model of Φ. To this purpose, introduce a fresh variable x_d for every $d \in \Delta^{\mathcal{I}}$ and consider the set $\Gamma_{\mathcal{I}}$ consisting of all atoms $P^{f^1,\dots,f^n}(x_{d_1}, \dots, x_{d_n})$ such that $P^{f^1,\dots,f^n}(d_1, \dots, d_n)$ is satisfied in $\mathcal{I}$, where $d_1, \dots, d_n$ ranges over all elements of $\mathcal{I}$ and $f^1, \dots, f^n$ over all feature symbols. Due to our construction of $\Psi^{\mathfrak{D}}$ and the fact that $\mathcal{I}$ is a model of this set, we know that all finite subsets of $\hat{\Gamma}_{\mathcal{I}}$ are satisfiable in $\mathfrak{D}$. Since $\hat{\Gamma}_{\mathcal{I}}$ is countable, homomorphism ω-compactness implies that there exists a solution h of it in $\mathfrak{D}$. For all feature symbols f and elements $d \in \Delta^{\mathcal{I}}$ for which the variable f_{x_d} occurs in $\hat{\Gamma}_{\mathcal{I}}$, we define $f^{\mathfrak{F}}(d) := h(f_{x_d})$. Otherwise, we choose an arbitrary value for $f^{\mathfrak{F}}(d)$ if $\mathrm{Def}_f(d)$ is true in $\mathcal{I}$, and leave $f^{\mathfrak{F}}(d)$ undefined otherwise. The fact that, together with this interpretation of the features $\mathfrak{F}$, the FOL interpretation $\mathcal{I}$ is indeed a model of Φ, is an immediate consequence of the following two claims:

1. $\mathrm{Def}_f(d)$ is true in $\mathcal{I}$ iff $\mathrm{Def}(f)(d)$ is true in $(\mathcal{I}, \mathfrak{F})$;
2. $P^{f_1,\dots,f_n}(d_1,\dots,d_n)$ is true in $\mathcal{I}$ iff $P(f_1,\dots,f_n)(d_1,\dots,d_n)$ is true in $(\mathcal{I}, \mathfrak{F})$.

To show (1), assume that $\mathrm{Def}_f(d)$ is true in $\mathcal{I}$. Then $f^{\mathfrak{F}}(d)$ is defined either by the solution h of the constraint system $\hat{\Gamma}_{\mathcal{I}}$ in $\mathfrak{D}$ or it has received some arbitrary value. If $\mathrm{Def}_f(d)$ is not true in $\mathcal{I}$, then $f^{\mathfrak{F}}(d)$ cannot have been defined in terms of h, since otherwise an expression $P^{f_1,\dots,f_n}(d_1,\dots,d_n)$ that is true in $\mathcal{I}$ would have to exist such that $f = f_i$ and $d = d_i$. But then $\Psi^{\mathfrak{D}}$ would have enforced $\mathrm{Def}_f(d)$ to be true in $\mathcal{I}$, leading to a contradiction. In addition, since $\mathrm{Def}_f(d)$ is not true in $\mathcal{I}$, no arbitrary value is assigned to $f^{\mathfrak{F}}(d)$. Thus $f^{\mathfrak{F}}(d)$ is undefined.

Regarding (2), first assume $P^{f_1,\dots,f_n}(d_1,\dots,d_n)$ is true in $\mathcal{I}$. This implies that the formula $P^{f_1,\dots,f_n}(x_{d_1},\dots,x_{d_n})$ belongs to $\Gamma_{\mathcal{I}}$. Since $\mathfrak{F}$ was defined using a solution of $\hat{\Gamma}_{\mathcal{I}}$, we know that $P(f_1^{\mathfrak{F}}(d_1),\dots,f_n^{\mathfrak{F}}(d_n))$ holds in $\mathfrak{D}$, and thus $P(f_1,\dots,f_n)(d_1,\dots,d_n)$ is true in $(\mathcal{I}, \mathfrak{F})$.

Conversely, assume that $P^{f_1,\dots,f_n}(d_1,\dots,d_n)$ is not true in $\mathcal{I}$, which means that its negation is true in $\mathcal{I}$. Since $\mathcal{I}$ is a model of $\Psi^{\mathfrak{D}}$, this implies that $\phi_{\neg P}^{f_1,\dots,f_n}(d_1,\dots,d_n)$ is true in $\mathcal{I}$, or $\neg\,\mathrm{Def}_{f_i}(d_i)$ is true in $\mathcal{I}$ for some $1 \leqslant i \leqslant n$.

Clearly, if $\neg\,\mathrm{Def}_{f_i}(d_i)$ is true in $\mathcal{I}$ for some $1 \leqslant i \leqslant n$ it follows from the first claim that $\mathrm{Def}(f_i)(d_i)$ is false in $(\mathcal{I}, \mathfrak{F})$, hence that $f_i^{\mathfrak{F}}(d_i)$ is undefined. As a consequence, the formula $P(f_1,\dots,f_n)(d_1,\dots,d_n)$ cannot be true in $(\mathcal{I}, \mathfrak{F})$.

Next, assume that $\phi_{\neg P}^{f_1,\dots,f_n}(d_1,\dots,d_n)$ is true in $\mathcal{I}$ and w.l.o.g. that $\phi_{\neg P}(x_1,\dots,x_n)$ is in disjunctive normal form. Then, $\phi_{\neg P}^{f_1,\dots,f_n}(x_1,\dots,x_n)$ contains a disjunct that is a conjunction of atomic expressions of the form $R^{f_{i_1},\dots,f_{i_k}}(x_{i_1},\dots,x_{i_k})$ such that, for each of the atomic expressions $R^{f_{i_1},\dots,f_{i_k}}(d_{i_1},\dots,d_{i_k})$ in the disjunct, this expression is true in $\mathcal{I}$. Following the steps used in the only-if direction, we obtain that $(f_{i_1}^{\mathfrak{F}}(d_{i_1}),\dots,f_{i_k}^{\mathfrak{F}}(d_{i_k})) \in R^D$ holds for each of these atomic expressions. This implies that $\phi_{\neg P}(f_1^{\mathfrak{F}}(d_1),\dots,f_n^{\mathfrak{F}}(d_n))$ holds in $\mathfrak{D}$. By definition of $\phi_{\neg P}$ we conclude that $(f_1^{\mathfrak{F}}(d_1),\dots,f_n^{\mathfrak{F}}(d_n)) \notin P^D$, and hence that $P(f_1,\dots,f_n)(d_1,\dots,d_n)$ is not true in $(\mathcal{I}, \mathfrak{F})$. This concludes the proof of (2).

"⇒" Assume that Φ is satisfiable in FOL($\mathfrak{D}$) by the interpretation $\mathcal{I}$ of the FOL part and the interpretation $\mathfrak{F}$ of the features. We extend $\mathcal{I}$ to an interpretation $\mathcal{I}^{\mathsf{FOL}}$ that also takes the new predicates Def_f and $P^{f_1,\dots,f_n}$ into account:

- $d \in \mathrm{Def}_f^{\mathcal{I}^{\mathsf{FOL}}}$ iff $f^{\mathfrak{F}}(d)$ is defined,
- $(d_1,\dots,d_n) \in (P^{f_1,\dots,f_n})^{\mathcal{I}^{\mathsf{FOL}}}$ iff $(f_1^{\mathfrak{F}}(d_1),\dots,f_n^{\mathfrak{F}}(d_n)) \in P^{\mathfrak{D}}$.

Since $(\mathcal{I}, \mathfrak{F})$ makes Φ true, it is easy to see that $\mathcal{I}^{\mathsf{FOL}}$ is a model of Φ^{FOL}. In addition, it is a model of $\Psi^{\mathfrak{D}}$ due to the semantics of CD-restrictions in FOL($\mathfrak{D}$) and the fact that the complement of P in $\mathfrak{D}$ has the positive, quantifier-free definition $\phi_{\neg P}$ over $\mathfrak{D}$. □

Note that this theorem only shows that our translation from FOL($\mathfrak{D}$) to FOL preserves satisfiability. It does *not* imply that the abstract models of Φ coincide with the models of $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$. In particular, the model we construct for Φ in our proof is always at most countable, whereas $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$ may also have uncountable models. Nevertheless, thanks to this theorem, we can transfer most of the properties of FOL introduced above to FOL($\mathfrak{D}$).

Corollary 7.6. *If $\mathfrak{D}$ is strongly positive and homomorphism ω-compact, then* $\mathsf{FOL}(\mathfrak{D})$ *is countably compact and satisfies the downward Löwenheim-Skolem property. Homomorphism ω-compactness is also a necessary condition for countable compactness. In general,* $\mathsf{FOL}(\mathfrak{D})$ *need not satisfy the upward Löwenheim-Skolem property. If the finite unsatisfiable constraint systems for $\mathfrak{D}$ are r.e., then so are the unsatisfiable sentences of* $\mathsf{FOL}(\mathfrak{D})$.

Proof sketch. Compactness follows from Theorem 7.5. In fact, if Φ is unsatisfiable, then this theorem and compactness of FOL yield a finite subset Ψ of $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$ that is unsatisfiable. Then translating $\Psi \cap \Phi^{\mathsf{FOL}}$ back to $\mathsf{FOL}(\mathfrak{D})$ yields an unsatisfiable finite subset of Φ. The downward Löwenheim-Skolem property follows from the construction of the abstract model $\mathcal{I}$ in the if-direction of Theorem 7.5, which is at most countable.

Assume that the countable constraint system Γ for $\mathfrak{D}$ is a counterexample to the homomorphism ω-compactness of $\mathfrak{D}$. Then, the countable set of $\mathsf{FOL}(\mathfrak{D})$ sentences

$$\Phi_\Gamma := \{\forall x.(P(f_{x_1}, \dots, f_{x_n})(x, \dots, x)) \mid P(x_1, \dots, x_n) \in \Gamma\}$$

is a counterexample to countable compactness of $\mathsf{FOL}(\mathfrak{D})$.

Next, consider the concrete domain $\mathfrak{Q}_= := (\mathbb{Q}, =, \neq)$, which is strongly positive and easily seen to be homomorphism ω-compact. The $\mathsf{FOL}(\mathfrak{Q}_=)$ sentence

$$\phi_{\mathsf{up}} := \forall x, y.\, \mathrm{Def}(f)(x) \wedge (x \neq y \rightarrow \neq(f, f)(x, y))$$

states that f is an injective function from the domain of an abstract model of ϕ_{up} into $\mathbb{Q}$. Thus, no abstract model of ϕ_{up} can have an uncountable domain, as $\mathbb{Q}$ is is countable.

Finally, assume that $\Phi = \{\phi\}$ for an $\mathsf{FOL}(\mathfrak{D})$ sentence ϕ. The assumption that the finite unsatisfiable constraint systems for $\mathfrak{D}$ are r.e. entails that the set $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$ is r.e. as well. We can now dovetail a partial decision procedure for unsatisfiability of finite sets of FOL sentences with the enumeration of $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$ to get a procedure that terminates iff $\Phi^{\mathsf{FOL}} \cup \Psi^{\mathfrak{D}}$ is unsatisfiable. Together with Theorem 7.5 this shows that unsatisfiability of $\mathsf{FOL}(\mathfrak{D})$ sentences is partially decidable, and thus r.e. □

We observe that the positive results of Corollary 7.6 concerning countable compactness, the downward Löwenheim-Skolem property, and recursive enumerability transfer from $\mathfrak{D}$ to all of its *reducts*, i.e., all concrete domains $\mathfrak{D}'$ with the same domain as $\mathfrak{D}$ and a subset of the relations of $\mathfrak{D}$. The reason is that $\mathsf{FOL}(\mathfrak{D}')$ is then a sublogic of $\mathsf{FOL}(\mathfrak{D})$. Since $(\mathbb{Q}, <, =, >)$ is strongly positive and homomorphism ω-compact, then $\mathsf{FOL}(\mathfrak{Q}_>)$ where $\mathfrak{Q}_> := (\mathbb{Q}, >)$ is countably compact and has the downward Löwenheim-Skolem property – though, as we show in Theorem 7.16, it does not have the upward Löwenheim-Skolem property.

For $\mathcal{ALC}$ with a concrete domain, we can strengthen the result of Corollary 7.6.

Corollary 7.7. *Let $\mathfrak{D}$ be a strongly positive, homomorphism ω-compact concrete domain and $\mathcal{L}$ be either $\mathcal{ALC}(\mathfrak{D})$, $\mathcal{ALC}_{\mathsf{V+}}(\mathfrak{D})$ or $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{D})$. Then $\mathcal{L}$ is countably compact and satisfies the upward and the downward Löwenheim-Skolem property. Homomorphism ω-compactness is also a necessary condition for countable compactness.*

Proof sketch. The downward Löwenheim-Skolem property and countable compactness are an immediate consequence of the fact that $\mathcal{L}$ can be expressed in $\mathsf{FOL}(\mathfrak{D})$. Regarding necessity of homomorphism ω-compactness, it is easy to see that a counterexample to this property for $\mathfrak{D}$ can also be turned into a counterexample to countable compactness of $\mathcal{L}$, similar to the construction for $\mathsf{FOL}(\mathfrak{D})$. The upward Löwenheim-Skolem property is an immediate consequence of the fact that, like $\mathcal{ALC}$, its extension $\mathcal{L}$ is closed under disjoint unions[1]. □

Craig interpolation and concrete domains. In first-order logic, the *Craig Interpolation Property (CIP)* [40] ensures that an entailment $\phi \models \psi$ between sentences ϕ, ψ holds iff there is a sentence χ whose signature is shared by both ϕ and ψ such that $\phi \models \chi$ and $\chi \models \psi$. For first-order languages extended with concrete domains, we define the set of available formulae based on a first-order signature σ together with a set $\mathcal{F}$ of feature symbols. Correspondingly, we introduce an abstract and a concrete notion of Craig interpolant for sentences in these languages.

Definition 7.8. *Let ϕ, ψ be* $\mathsf{FOL}(\mathfrak{D})$ *sentences. The* $\mathsf{FOL}(\mathfrak{D})$ *sentence χ is an* abstract Craig interpolant *of ϕ and ψ if ϕ entails χ, χ entails ψ and every symbol in the first-order signature of χ occurs in both ϕ and ψ. We say that χ is a* concrete Craig interpolant *of ϕ and ψ if every feature symbol used in χ occurs in both ϕ and ψ. We say that* $\mathsf{FOL}(\mathfrak{D})$ *has the* abstract (concrete) Craig interpolation property *if, whenever $\phi \models \psi$ holds for* $\mathsf{FOL}(\mathfrak{D})$ *sentence ϕ and ψ, there is an abstract (concrete) Craig interpolant χ of ϕ and ψ.*

Using Theorem 7.5 again, we can show that $\mathsf{FOL}(\mathfrak{D})$ satisfies the abstract Craig interpolation property if $\mathfrak{D}$ satisfies the conditions required by this theorem.

Theorem 7.9. *Let $\mathfrak{D}$ be a strongly positive, homomorphism ω-compact concrete domain. Then,* $\mathsf{FOL}(\mathfrak{D})$ *satisfies the abstract Craig interpolation property.*

Proof. Let ϕ and ψ be $\mathsf{FOL}(\mathfrak{D})$ sentences such that ϕ entails ψ. Then, the $\mathsf{FOL}(\mathfrak{D})$ sentence $\phi \wedge \neg\psi$ must be unsatisfiable. By Theorem 7.5, we deduce that the set of FOL sentences $\{\phi^{\mathsf{FOL}} \wedge \neg\psi^{\mathsf{FOL}}\} \cup \Psi^{\mathfrak{D}}$ is also unsatisfiable. By compactness of first-order logic, there exists a finite subset $\Phi^{\mathfrak{D}} \subseteq \Psi^{\mathfrak{D}}$ such that the FOL sentence $\phi^{\mathsf{FOL}} \wedge \neg\psi^{\mathsf{FOL}} \wedge \phi_{\mathfrak{D}}$ with $\phi_{\mathfrak{D}} := \bigwedge \Phi^{\mathfrak{D}}$ is unsatisfiable, that is, $\phi^{\mathsf{FOL}} \wedge \phi_{\mathfrak{D}} \models \psi^{\mathsf{FOL}}$. Using the Craig interpolation property of FOL, we obtain an interpolant χ^{FOL} whose signature is in the intersection of the signatures of $\phi^{\mathsf{FOL}} \wedge \phi_{\mathfrak{D}}$ and ψ^{FOL}, such that $\phi^{\mathsf{FOL}} \wedge \phi_{\mathfrak{D}} \models \chi^{\mathsf{FOL}}$ and $\chi^{\mathsf{FOL}} \models \psi^{\mathsf{FOL}}$. In particular, every predicate symbol in χ^{FOL} is either a predicate symbol that occurs in the intersection of the first-order signatures of ϕ and ψ, or a symbol of the form $P^{f_1,\dots,f_n}$ or Def_f introduced by the translation into FOL.

Let χ be the $\mathsf{FOL}(\mathfrak{D})$ sentence obtained by replacing every atom in χ^{FOL} built using the latter symbols with the corresponding definedness or concrete domain predicate. Then every predicate symbol in the first-order signature of χ occurs in both ϕ and ψ. Thus, χ is an abstract interpolant of ϕ and ψ if we can show that it is entailed by ϕ and entails ψ. By monotonicity of FOL we can deduce from $\phi^{\mathsf{FOL}} \wedge \phi_{\mathfrak{D}} \models \chi^{\mathsf{FOL}}$ and $\chi^{\mathsf{FOL}} \models \psi^{\mathsf{FOL}}$ that the sets of sentences $\{\phi^{\mathsf{FOL}} \wedge \neg\chi^{\mathsf{FOL}}\} \cup \Psi^{\mathfrak{D}}$ and $\{\chi^{\mathsf{FOL}} \wedge \neg\psi^{\mathsf{FOL}}\} \cup \Psi^{\mathfrak{D}}$ are both unsatisfiable. By Theorem 7.5, this implies that $\phi \wedge \neg\chi$ and $\chi \wedge \neg\psi$ are unsatisfiable, and thus that $\phi \models \chi$ and $\chi \models \psi$. □

Note that this proof cannot be used to show the concrete Craig interpolation property for $\mathsf{FOL}(\mathfrak{D})$. In fact, the presence of $\phi_{\mathfrak{D}}$ in the entailment $\phi^{\mathsf{FOL}} \wedge \phi_{\mathfrak{D}} \models \psi^{\mathsf{FOL}}$ means that the interpolant χ^{FOL} may contain symbols of the form $P^{f_1,\dots,f_n}$ or Def_f that contain feature symbols not occurring in ϕ. Consequently, χ may contain concrete domain or definedness predicates involving feature symbols not contained in ϕ.

[1]This can be shown e.g. using the notion of concrete bisimulation introduced in Chapter 8

It is unclear whether the assumptions of Theorem 7.9 are sufficient to show that $\mathsf{FOL}(\mathfrak{D})$ satisfies the concrete Craig interpolation property. Our hypothesis is that we may obtain results on this concrete version, by additionally assuming that the concrete domain $\mathfrak{D}$ satisfies an appropriate interpolation property [35].

7.3 First-order (non-)definability and decidability

In Example 7.1 we have seen an instance of a unary concrete domain $\mathfrak{N}_{\mathsf{par}}$ and an $\mathcal{ALC}(\mathfrak{N}_{\mathsf{par}})$ TBox $\mathcal{T}$ such that $\mathcal{T}$ is abstractly equivalent to an $\mathcal{ALC}$ TBox $\mathcal{T}'$. The first part of this subsection generalizes this result to all unary concrete domains $\mathfrak{D}$ that are strongly positive. To be more precise, we show that, in this setting, every $\mathsf{FOL}(\mathfrak{D})$ sentence has an abstract projective definition in FOL, and likewise every $\mathcal{ALC}(\mathfrak{D})$ TBox is abstractly projectively equivalent to an $\mathcal{ALC}$ TBox. As a byproduct of these results, we are also able to show an interpolation result for $\mathcal{ALC}(\mathfrak{D})$. Under the additional assumption that CSP($\mathfrak{D}$) is decidable, we can prove that the extension of the guarded or two-variable fragments with counting of FOL with such a concrete domain remain decidable.

Example 7.4 shows an $\mathcal{ALC}(\mathfrak{Q}_>)$ TBox $\mathcal{T}$ that has no abstract projective definition in FOL, where $\mathfrak{Q}_>$ is homomorphism ω-compact. In the second part of this subsection, we generalize this result from $\mathfrak{Q}_>$ to countable concrete domains $\mathfrak{D}$ in which (in)equality is appropriately definable. We show that adding such concrete domains $\mathfrak{D}$ to FOL destroys the upward Löwenheim-Skolem property even if we restrict to the two-variable fragment of $\mathsf{FOL}(\mathfrak{D})$. We also prove that reasoning in such a logic is undecidable, which contrasts with the case of first-order logic with two variables, where reasoning is decidable in NExpTime [53].

Unary concrete domains

We recall that a concrete domain is *unary* if it contains only unary relations. Assume that $\mathfrak{D}$ is a strongly positive unary concrete domain. Let ϕ be an $\mathsf{FOL}(\mathfrak{D})$ sentence and $\Phi := \{\phi\}$. The rewriting approach described in Section 7.2 produces a singleton set Φ^{FOL} consisting of an FOL sentence ϕ^{FOL} and a set $\Psi^{\mathfrak{D}} := \Psi_1 \cup \Psi_2$ of FOL sentences such that

- Ψ_1 contains finitely many sentences $\forall x.P^f(x) \rightarrow \mathrm{Def}_f(x)$ and finitely many sentences

$$\forall x.\neg P^f(x) \rightarrow \phi^f_{\neg P}(x) \vee \neg\,\mathrm{Def}_f(x),$$

- Ψ_2 contains finitely many sentences of the form $\forall x.\Gamma \rightarrow \bot$ where Γ is a set of atoms $\{P_1^f(x), \dots, P_n^f(x)\}$ appearing in Ψ_1 s.t. $\hat{\Gamma}$ is unsatisfiable in $\mathfrak{D}$.

The first point is justified by the fact that we can restrict our attention to the concrete domain predicates and feature symbols that occur in ϕ. Regarding the last point, we notice that, in a setting where all concrete domain predicates are unary, constraints of the form $P^f(x)$ and $Q^g(y)$, where $f \neq g$ or $x \neq y$, cannot influence each other. Thus, one can restrict the attention to constraint systems built using a single feature symbol f and variable x. In fact, any unsatisfiable constraint systems must contain an unsatisfiable one of this form. Since we can again restrict the attention to the concrete domain predicates and feature symbols occurring in ϕ, and the name of single variable is irrelevant, there are only finitely many sentences of this form. Overall, this rewriting approach yields an FOL sentence $\psi := \phi^{\mathsf{FOL}} \wedge \bigwedge \Psi^{\mathfrak{D}}$. We cannot directly apply

Theorem 7.5 to conclude that ϕ and ψ are equisatisfiable since we have not assumed that $\mathfrak{D}$ is homomorphism ω-compact. Even without this assumption, we obtain the stronger result that ψ is a first-order abstract projective definition of ϕ.

Corollary 7.10. *Let $\mathfrak{D}$ be a strongly positive unary concrete domain. Then, every* $\mathsf{FOL}(\mathfrak{D})$ *sentence has an abstract projective definition in* FOL.

Proof. Let ϕ be a $\mathsf{FOL}(\mathfrak{D})$ sentence and ψ the FOL sentence obtained by the rewriting process previously described. First, we show that every model of ψ is an abstract model of ϕ. Let $\mathcal{I}$ be a model of ψ. Since $\mathfrak{D}$ is unary, the constraint system $\Gamma_{\mathcal{I}}$ considered in the proof of Theorem 7.5 contains all expressions $P^f(x_d)$ such that $P^f(d)$ holds in $\mathcal{I}$ for f a feature symbol and $d \in \Delta^{\mathcal{I}}$. For every feature symbol f and $d \in \Delta^{\mathcal{I}}$, let $\Gamma_{d,f}$ be the subsystem of $\Gamma_{\mathcal{I}}$ containing all and only expressions of the form $P^f(x_d)$. We notice that each of these subsystems is finite, and that they partition $\Gamma_{\mathcal{I}}$. In particular, $\Gamma_{\mathcal{I}}$ is satisfiable in $\mathfrak{D}$ iff $\Gamma_{d,f}$ is satisfiable in $\mathfrak{D}$ for every f and $d \in \Delta^{\mathcal{I}}$. The satisfiability of each such $\Gamma_{d,f}$ in $\mathfrak{D}$ is a consequence of the fact that $\mathcal{I}$ is a model of ψ, and thus of $\Psi^{\mathfrak{D}}$. Otherwise, $\Psi^{\mathfrak{D}}$ would contain the sentence $\forall x.\Gamma_{d,f} \rightarrow \bot$ and this would lead to a contradiction. We conclude that $\Gamma_{\mathcal{I}}$ has a solution h in $\mathfrak{D}$, which we use as in the proof of Theorem 7.5 to define an interpretation $\mathfrak{F}$ of feature symbols such that $(\mathcal{I}, \mathfrak{F})$ is a model of ϕ.

Second, we must show that every abstract model of ϕ can be extended to a model of ψ by interpreting the new predicates of the form P^f and Def_f appropriately. This can be done exactly as in the proof of Theorem 7.5. □

Recall that, in the proof of Theorem 7.5, we used the downward Löwenheim-Skolem property of first-order logic to ensure that the constraint system $\Gamma_{\mathcal{I}}$ is countable, a necessary requirement to be able to apply homomorphism ω-compactness. In the proof of Corollary 7.10, this was not possible since we had to show that the given model of ψ is an abstract model of ϕ. Fortunately, the fact that we can reduce satisfiability of $\Gamma_{\mathcal{I}}$ to that of the finite systems $\Gamma_{d,f}$ allowed us to dispense with this step and the requirement that $\mathfrak{D}$ is homomorphism ω-compact.

For $\mathcal{ALC}(\mathfrak{D})$ TBoxes $\mathcal{T}$ we can strengthen Corollary 7.10 by introducing a TBox $\mathcal{T}^{\mathfrak{D}}$ that takes on the role of $\Psi^{\mathfrak{D}}$ in the $\mathsf{FOL}(\mathfrak{D})$ setting. First, we introduce fresh concept names P^f and Def_f for every feature name f and unary predicate P of $\mathfrak{D}$ occurring in a concrete domain restriction of $\mathcal{T}$. If $\phi_{\neg P}(x)$ is the positive, quantifier-free definition of the complement of P over $\mathfrak{D}$, we denote with $[\![\neg P]\!]^f_{\mathsf{FOL}}$ the $\mathcal{ALC}$ concept obtained by replacing every predicate $Q(x)$ in that formula with the concept name Q^f and replacing conjunction and disjunction with the corresponding DL constructors. We denote with $\mathcal{T}^{\mathsf{FOL}}$ the $\mathcal{ALC}$ TBox obtained from $\mathcal{T}$ by replacing $\exists f.P(x)$ with P_f, $\exists rf.P(x)$ with $\exists r.P_f$, $\forall f.P(x)$ with $\neg\,\mathrm{Def}_f \sqcup P^f$ and $\forall rf.P(x)$ with $\forall r.(\neg\,\mathrm{Def}_f \sqcup P^f)$. The $\mathcal{ALC}$ TBox $\mathcal{T}^{\mathfrak{D}} := \mathcal{T}_1 \cup \mathcal{T}_2$ consists of the following CIs:

- $\mathcal{T}_1$ contains $P^f \sqsubseteq \mathrm{Def}_f$ and $\neg P^f \sqsubseteq [\![\neg P]\!]^f_{\mathsf{FOL}} \sqcup \neg\,\mathrm{Def}_f$ for every feature name f and unary relation P of $\mathfrak{D}$ occurring in $\mathcal{T}$,
- $\mathcal{T}_2$ contains $\bigsqcap \Gamma \sqsubseteq \bot$ for every feature name f and every finite set Γ of concept names P^f occurring in $\mathcal{T}_1$ s.t. the constraint system $\{P(x) \mid P^f \in \Gamma\}$ is unsatisfiable in $\mathfrak{D}$.

Then, $\mathcal{T}^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}}$ plays the role of the sentence ψ in the proof of Corollary 7.10, that is, it is an abstract projective definition of $\mathcal{T}$.

Corollary 7.11. *Let $\mathfrak{D}$ be a strongly positive unary concrete domain. Then, every $\mathcal{ALC}(\mathfrak{D})$ TBox has an abstract projective definition in $\mathcal{ALC}$.*

Interpolation results

In the context of DLs, the following notion of interpolant related to CIs and TBoxes, akin to that of Craig interpolant for FOL, has been used to show that an analogous version of the Craig interpolation property holds for a number of expressive logics, including $\mathcal{ALC}$ [37].

Definition 7.12. *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be TBoxes and C_1, C_2 be concepts. For $i = 1, 2$, let N_C^i and N_R^i be the sets of concept and role names that occur in $\mathcal{T}_i$ or C_i. The concept I is a* Craig interpolant *for $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_1 \sqsubseteq C_2$ if $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_1 \sqsubseteq I$, $\mathcal{T}_1 \cup \mathcal{T}_2 \models I \sqsubseteq C_2$ and I only uses concept names in $N_C^1 \cap N_C^2$ and role names in $N_R^1 \cap N_R^2$.*

For DLs with concrete domains, we call an interpolant of the form above an *abstract* Craig interpolant, and we say that a Craig interpolant is *concrete* if every feature name occurring in the interpolant I is constrained in the same way as concept and role names in Definition 7.12. Thanks to the encoding described earlier in this section and Corollary 7.11, we can adapt the proof of Theorem 7.9 to show that, if $\mathfrak{D}$ is unary and strongly positive, then the existence of an abstract Craig interpolant is always guaranteed in $\mathcal{ALC}(\mathfrak{D})$. When this is the case, we say that $\mathcal{ALC}(\mathfrak{D})$ satisfies the *abstract Craig interpolation property*.

But first, we must show a technical lemma that relates subsumption in $\mathcal{ALC}(\mathfrak{D})$ to subsumption in the corresponding first-order translation.

Lemma 7.13. *Let $\mathfrak{D}$ be a strongly positive unary concrete domain, let $\mathcal{T}$ be an $\mathcal{ALC}(\mathfrak{D})$ TBox and C_1, C_2 $\mathcal{ALC}(\mathfrak{D})$ concepts. Then, $\mathcal{T} \models C_1 \sqsubseteq C_2$ iff $\mathcal{T}^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}} \models C_1^{\mathsf{FOL}} \sqsubseteq C_2^{\mathsf{FOL}}$.*

Proof. We observe that $\mathcal{T} \models C_1 \sqsubseteq C_2$ holds iff the TBox $\mathcal{T}' := \mathcal{T} \cup \{\top \sqsubseteq \exists r.(C_1 \sqcap \neg C_2)\}$ with r a fresh role name is inconsistent. To see this, first, assume that $\mathcal{T}$ does not entail $C_1 \sqsubseteq C_2$ and thus that there exists a model $\mathcal{I}$ of $\mathcal{T}$ such that $d \in (C_1 \sqcap \neg C_2)^{\mathcal{I}}$ for some $d \in \Delta^{\mathcal{I}}$. Then, the interpretation $\mathcal{J}$ obtained by expanding $\mathcal{I}$ with $r^{\mathcal{J}} := I \times \{d\}$ is a model of $\mathcal{T}'$, which is thus consistent. Vice versa, if $\mathcal{T}'$ is consistent with model $\mathcal{I}$, then $\mathcal{I}$ is a model of $\mathcal{T}$ in which $(C_1 \sqcap \neg C_2)^{\mathcal{I}} \neq \emptyset$, thus $C_1 \sqsubseteq C_2$ does not hold.

Similarly, we can show that $\mathcal{T}^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}} \models C_1^{\mathsf{FOL}} \sqsubseteq C_2^{\mathsf{FOL}}$ iff $\mathcal{T}'' := \mathcal{T}^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}} \cup \{\top \sqsubseteq \exists r.(C_1^{\mathsf{FOL}} \sqcap \neg C_2^{\mathsf{FOL}})\}$ with r a fresh role name is inconsistent. We notice that $\mathcal{T}''$ is an abstract projective definition of $\mathcal{T}'$ as explained before Corollary 7.11. Therefore, $\mathcal{T}'$ is inconsistent iff $\mathcal{T}''$ is inconsistent. This allows us to conclude that $\mathcal{T}$ entails $C_1 \sqsubseteq C_2$ iff $\mathcal{T}^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}}$ entails $C_1^{\mathsf{FOL}} \sqsubseteq C_2^{\mathsf{FOL}}$. □

Theorem 7.14. *Let $\mathfrak{D}$ be a strongly positive unary concrete domain. Then, $\mathcal{ALC}(\mathfrak{D})$ satisfies the abstract Craig interpolation property.*

Proof. Assume that $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_1 \sqsubseteq C_2$ with $\mathcal{T}_1$, $\mathcal{T}_2$ $\mathcal{ALC}(\mathfrak{D})$ TBoxes and C_1, C_2 $\mathcal{ALC}(\mathfrak{D})$ concepts. By Lemma 7.13 we obtain that $\mathcal{T}_1^{\mathsf{FOL}} \cup \mathcal{T}_2^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}} \models C_1^{\mathsf{FOL}} \sqsubseteq C_2^{\mathsf{FOL}}$, where all TBoxes and concepts belong to $\mathcal{ALC}$. Since $\mathcal{ALC}$ satisfies the Craig interpolation property, we can find a Craig interpolant I_{Craig} of $\mathcal{T}_1^{\mathsf{FOL}} \cup \mathcal{T}_2^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}} \models C_1^{\mathsf{FOL}} \sqsubseteq C_2^{\mathsf{FOL}}$ as in Definition 7.12, i.e.

$$\mathcal{T}_1^{\mathsf{FOL}} \cup \mathcal{T}_2^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}} \models C_1^{\mathsf{FOL}} \sqsubseteq I_{\mathsf{Craig}}$$
$$\mathcal{T}_1^{\mathsf{FOL}} \cup \mathcal{T}_2^{\mathsf{FOL}} \cup \mathcal{T}^{\mathfrak{D}} \models I_{\mathsf{Craig}} \sqsubseteq C_2^{\mathsf{FOL}}.$$

In this case, let N_C^i be the set of concept names occurring in C_i or $\mathcal{T}_i' := \mathcal{T}_i \cup \mathcal{T}_{\mathfrak{D}}$ and similarly define the set of role names N_R^i for $i = 1, 2$. Then, I_{Craig} uses names from $N_C^1 \cap N_C^2$ and $N_R^1 \cap N_R^2$.

Recall that, for a given unary relation Q of $\mathfrak{D}$, $[\![\neg Q]\!]^f_{\mathsf{FOL}}$ basically is the $\mathcal{ALC}$ representation of $\exists f.\neg Q$. Let $[\![\neg Q]\!]^f_{\mathfrak{D}}$ be the $\mathcal{ALC}(\mathfrak{D})$ concept whose first-order translation is $[\![\neg Q]\!]^f_{\mathsf{FOL}}$, i.e., $[\![\neg Q]\!]^f_{\mathfrak{D}}$ is obtained from $[\![\neg Q]\!]^f_{\mathsf{FOL}}$ by replacing every predicate P^f occurring in it with $\exists f.P(x)$. We need this concept to translate definedness predicates occurring in I_{Craig} back to $\mathcal{ALC}(\mathfrak{D})$.

Let I be the $\mathcal{ALC}(\mathfrak{D})$ concept obtained from I_{Craig} by substituting every concept name Def_f in I_{Craig} with the concept $\exists f.Q(x) \sqcup [\![\neg Q]\!]^f_{\mathfrak{D}}$, where Q is an arbitrary unary relation over $\mathfrak{D}$, and every concept name P^f with the CD-restriction $\exists f.P(x)$. Then, every feature name occurring in I occurs in the union of the sets N_F^i of feature names occurring in $\mathcal{T}_i$ or C_i for $i = 1, 2$, while concept and role names in I belong to the intersection of the respective signature sets. Using Lemma 7.13 again, we obtain that $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_1 \sqsubseteq I$ and $\mathcal{T}_1 \cup \mathcal{T}_2 \models I \sqsubseteq C_2$. Therefore, we can conclude that I is an abstract Craig interpolant for $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_1 \sqsubseteq C_2$.

To see that we can really apply Lemma 7.13 here, we must know that I^{FOL} is equivalent to I_{Craig} in the presence of $\mathcal{T}^{\mathfrak{D}}$. This boils down to showing that $(\exists f.Q(x) \sqcup [\![\neg Q]\!]^f_{\mathfrak{D}})^{\mathsf{FOL}} = Q^f \sqcup [\![\neg Q]\!]^f_{\mathsf{FOL}}$ is equivalent w.r.t. $\mathcal{T}^{\mathfrak{D}}$ to Def_f, which is an easy consequence of the fact that $\mathcal{T}^{\mathfrak{D}}$ contains the CIs $Q^f \sqsubseteq \mathrm{Def}_f$ and $\neg Q^f \sqsubseteq [\![\neg Q]\!]^f_{\mathsf{FOL}} \sqcup \neg\,\mathrm{Def}_f$. □

Decidability results

Note that, in the setting introduced in this subsection, the $\mathsf{FOL}(\mathfrak{D})$ sentence $\Psi^{\mathfrak{D}}$ belongs both to the guarded and the two-variable fragment with counting of first-order logic, which are known to be decidable [4, 54, 87, 90]. Therefore, if the sentence ϕ falls into one of these fragments, defined analogously to their first-order counterparts, it follows that the abstract projective definition ψ of ϕ used in Corollary 7.10 also falls into this fragment. To ensure that satisfiability of $\mathsf{FOL}(\mathfrak{D})$ sentences belonging to one of these fragments is decidable, it is necessary to guarantee that $\Psi^{\mathfrak{D}}$ can effectively be computed. This is the case if checking satisfiability of a finite constraint system for $\mathfrak{D}$ is decidable.

Corollary 7.15. *Let $\mathfrak{D}$ be a strongly positive unary concrete domain. If constraint satisfiability for $\mathfrak{D}$ is decidable, then satisfiability of sentences in the guarded or the two-variable fragment with counting of* $\mathsf{FOL}(\mathfrak{D})$ *is decidable.*

The first-order translations of many DLs considered in the literature actually belong to the guarded or the two-variable fragment with counting. Since, in the unary case, the translations of CD-restrictions into $\mathsf{FOL}(\mathfrak{D})$ given in (7.1) also belong to these fragments, the above corollary yields decidability results for a great number of DLs extended with unary and decidable concrete domains. However, this does not cover the decidability result for $\mathcal{SHOQ}$ extended with unary concrete domains in [63] since the transitivity of roles specifiable in that DL cannot be expressed in the guarded or the two-variable fragment with counting.

(In)equality, non-definability, undecidability

We have seen above that the restriction to a unary concrete domain $\mathfrak{D}$ ensures that every $\mathsf{FOL}(\mathfrak{D})$ sentence has an abstract projective definition in FOL. This also implies that $\mathsf{FOL}(\mathfrak{D})$ satisfies the upward Löwenheim-Skolem property. Without the restriction to predicates of arity 1, this need no longer be the case. In fact, Example 7.4 demonstrates that, if we consider the concrete domain $\mathfrak{Q}_>$, then there is an $\mathcal{ALC}(\mathfrak{Q}_>)$ TBox $\mathcal{T}$ that does not have an abstract projective definition in FOL. In addition, the proof of Corollary 7.6 shows that, for the concrete domain $\mathfrak{Q}_=$, the logic

$\mathsf{FOL}(\mathfrak{Q}_=)$ does not satisfy the upward Löwenheim-Skolem property. In the following, we extend these negative results from single examples to a large class of concrete domains.

Analyzing the two concrete examples, we see that they crucially depend on the fact that (in)equality can be expressed in the concrete domain under consideration.

Theorem 7.16. *Let $\mathfrak{D}$ be a jointly diagonal, at most countable concrete domain. Then,* $\mathsf{FOL}(\mathfrak{D})$ *does not have the upward Löwenheim-Skolem property.*

Proof. Assume that $\psi_=(x,y)$ is the quantifier-free formula that expresses equality between elements of D. Let $\psi_=^f(x,y)$ be the $\mathsf{FOL}(\mathfrak{D})$ formula obtained by replacing every atom $P(x_1,\dots,x_n)$ in $\psi_=$ with $P(f,\dots,f)(x_1,\dots,x_n)$. Similarly to the proof of Corollary 7.6, we can define a $\mathsf{FOL}(\mathfrak{D})$ sentence ϕ_{up} which enforces that the interpretation of f is a total and injective function from the domain of its abstract models into D, as follows:

$$\phi_{\mathsf{up}} := \forall x,y.\,\mathrm{Def}(f)(x) \wedge (x \neq y \rightarrow \neg\psi_=^f(x,y)).$$

Thus, no abstract model of ϕ_{up} can have an uncountable domain. □

In Corollary 7.7 we use closure under disjoint union of models of $\mathcal{ALC}(\mathfrak{D})$ TBoxes to show that $\mathcal{ALC}(\mathfrak{D})$ has the upward Löwenheim-Skolem property. However, the fact that such a TBox then always has an uncountable model is not sufficient to apply the argument used in Example 7.4 to show that there exists an $\mathcal{ALC}(\mathfrak{D})$ TBox that has no abstract projective first-order definition. In fact, such an uncountable model could be the uncountable disjoint union of countable models, and injectivity of the feature name f can possibly only be enforced on the countable sub-models. This is why we needed the formula τ in the proof given in that example, which states that any two distinct elements of the interpretation domain are linked by the role r. We show how to use the JD condition to adapt the idea underlying this proof to our more general setting.

Theorem 7.17. *Let $\mathfrak{D}$ be a jointly diagonal, at most countable concrete domain. Then, there is an $\mathcal{ALC}(\mathfrak{D})$ TBox that has no abstract projective definition in first-order logic.*

Proof. Let equality over $\mathfrak{D}$ be defined by the quantifier-free, equality-free formula $\psi_=(x,y)$, which is w.l.o.g. assumed to be in negation normal form.

We define the $\mathcal{ALC}(\mathfrak{D})$ concept $C_=$ obtained by replacing each non-negated atom $P(x_1,\dots,x_k)$ in $\psi_=(x,y)$ with the concept $\exists f,\dots,f.P(x_1,\dots,x_k)$ and every negated atom $\neg P(x_1,\dots,x_k)$ in $\psi_=(x,y)$ with $\neg\forall f,\dots,f.P(x_1,\dots,x_k)$.[2] We can force the interpretation of the feature name f to be a total function (in the spirit of the CI (7.2) used in Example 7.4) with the $\mathcal{ALC}(\mathfrak{D})$ TBox $\mathcal{T}_{\mathsf{tot}} := \{\top \sqsubseteq C_=\}$. Note that, in addition to requiring that every element of the abstract domain must have an f-value, this TBox only states that this f-value is equal to itself, which is trivially satisfied. This construction is needed since, in contrast to $\mathsf{FOL}(\mathfrak{D})$, the DL $\mathcal{ALC}(\mathfrak{D})$ is not equipped with definedness restrictions.

We derive from $\neg\psi_=(x,y)$ a quantifier-free, equality-free formula $\psi_{\neq}(x,y)$ in negation-normal form that defines inequality over $\mathfrak{D}$. For every non-negated atom $P(x_1,\dots,x_k)$ that occurs in $\psi_{\neq}(x,y)$ we introduce a fresh role name $r_{P(x_1,\dots,x_k)}$, a mapping $\lambda_{x_1,\dots,x_k}$ that assigns to a tuple of individuals $(d,e) \in D \times D$ the tuple $(d_1,\dots,d_k) \in D^k$ s.t. $d_i = d$ if $x_i = x$ and $d_i = e$ if $x_i = y$, and a CI $\top \sqsubseteq \forall p_1,\dots,p_k.P(x_1,\dots,x_k)$ where $p_i = f$ if $x_i = x$ and $p_i = r_{P(x_1,\dots,x_k)}f$ if $x_i = y$. For negated atoms $\neg P(x_1,\dots,x_k)$, we introduce a fresh role name $r_{\neg P(x_1,\dots,x_k)}$, we define $\lambda_{x_1,\dots,x_k}$

[2]Since $\psi_=(x,y)$ is quantifier-free and has $\{x,y\}$ as its set of free variables, each variable x_i occurring in these literals is either x or y.

as in the non-negated case and introduce a CI $\exists p_1, \dots, p_k.P(x_1, \dots, x_k) \sqsubseteq \bot$ where feature paths are assigned as in the non-negated case. Note that, from a semantic point of view, this CI is equivalent to the expression $\top \sqsubseteq \forall p_1, \dots, p_k.\neg P(x_1, \dots, x_k)$, though from a syntactic point of view $\forall p_1, \dots, p_k.\neg P(x_1, \dots, x_k)$ is not an admissible CD-restriction since $\neg P$ is not a relation of $\mathfrak{D}$. For this reason, we had to represent the intended CI by its contrapositive. We call $\mathcal{T}_{\neq}$ the TBox containing all the CIs introduced this way.

Let $\mathcal{T} := \mathcal{T}_{\mathsf{tot}} \cup \mathcal{T}_{\neq}$ and assume, by contradiction, that $\mathcal{T}$ is abstractly projectively equivalent to a first-order sentence ϕ. The interpretation $\mathcal{I}$ with countable domain $\Delta^{\mathcal{I}} := D$ and

$$r^{\mathcal{I}}_{P(x_1,\dots,x_k)} := \{(d, e) \in D \times D \mid \lambda_{x_1,\dots,x_k}(d, e) \in P^D\} \text{ and}$$
$$r^{\mathcal{I}}_{\neg P(x_1,\dots,x_k)} := \{(d, e) \in D \times D \mid \lambda_{x_1,\dots,x_k}(d, e) \notin P^D\}$$

for each of the role names introduced above is an abstract model of $\mathcal{T}$, where we interpret the feature name f using the identity on D. Then, $\mathcal{I}$ can be extended to a model $\mathcal{I}^{\mathsf{FOL}}$ of ϕ. Using the upward Löwenheim-Skolem property of FOL, we find an uncountable interpretation $\mathcal{J}$ that is elementary equivalent to $\mathcal{I}^{\mathsf{FOL}}$ in first-order logic (apply the property to the first-order theory of $\mathcal{I}^{\mathsf{FOL}}$, which is trivially satisfied by $\mathcal{I}^{\mathsf{FOL}}$). This implies that $\mathcal{J}$ satisfies ϕ; by assumption, we can thus find an interpretation $f^{\mathfrak{F}}$ of f such that $(\mathcal{J}, \mathfrak{F})$ is a model of $\mathcal{T}$.

Let d, e be two distinct elements of $\Delta^{\mathcal{J}}$. Assuming that $\psi^r_{\neq}(x, y)$ is the *positive* formula obtained by replacing every non-negated occurrence of $P(x_1, \dots, x_k)$ in $\psi_{\neq}(x, y)$ with $r_{P(x_1,\dots,x_k)}(x, y)$ and every negated atom $\neg P(x_1, \dots, x_k)$ with $r_{\neg P(x_1,\dots,x_k)}$, we observe that both $\mathcal{I}$ and $\mathcal{I}^{\mathsf{FOL}}$ satisfy the first-order sentence

$$\phi_{\mathsf{inj}} := \forall x, y.(x \neq y) \leftrightarrow \psi^r_{\neq}(x, y).$$

Since $\mathcal{J}$ and $\mathcal{I}^{\mathsf{FOL}}$ are elementary equivalent, $\mathcal{J}$ also satisfies ϕ_{inj} and thus $(d, e) \in (\psi^r_{\neq})^{\mathcal{J}}$.

Since $(\mathcal{J}, \mathfrak{F})$ is a model of $\mathcal{T}_{\mathsf{tot}}$, both $f^{\mathfrak{F}}(d)$ and $f^{\mathfrak{F}}(e)$ must be defined. The fact that $(\mathcal{J}, \mathfrak{F})$ is a model of $\mathcal{T}_{\neq}$ ensures that $(d, e) \in (r_{P(x_1,\dots,x_k)})^{\mathcal{J}}$ implies $\lambda_{x_1,\dots,x_k}(f^{\mathfrak{F}}(d), f^{\mathfrak{F}}(e)) \in P^D$ for every non-negated atom $P(x_1, \dots, x_k)$ occurring in $\psi_{\neq}(x, y)$ and similarly $(d, e) \in (r_{\neg P(x_1,\dots,x_k)})^{\mathcal{J}}$ implies that $\lambda_{x_1,\dots,x_k}(f^{\mathfrak{F}}(d), f^{\mathfrak{F}}(e)) \notin P^D$ for negated atoms $\neg P(x_1, \dots, x_k)$ in $\psi_{\neq}(x, y)$. Then, $\psi_{\neq}(f^{\mathfrak{F}}(d), f^{\mathfrak{F}}(e))$ holds in $\mathfrak{D}$, and consequently $f^{\mathfrak{F}}(d) \neq f^{\mathfrak{F}}(e)$, which implies that $f^{\mathfrak{F}}$ is an injective function. This leads to a contradiction since we know that D is at most countable, but $\Delta^{\mathcal{J}}$ is uncountable, and $f^{\mathfrak{F}}$ is supposed to be an injective function from $\Delta^{\mathcal{J}}$ to D. We conclude that $\mathcal{T}$ is not abstractly projectively equivalent to any first-order sentence. □

Let us point out that the assumptions made in this theorem are not very restrictive. As already mentioned above, JD is part of the definition of ω-admissibility (cf. Definition 2.20). In fact, what we showed is that there are concrete domains $\mathfrak{D}$ such that reasoning in $\mathcal{ALC}(\mathfrak{D})$ is decidable and the abstract expressive power of $\mathcal{ALC}(\mathfrak{D})$ is not contained in that of first-order logic. In Chapter 6, we showed that satisfiability of a concept w.r.t. a TBox written in $\mathcal{ALC}(\mathfrak{Q})$ with $\mathfrak{Q} := (\mathbb{Q}, <, =, >)$ is decidable in exponential time, because the concrete domain $\mathfrak{Q}$ is ω-admissible and its CSP is decidable in polynomial time. Clearly, $\mathfrak{Q}$ is JD since equality is available as one of its relations, and its domain (the set of rational numbers) is countably infinite.

Example 7.18. *We illustrate how to apply Theorem 7.17 to concrete domains whose relations have arity greater than 2. Let $\mathfrak{Q}_{\mathsf{max}} := (\mathbb{Q}, \mathsf{max})$ be the concrete domain where the ternary relation* max *relates x, y and z iff z is the maximum of x and y w.r.t. the standard ordering $<$ on the rational numbers. This concrete domain is jointly diagonal, since equality can be expressed by the formula* $\mathsf{max}(x, y, x) \wedge \mathsf{max}(x, y, y)$.

Using the notation employed to prove Theorem 7.17, let $\mathcal{T}_{\mathsf{tot}} := \{\top \sqsubseteq C_{=}\}$ *with*

$$C_{=} := \exists f, f, f.\mathsf{max}(x_1, x_2, x_1) \sqcap \exists f, f, f.\mathsf{max}(x_1, x_2, x_2).$$

Inequality over $\mathfrak{Q}_{\mathsf{max}}$ *is defined by the formula* $\psi_{\neq}(x,y) := \neg\mathsf{max}(x,y,x) \vee \neg\mathsf{max}(x,y,y)$. *We introduce the two role names* $r := r_{\neg\mathsf{max}(x,y,x)}$ *and* $r' := r_{\neg\mathsf{max}(x,y,y)}$ *and add to* $\mathcal{T}_{\neq}$ *the two CIs* $\exists f, rf, f.\mathsf{max}(x_1, x_2, x_1) \sqsubseteq \bot$ *and* $\exists f, r'f, r'f.\mathsf{max}(x_1, x_2, x_2) \sqsubseteq \bot$. *Then,* $f^{\mathfrak{F}}$ *is an injective function with values in* $\mathbb{Q}$ *for every model* $(\mathcal{I}, \mathfrak{F})$ *of* $\mathcal{T}$. *Thanks to* $\mathcal{T}_{\mathsf{tot}}$ *we know that* $f^{\mathfrak{F}}$ *is defined for every element of* $\mathcal{I}$. *Due to* $\mathcal{T}_{\neq}$, $(f^{\mathfrak{F}}(d), f^{\mathfrak{F}}(e), f^{\mathfrak{F}}(d)) \notin \mathsf{max}^{\mathfrak{Q}_{\mathsf{max}}}$ *holds if* $(d,e) \in r^{\mathcal{I}}$ *and similarly* $(f^{\mathfrak{F}}(d), f^{\mathfrak{F}}(e), f^{\mathfrak{F}}(e)) \notin \mathsf{max}^{\mathfrak{Q}_{\mathsf{max}}}$ *if* $(d,e) \in (r')^{\mathcal{I}}$. *The interpretations considered in the proof of Theorem 7.17 satisfy the sentence* $\forall x, y.(x \neq y \leftrightarrow r(x,y) \vee r'(x,y))$, *which together with the considerations above imply that if* $d \neq e$ *then* $f^{\mathfrak{F}}(d) \neq f^{\mathfrak{F}}(e)$ *thus that* $f^{\mathfrak{F}}$ *is an injective function.*

If $\mathcal{T}$ *was abstractly equivalent to a first-order logic sentence* ϕ, *then, following the proof of Theorem 7.17, we would be able to find an uncountable model of* ϕ, *and thus an uncountable abstract model of* $\mathcal{T}$, *that maps injectively into a countable structure, which clearly is a contradiction.*

Undecidability results

In the remainder of this section, we focus on the two-variable fragment FOL_2 of first-order logic and assume that no function symbols are allowed. Then, we know that this logic has the *finite model property* [83] and is decidable in NExpTime [53]. In contrast, if $\mathfrak{D}$ is jointly diagonal, then the finite model property need not hold for the extension $\mathsf{FOL}_2(\mathfrak{D})$ of FOL_2 with concrete domains. A counterexample is the $\mathsf{FOL}_2(\mathfrak{Q}_>)$ sentence $\forall x.\exists y.{>}(f,f)(x,y)$, whose models always contain an infinitely descending chain of f-values f_0, f_1, ...associated to individuals d_0, d_1, ..., which means that their domain must be infinite.

In general, the abstract expressive power of $\mathsf{FOL}_2(\mathfrak{D})$ is not contained in that of FOL_2 and not even in that of full first-order logic: indeed, the sentence ϕ_{up} used in the proof of Theorem 7.16 is contained in $\mathsf{FOL}_2(\mathfrak{D})$, and so this logic does not have the upward Löwenheim-Skolem property. We are further able to show that, if $\mathfrak{D}$ is additionally infinite, then reasoning in $\mathsf{FOL}_2(\mathfrak{D})$ is undecidable. Our undecidability proof is based on a reduction from the tiling problem [29].

Definition 7.19. *A* tiling problem $P := (T, H, V)$ *consists of a finite set* T *of* tile types *and binary relations* $H, V \subseteq T \times T$ *respectively called* horizontal *and* vertical matching *conditions. The function* $\pi\colon \mathbb{N} \times \mathbb{N} \to T$ *is a* solution *of* P *if for all* $i, j \in \mathbb{N}$ *it holds that* $(\pi(i,j), \pi(i+1,j)) \in H$ *and* $(\pi(i,j), \pi(i,j+1)) \in V$.

We show how to reduce the solvability of a tiling problem P to the satisfiability of a finite set Φ_P of $\mathsf{FOL}_2(\mathfrak{D})$ sentences. The signature used to define Φ_P contains a unary predicate $A_t(x)$ for every tile type $t \in T$, two binary predicates $H(x,y)$ and $V(x,y)$ meant to capture the matching conditions of P and four feature symbols f_H, f_V, g_H and g_V, which are used to enforce a grid structure on the two predicates H and V. We assume that $\psi_{=}(x,y)$ is the quantifier-free first-order formula expressing equality over $\mathfrak{D}$ and define $\psi_{=}(f,g)(x,y)$ in the same way as $\phi_{\neg P}(f_1, \dots, f_n)(x_1, \dots, x_n)$ from $\phi_{\neg P}(x_1, \dots, x_n)$ in Section 7.2.

We add the following sentences to guarantee that every individual in a model of Φ_P has a horizontal and vertical match and exactly one tile type, and that horizontal and vertical matches follow the matching conditions stated in the tiling problem:

$$\forall x.\exists y.H(x,y) \wedge \forall x.\exists y.V(x,y) \tag{mat}$$

$$\forall x.(\bigvee_{t\in T}(A_t(x)) \wedge \bigwedge_{t\neq t'\in T}(\neg A_t(x) \vee \neg A_{t'}(x))) \tag{til-T}$$

$$\forall x,y.(H(x,y) \to \bigvee_{(t,t')\in H}(A_t(x) \wedge A_{t'}(y))) \tag{til-H}$$

$$\forall x,y.(V(x,y) \to \bigvee_{(t,t')\in V}(A_t(x) \wedge A_{t'}(y))) \tag{til-V}$$

We add sentences that force the partial functions associated to f_H, f_V by an interpretation $\mathfrak{F}$ to be total and the mapping $x \mapsto (f_H^{\mathfrak{F}}(x), f_V^{\mathfrak{F}}(x))$ to be injective:

$$\forall x.(\mathrm{Def}(f_H)(x) \wedge \mathrm{Def}(f_V)(x)), \tag{tot}$$

$$\forall x,y.(x = y \leftrightarrow (\psi_=(f_H,f_H)(x,y) \wedge \psi_=(f_V,f_V)(x,y))) \tag{inj}$$

We add a sentence that implicitly defines and constrains the values of g_H, g_V associated to a given individual:

$$\forall x.(\neg\psi_=(f_H,g_H)(x,x) \wedge \neg\psi_=(f_V,g_V)(x,x)) \tag{irr}$$

Finally, we relate the binary relations H, V and the feature symbols:

$$\begin{aligned}\forall x,y.\ H(x,y) \leftrightarrow (&\psi_=(g_V,g_V)(x,y) \wedge \psi_=(f_V,f_V)(x,y) \\ &\wedge \psi_=(g_H,f_H)(x,y))\end{aligned} \tag{fun-H}$$

$$\begin{aligned}\forall x,y.\ V(x,y) \leftrightarrow (&\psi_=(g_H,g_H)(x,y) \wedge \psi_=(f_H,f_H)(x,y) \\ &\wedge \psi_=(g_V,f_V)(x,y))\end{aligned} \tag{fun-V}$$

We notice that in every abstract model $\mathcal{I}$ of Φ_P the binary relations $H^{\mathcal{I}}$ and $V^{\mathcal{I}}$ are functional and irreflexive. Indeed, assume that $\mathfrak{F}$ is an interpretation of feature symbols s.t. $(\mathcal{I},\mathfrak{F})$ is a model of Φ_P. If $(x,y) \in H^{\mathcal{I}}$ and $(x,z) \in H^{\mathcal{I}}$ then by (fun-H)

$$f_H^{\mathfrak{F}}(y) = g_H^{\mathfrak{F}}(x) = f_H^{\mathfrak{F}}(z) \text{ and } f_V^{\mathfrak{F}}(y) = f_V^{\mathfrak{F}}(x) = f_V^{\mathfrak{F}}(z)$$

and so by (inj) we derive that $y = z$; combining the previous identities with (irr) and (inj), we deduce that $f_H^{\mathfrak{F}}(x) \neq f_H^{\mathfrak{F}}(y)$ and so $x \neq y$. Similarly, we show that $V^{\mathcal{I}}$ is functional and irreflexive in every abstract model $\mathcal{I}$ of Φ_P. The last, crucial property enjoyed by the abstract models of Φ_P is that the horizontal and vertical matching relations H and V commute.

Lemma 7.20. *If $\mathcal{I}$ is an abstract model of Φ_P then $H^{\mathcal{I}} \circ V^{\mathcal{I}}$ and $V^{\mathcal{I}} \circ H^{\mathcal{I}}$ coincide and are functional.*

Proof. If $x \in I$, let $u,v,y,z \in I$ be those individuals that satisfy $(x,u) \in H^{\mathcal{I}}$, $(u,v) \in V^{\mathcal{I}}$, $(x,y) \in V^{\mathcal{I}}$ and $(y,z) \in H^{\mathcal{I}}$, whose existence and uniqueness is guaranteed by (mat) and our previous observation. We show that $v = z$.

If $(\mathcal{I},\mathfrak{F})$ is a model of Φ_P, we obtain the following identities:

$$f_H^{\mathfrak{F}}(v) = f_H^{\mathfrak{F}}(u) = g_H^{\mathfrak{F}}(x) = g_H^{\mathfrak{F}}(y) = f_H^{\mathfrak{F}}(z), \tag{7.3}$$

$$f_V^{\mathfrak{F}}(v) = g_V^{\mathfrak{F}}(u) = g_V^{\mathfrak{F}}(x) = f_V^{\mathfrak{F}}(y) = f_V^{\mathfrak{F}}(z). \tag{7.4}$$

Thus, we conclude that $z = v$ using (inj). □

Now that we established the crucial properties of the abstract models of Φ_P, we are ready to show the correctness of the reduction.

Lemma 7.21. *Let $\mathfrak{D}$ be an infinite, jointly diagonal concrete domain. Then, Φ_P is satisfiable iff P has a solution.*

Proof. Let $\mathcal{I}$ be an abstract model of Φ_P. We define the mapping $\pi\colon \mathbb{N} \times \mathbb{N} \to I$ inductively, as follows. First, let $\pi(0,0)$ be an arbitrary individual in I, which exists since this set must be non-empty. Assuming that for $i, j \in \mathbb{N}$ the value $\pi(i, j) := x$ is defined, we define $\pi(i+1, j)$ as the unique $y \in I$ such that $(x, y) \in H^{\mathcal{I}}$ and $\pi(i, j+1)$ as the unique $z \in I$ such that $(x, z) \in V^{\mathcal{I}}$. Lemma 7.20 guarantees that π is well-defined: indeed, the individual $\pi(i+1, j+1)$ is supposed to be both the unique $H^{\mathcal{I}}$-successor of $\pi(i, j+1)$ and the unique $V^{\mathcal{I}}$-successor of $\pi(i, j+1)$, and the lemma ensures that these are indeed the same elements. Clearly, for all $i, j \in \mathbb{N}$

$$(\pi(i, j), \pi(i+1, j)) \in H^{\mathcal{I}} \text{ and } (\pi(i, j), \pi(i, j+1)) \in V^{\mathcal{I}}. \tag{7.5}$$

Using π, we define $\pi_P\colon \mathbb{N} \times \mathbb{N} \to T$ so that $\pi_P(i, j) := t$ iff $\pi(i, j) \in A_t^{\mathcal{I}}$. Then, the fact that $\mathcal{I}$ satisfies (til-T), (til-H) and (til-V) ensures that π_P is a solution to the tiling problem P.

Next, let π be a solution of P. We define the interpretation $\mathcal{I}_\pi$ with domain $\mathbb{N} \times \mathbb{N}$ as follows. For each tile type $t \in T$, we set $A_t^{\mathcal{I}_\pi}$ as the set of elements (i, j) for which $\pi(i, j) = t$. For each element (i, j) in the domain, we add $((i, j), (i+1, j))$ to $H^{\mathcal{I}_\pi}$ and $((i, j), (i, j+1))$ to $V^{\mathcal{I}_\pi}$.

Since $\mathfrak{D}$ is infinite, we can assume that there is an injective function $h\colon \mathbb{N} \to D$. Using this mapping, we set $f_H^{\mathfrak{F}}(i, j) := h(i)$, $f_V^{\mathfrak{F}}(i, j) := h(j)$, $g_H^{\mathfrak{F}}(i, j) := h(i+1)$ and $g_V^{\mathfrak{F}}(i, j) := h(j+1)$. It is straightforward to verify that $\mathcal{I}_\pi$ is an abstract model of Φ_P. □

The problem of checking if a tiling problem has a solution is undecidable [29] and we thus conclude that our decision problem is undecidable, too, as a consequence of Lemma 7.21.

Theorem 7.22. *Let $\mathfrak{D}$ be an infinite and jointly diagonal concrete domain. Then, the satisfiability problem for the two-variable fragment of* $\mathsf{FOL}(\mathfrak{D})$ *is undecidable.*

Summary

We introduced the notion of *abstract expressive power* of logics with concrete domains and established sufficient conditions on the concrete domain that ensure that the resulting extensions of FOL and $\mathcal{ALC}$ satisfy (countable) compactness or other important first-order properties. We analyzed abstract (non-)definability and leveraged some related results to derive (un-)decidability results for several fragments of $\mathsf{FOL}(\mathfrak{D})$. These results are summarised in the following tables.

	$\mathsf{FOL}(\mathfrak{D})$	$\mathcal{ALC}(\mathfrak{D})$
$\mathfrak{D}$ is strongly positive and homomorphism ω-compact		
Downward Löwenheim-Skolem	yes	yes
Countable Compactness	yes	yes
Upward Löwenheim-Skolem	—	yes
Craig Interpolation	abstract	—
$\mathfrak{D}$ is strongly positive and unary		
Abstract definability	FOL (projective)	$\mathcal{ALC}$ (projective)
Craig Interpolation	abstract	abstract
$\mathfrak{D}$ is countably infinite and jointly diagonal		
Abstract FOL-definability	no	no
Upward Löwenheim-Skolem	no	—

The table above lists the results obtained w.r.t. the first-order properties of $\mathsf{FOL}(\mathfrak{D})$ and $\mathcal{ALC}(\mathfrak{D})$, according to what conditions on the concrete domain $\mathfrak{D}$ are assumed.

	$\mathfrak{D}$ strongly positive and unary	$\mathfrak{D}$ infinite and jointly diagonal
Satisfiability of $\mathsf{GF}_2(\mathfrak{D})$	decidable	—
Satisfiability of $\mathsf{FOL}_2(\mathfrak{D})$	decidable	undecidable
Satisfiability of $\mathcal{C}^2(\mathfrak{D})$	decidable	undecidable

This second table contains the (un-)decidability results derived in this chapter.

8 The Expressive Power of DLs with Concrete Domains

The focus of Chapter 7 was on the comparison of the expressive power of logics with and without concrete domain reasoning, through the usage of the notion of *abstract* expressive power. In this section, instead, we focus solely on logics with concrete domains. First, we define a notion of *concrete* bisimulation which also accounts for the presence of feature names and CD-restrictions, unlike the bisimulations considered so far. We show how to apply concrete bisimulations to show two types of non-expressivity results for DLs with concrete domains: the first is concerned with concepts in two extensions of $\mathcal{ALC}$ where the concrete domains $\mathfrak{D}$ and $\mathfrak{D}'$ have the same domain set, but different relations, while the second involves concepts in extensions of $\mathcal{ALC}$ that use the same concrete domain, but different kinds of CD-restrictions. After that, we characterize $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is $\mathbb{C}$-invariant under concrete bisimulation, where $\mathbb{C}$ is a class of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ that satisfies certain properties similar to those investigated in Chapter 4.

This work contained in this chapter is based on unpublished material, which is currently being prepared for a conference submission.

Back-and-forth with concrete domains. We turn our attention to the expressive power of $\mathcal{ALC}(\mathfrak{D})$. Here, the back-and-forth conditions associated to a bisimulation can be less restrictive are taken from the notion of $\mathcal{ALC}$ bisimulation (see Chapter 2). On the other hand, we must account for the presence of CD-restrictions and add appropriate conditions over feature values of an individual and its role successors. Thus, we propose the following notion of bisimulation relating members of a class $\mathbb{C}$ of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ using the relations of the concrete domain $\mathfrak{D}$.

Definition 8.1. *Let $\mathfrak{D}$ be a concrete domain and $\mathcal{I}$, $\mathcal{J}$ interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ that assign elements of $\mathfrak{D}$ to features from $\mathsf{N_F}$. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a $\mathfrak{D}$* bisimulation *between $\mathcal{I}$ and $\mathcal{J}$ if for all $A \in \mathsf{N_C}$, all $r \in \mathsf{N_R}$, all k-ary relations P of $\mathfrak{D}$, and all feature paths $p_1, \dots, p_k$ over $\mathsf{N_R}$ and $\mathsf{N_F}$:*

atomic *if $(d, e) \in \rho$ then $d \in A^{\mathcal{I}}$ iff $e \in A^{\mathcal{J}}$;*

forth *if $(d, e) \in \rho$ and $d' \in r^{\mathcal{I}}(d)$, then there is $e' \in r^{\mathcal{J}}(e)$ such that $(d', e') \in \rho$;*

back *if* $(d, e) \in \rho$ *and* $e' \in r^{\mathcal{J}}(e)$*, then there is* $d' \in r^{\mathcal{I}}(d)$ *such that* $(d', e') \in \rho$*.*

features *if* $(d, e) \in \rho$*, then there is* $(v_1, \dots, v_k) \in P^D$ *with* $v_1 \in p_1^{\mathcal{I}}(d)$*, ...,* $v_k \in p_k^{\mathcal{I}}(d)$ *iff there is* $(w_1, \dots, w_k) \in P^D$ *with* $w_1 \in p_1^{\mathcal{J}}(e)$*, ...,* $w_k \in p_k^{\mathcal{J}}(e)$*.*

Bisimilarity between individuals and $\mathbb{C}$*-invariance w.r.t.* $\mathfrak{D}$ *bisimulation are defined similarly to how it was done in Definition 4.2 w.r.t. Pr bisimulation.*

A result analogous to Theorem 4.3 holds for $\mathcal{ALC}(\mathfrak{D})$ concepts if the concrete domain $\mathfrak{D}$ is weakly closed under negation.

Theorem 8.2. *If* $\mathfrak{D}$ *is WCUN and* $\mathbb{C}$ *is a class of interpretations of* $\mathsf{N_C}$, $\mathsf{N_R}$ *and* $\mathsf{N_F}$ *that assign elements of* $\mathfrak{D}$ *to features from* $\mathsf{N_F}$*, then every* $\mathcal{ALC}(\mathfrak{D})$ *concept is* $\mathbb{C}$*-invariant under* $\mathfrak{D}$ *bisimulation.*

Proof. The proof by structural induction on the concept C proceeds like the one for $\mathcal{ALC}$ in [22], except for the cases where C is an CD-restriction. We only consider these cases explicitly here. Thus, let ρ be a $\mathfrak{D}$ bisimulation between $\mathcal{I}$, $\mathcal{J}$ with $(d, e) \in \rho$. We show that d and e satisfy the same CD-restrictions.

If $C := \exists p_1, \dots, p_k.P$ then $d \in C^{\mathcal{I}}$ implies the existence of $v_1 \in p_1^{\mathcal{I}}(d)$,...,$v_k \in p_k^{\mathcal{I}}(d)$ such that $(v_1, \dots, v_k) \in P^D$. Since ρ satisfies features, there must be $w_1 \in p_1^{\mathcal{J}}(e)$, ..., $w_k \in p_k^{\mathcal{J}}(e)$ such that $(w_1, \dots, w_k) \in P^D$, hence $e \in C^{\mathcal{J}}$. Similarly, we can show that $e \in C^{\mathcal{J}}$ implies $d \in C^{\mathcal{I}}$.

If $C := \forall p_1, \dots, p_k.P$, then $d \in C^{\mathcal{I}}$ implies that $(v_1, \dots, v_k) \in P^D$ holds for all values $v_1 \in p_1^{\mathcal{I}}(d)$, ..., $v_k \in p_k^{\mathcal{I}}(d)$. Since $\mathfrak{D}$ is WCUN, this holds iff there are relations P_1, ..., P_{n_P} of $\mathfrak{D}$ such that $(v_1, \dots, v_k) \notin P_i^D$ for $i = 1, \dots, n_P$. Using the features condition of ρ, we deduce that $(w_1, \dots, w_k) \notin P_i^D$ holds for all $w_1 \in p_1^{\mathcal{J}}(e)$, ..., $w_k \in p_k^{\mathcal{J}}(e)$ and $i = 1, \dots, n_P$. By WCUN it follows that $(w_1, \dots, w_k) \in P_i^D$, and we conclude that $e \in C^{\mathcal{J}}$. The proof of the other direction is symmetric. Therefore, d and e satisfy the same CD-restrictions. □

A non-expressivity result.

We can use the notion of $\mathfrak{D}$ bisimulation to show that $\mathcal{ALC}(\mathfrak{D})$ cannot express certain concepts of the DL $\mathcal{ALC}(\mathfrak{D}')$, where $\mathfrak{D}'$ has the same domain set as $\mathfrak{D}$, but different relations. Coming back to the example in the introduction, we compare the expressive power of $\mathfrak{Q}_{+1}$ and $\mathfrak{Q}_{+2}$, both having domain set $\mathbb{Q}$, where the former has a binary relation $+_1$ relating $q \in \mathbb{Q}$ and $q + 1$ (and the complementary relation $\neq_{+_1}$) and the latter has a binary relation $+_2$ relating q and $q+2$ (and the complementary relation $\neq_{+_2}$).

These two DLs have the same *abstract expressive power*. In fact, we can interchange CD-restrictions using relations $+_1$ and $\neq_{+_1}$ with restrictions of the same kind (existential or universal) using relations $+_2$ and $\neq_{+_2}$. Abstract models of a concept in one of these DLs are then the same as of the corresponding concept in the other DL: in one direction, we just double the feature values, and in the other we halve them. Nevertheless, we can show that their *concrete expressive power*, which takes the feature values into account, is incomparable.

Proposition 8.3. *Let* $\mathbb{C}$ *be* $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$*, or* $\mathbb{C}_{\mathsf{fin}}$*. There are* $\mathcal{ALC}(\mathfrak{Q}_{+_1})$ *concepts that are not* $\mathbb{C}$*-equivalent to any* $\mathcal{ALC}(\mathfrak{Q}_{+_2})$ *concept (and vice versa).*

Proof. First, consider the $\mathcal{ALC}(\mathfrak{Q}_{+_1})$ concept $C := \exists rf, rf.+_1$ and assume by contradiction that it is $\mathbb{C}_{\mathsf{all}}$-equivalent to some $\mathcal{ALC}(\mathfrak{Q}_{+_2})$ concept D. Let us consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted in Figure 8.2. Then, $a \in C^{\mathcal{I}}$ and by equivalence $a \in D^{\mathcal{I}}$, while $a_1 \notin C^{\mathcal{J}}$ and so

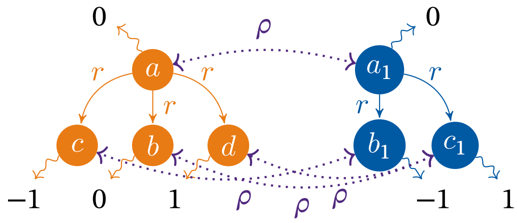


Figure 8.1: A concrete bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$.

$a_1 \notin D^{\mathcal{J}}$ by equivalence. This leads to a contradiction, since the relation ρ between $\mathcal{I}$ and $\mathcal{J}$ is a $\mathfrak{Q}_{+_2}$ bisimulation relating a and a_1, and by Theorem 8.2 this means that $a \in D^{\mathcal{I}}$ iff $a_1 \in D^{\mathcal{J}}$. Therefore, we conclude that C and D cannot be equivalent w.r.t. any class of interpretations that contains the two interpretations of Figure 8.2. Vice versa, we show with a similar argument that $\exists rf, rf.+_2$ cannot be expressed in $\mathcal{ALC}(\mathfrak{Q}_{+_1})$. □

We can also use $\mathfrak{D}$ bisimulations to show that some extended CD-restrictions cannot be simulated by normal CD-restrictions. Here, we show that $\mathcal{ALC}(\mathfrak{Q})$ is less expressive than its extension $\mathcal{ALC}_{pp}(\mathfrak{Q})$ where we allow CD-restrictions of the form $\exists p_1, \ldots, p_k.\phi(x_1, \ldots, x_k)$ with $\phi(x_1, \ldots, x_k)$ a conjunction of atomic formulae $P(y_1, \ldots, y_n)$ where $y_1, \ldots, y_n \in \{x_1, \ldots, x_k\}$ and P is a relation of $\mathfrak{Q}$.

Proposition 8.4. *Let $\mathbb{C}$ be the class of all interpretations. There are $\mathcal{ALC}_{pp}(\mathfrak{Q})$ concepts that are not equivalent to any $\mathcal{ALC}(\mathfrak{Q})$ concept.*

Proof. Consider the $\mathcal{ALC}_{pp}(\mathfrak{Q})$ concept $C := \exists rf, rf, rf.(x < y \wedge y < z)$ and assume by contradiction that it is $\mathbb{C}$-equivalent to some $\mathcal{ALC}(\mathfrak{Q})$ concept D. Let us consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted in Figure 8.2. Then, $a \in C^{\mathcal{I}}$ and by equivalence $a \in D^{\mathcal{I}}$, while $a_1 \notin C^{\mathcal{J}}$ and so $a_1 \notin D^{\mathcal{J}}$ by equivalence. This leads to a contradiction, since the relation ρ between $\mathcal{I}$ and $\mathcal{J}$ is a $\mathfrak{Q}$ bisimulation relating a and a_1 and by Theorem 8.2 this means that $a \in D^{\mathcal{I}}$ iff $a_1 \in D^{\mathcal{J}}$. Therefore, we conclude that C and D are not $\mathbb{C}$-equivalent. □

8.1 The Expressive Power of $\mathcal{ALC}(\mathfrak{D})$ w.r.t. FOL($\mathfrak{D}$)

Our characterization of $\mathcal{ALC}(\mathfrak{D})$ as the fragment of FOL($\mathfrak{D}$) that is invariant under $\mathfrak{D}$ bisimulation mimics the corresponding result for $\mathcal{ALC}$ and propositional modal logic w.r.t. FOL as proved by Rosen [92]. In particular, it re-uses most of the constructions and results employed in Chapter 4 for $\mathcal{ALCSCC}$ and $\mathcal{ALCQ}t$. In the following, we assume that the concrete domain $\mathfrak{D}$ is WCUN and has finitely many relations; both conditions are always satisfied by ω-admissible concrete domains.

Recall that Lemma 4.12 turned out to be an important model-theoretic tool in that approach since it provided us with locality results for FOL formulae expressing $\mathcal{ALCSCC}$ concepts. The corresponding result also holds for FOL($\mathfrak{D}$). Note that notions like *finite disjoint union* and the corresponding notions of $\mathbb{C}$-invariance w.r.t. classes $\mathbb{C}$ of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ are obtained by extending Definition 2.31 to account for feature names in the obvious way. We define ℓ-neighborhoods in interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ by using the same notion of distance employed in Definition 4.10. This means that the distance of two individuals in an interpretation of this kind is not influenced by concrete domain predicates, but only by role names. The corresponding notion of ℓ-locality of a $\mathsf{FOL}(\mathfrak{D})$ formula and of $\mathbb{C}$-invariance w.r.t. classes $\mathbb{C}$ of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ are obtained by extending Definition 4.10 using this notion of neighborhood. In particular, the extension of $\mathbb{C}_{\mathsf{all}}$, $\mathbb{C}_{\mathsf{fb}}$, and $\mathbb{C}_{\mathsf{fin}}$ to interpretations taking feature names into account are defined in the obvious way, and these classes are localizable.

Lemma 8.5. *If $\mathbb{C}$ is localizable, then a $\mathsf{FOL}(\mathfrak{D})$ formula $\phi(x)$ of quantifier depth q that is $\mathbb{C}$-invariant under disjoint unions is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$.*

Proof. We adopt the same transformation used in Chapter 7 and [17, 15] to map $\phi(x)$ to a FOL formula $\phi^{\mathsf{FOL}}(x)$ of the same quantifier depth and $\mathcal{I} \in \mathbb{C}$ to a FOL interpretation $\mathcal{I}^{\mathsf{FOL}}$. Formally, we replace every atom $P(f_1, \dots, f_k)(t_1, \dots, t_k)$ in $\phi(x)$ with $P^{f_1,\dots,f_k}(t_1, \dots, t_k)$, where $P^{f_1,\dots,f_k}$ is a fresh k-ary predicate symbol for all k-ary relations P of $\mathfrak{D}$ and all $f_1, \dots, f_k \in \mathsf{N_F}$, and every atom of the form $\mathrm{Def}(f)(t)$ in $\phi(x)$ with $\mathrm{Def}_f(t)$ where Def_f is a new predicate symbol for $f \in \mathsf{N_F}$. No newly quantified variable is introduced in this transformation, and so $\phi^{\mathsf{FOL}}(x)$ has quantifier depth q, like $\phi(x)$. We associate to $\mathcal{I} \in \mathbb{C}$ an expansion $\mathcal{I}^{\mathsf{FOL}}$ by the following interpretation of the newly introduced predicates:

- $d \in (\mathrm{Def}_f)^{\mathcal{I}^{\mathsf{FOL}}}$ iff $f^{\mathcal{I}}(d)$ is defined
- $(d_1, \dots, d_k) \in (P^{f_1,\dots,f_k})^{\mathcal{I}^{\mathsf{FOL}}}$ iff $(f_1^{\mathcal{I}}(d_1), \dots, f_k^{\mathcal{I}}(d_k)) \in P^D$.

We denote with $\mathbb{C}^{\mathsf{FOL}}$ the resulting class of interpretations. By the semantics of $\mathsf{FOL}(\mathfrak{D})$, we obtain that for all $\mathsf{FOL}(\mathfrak{D})$ formula $\phi(x)$, all $\mathcal{I} \in \mathbb{C}$ and all $d \in \Delta^{\mathcal{I}}$

$$\mathcal{I} \models \phi(d) \text{ iff } \mathcal{I}^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}(d). \tag{$\star$}$$

We fix $d \in \Delta^{\mathcal{I}}$ and consider the ℓ-neighborhood $\mathcal{N}$ of d. Let $\mathcal{M}$ be the disjoint union of q copies $\mathcal{I}_1, \dots, \mathcal{I}_q$ of $\mathcal{I}$ and q copies $\mathcal{N}_1, \dots, \mathcal{N}_q$ of $\mathcal{N}$. We define $\mathcal{I}_\star$ as the disjoint union of $\mathcal{I}_0 := \mathcal{I}$ and $\mathcal{M}$, and $\mathcal{N}_\diamond$ as the disjoint union of $\mathcal{N}_0 := \mathcal{N}$ and $\mathcal{M}$. For each $e \in \Delta^{\mathcal{I}}$, $i = 0, \dots, q$ and $j = 1, \dots, q$ we denote with $(e, \mathcal{I}_i)_\star$ the individual in $\mathcal{I}_\star$ corresponding to $e \in \Delta^{\mathcal{I}_i}$ and with $(e, \mathcal{I}_j)_\diamond$ the individual in $\mathcal{N}_\diamond$ corresponding to $e \in \Delta^{\mathcal{I}_j}$. Similarly, if $e \in \Delta^{\mathcal{N}}$ then we introduce the notation $(e, \mathcal{N}_i)_\diamond$ and $(e, \mathcal{N}_j)_\star$.

Since $\mathbb{C}$ is localizable, we deduce that $\mathcal{I}_\star, \mathcal{N}_\diamond \in \mathbb{C}$. By $\mathbb{C}$-invariance under disjoint union of $\phi(x)$ we obtain that

$$\mathcal{I} \models \phi(d) \text{ iff } \mathcal{I}_\star \models \phi((d, \mathcal{I}_0)_\star) \quad\text{and}\quad \mathcal{N} \models \phi(d) \text{ iff } \mathcal{N}_\diamond \models \phi((d, \mathcal{N}_0)_\diamond),$$

and using ($\star$) we observe that

$$\mathcal{I}_\star \models \phi((d, \mathcal{I}_0)_\star) \text{ iff } \mathcal{I}_\star^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}((d, \mathcal{I}_0)_\star),$$
$$\mathcal{N}_\diamond \models \phi((d, \mathcal{N}_0)_\diamond) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}((d, \mathcal{N}_0)_\diamond).$$

We show that $(d, \mathcal{I}_0)_\star$ and $(d, \mathcal{N}_0)_\diamond$ are q-isomorphic. By Theorem 2.24, this implies that they satisfy the same FOL formulae of quantifier depth at most q and in particular that

$$\mathcal{I}_\star^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}((d, \mathcal{I}_0)_\star) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models \phi^{\mathsf{FOL}}((d, \mathcal{N}_0)_\diamond),$$

which together with all our previous observations implies that $\mathcal{I} \models \phi(d)$ iff $\mathcal{N} \models \phi(d)$, hence that $\phi(x)$ is ℓ-local.

Note that in this case, a partial isomorphism p between $\mathcal{I}_\star^{\mathsf{FOL}}$ and $\mathcal{N}_\diamond^{\mathsf{FOL}}$ must be defined not only in terms of $\mathsf{N_C}$ and $\mathsf{N_R}$ but also according to the newly introduced definedness and concrete predicates:

$$\mathcal{I}_\star^{\mathsf{FOL}} \models \mathrm{Def}_f(e) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models \mathrm{Def}_f(p(e)) \text{ and}$$
$$\mathcal{I}_\star^{\mathsf{FOL}} \models P^{f_1,\dots,f_k}(e_1,\dots,e_k) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models P^{f_1,\dots,f_k}(p(e_1),\dots,p(e_k))$$

must hold for all feature names $f, f_1, \dots, f_k$ and $e, e_1, \dots, e_k$ in $\mathcal{I}_\star^{\mathsf{FOL}}$. Nevertheless, we consider the *distance* of two individuals in $\mathcal{I}_\star^{\mathsf{FOL}}$ and $\mathcal{N}_\diamond^{\mathsf{FOL}}$ as introduced in Definition 4.10, i.e. in terms of elements connected by roles in $\mathsf{N_R}$ and thus consider neighborhoods in $\mathcal{I}_\star^{\mathsf{FOL}}$ and $\mathcal{N}_\diamond^{\mathsf{FOL}}$ according to this notion of distance.

Following Otto's construction in [86], we build a q-isomorphism $I_0, \dots, I_q$ such that for $i = 0, \dots, q$, $p \in I_{q-i}$ and all elements $e := (e', \mathcal{K})_\star$ of $\mathcal{I}_\star^{\mathsf{FOL}}$ for which p is defined we have that, having defined $\ell_i := (2^{q-i} - 1)$, the ℓ_i-neighborhoods of e and $p(e)$ are equal (up to renaming of the elements) and in particular that $p(e) = (e', \mathcal{K}')_\diamond$, where $\mathcal{K}$ and $\mathcal{K}'$ are any of the interpretations considered in the construction of $\mathcal{I}_\star$ and $\mathcal{N}_\diamond$. First, we set $I_q := \{\{(d, \mathcal{I}_0)_\star \mapsto (d, \mathcal{N}_0)_\diamond\}\}$. Since $\mathcal{N}$ is assumed to be the ℓ-neighborhood of d in $\mathcal{I}$ and $\ell = \ell_0$, it is clear that the mapping in I_q satisfies our requirement of equality of the neighborhoods. It is also trivial to see that this is a partial isomorphism w.r.t. $\mathsf{N_C}$, $\mathsf{N_R}$ and the newly introduced predicates, as a consequence of the fact that $f^{\mathcal{I}}(d) = f^{\mathcal{N}}(d)$ holds for all $f \in \mathsf{N_F}$. Assuming that we have defined I_{q-i} with $0 \leqslant i < q$, we show how to define $I_{q-(i+1)}$ so that i-forth Definition 2.23 is satisfied.

Let $p \in I_{q-i}$ and e an individual in $\mathcal{I}_\star^{\mathsf{FOL}}$. We show how to define a mapping p' that extends p by adding a value $p'(e)$ for e. First, we consider the case where every element e' for which $p(e')$ is defined has distance greater than $\ell_{i+1} + 1$ from e. If e is of the form $(d', \mathcal{N}_j)_\star$, we choose $p'(e)$ to be of the form $(d', \mathcal{N}_k)_\diamond$ for a value $1 \leqslant k \leqslant q$ such that no other element of the form $(d'', \mathcal{N}_k)_\diamond$ is in the image of p. This is always possible, since $\mathcal{N}_\diamond$ contains q copies of $\mathcal{N}$. Similarly, we treat the case where e is of the form $(d', \mathcal{I}_j)_\star$. Next, we consider the case where e has distance at most $\ell_{i+1} + 1$ from some element e' for which $p(e')$ is defined. By construction of I_{q-i} and the fact that $p \in I_{q-i}$, we deduce that e' and $p(e')$ have the same ℓ_i-neighborhoods up to renaming. Assuming that e' is of the form $(d', \mathcal{K})_\star$ with $\mathcal{K}$ of the form $\mathcal{I}_j$ or $\mathcal{N}_j$, we know that e is of the form $(d'', \mathcal{K})_\star$. Moreover, we know that $p(e')$ is of the form $(d'', \mathcal{K}')_\diamond$ with $\mathcal{K}'$ of the form $\mathcal{I}_j$ or $\mathcal{N}_j$, and we thus choose $p'(e)$ to be $(d'', \mathcal{K}')_\diamond$.

We verify that the ℓ_{i+1}-neighborhoods of e and $p'(e)$ are equal. Since for all other elements for which p' is defined this is a trivial consequence of $p \in I_{q-i}$, this is sufficient to conclude that p' satisfies this property for all the individuals on which is it defined. We distinguish two cases. In the first case, this is trivially a consequence of choosing the same individual w.r.t. the same original interpretation (either $\mathcal{N}$ or $\mathcal{I}$). In the second case, we have chosen the same individual (up to renaming) w.r.t. the identical ℓ_i-neighborhoods of two elements e' and $p(e')$ and both individuals have distance at most $\ell_{i+1} + 1$ from e' and $p(e')$ (respectively), which means that the ℓ_{i+1}-neighborhoods of e and $p'(e)$ are fully enclosed in the larger ℓ_i-neighborhoods of e' and $p(e')$ and thus are identical.

What is left is to prove that p' is a partial isomorphism w.r.t. $\mathsf{N_C}$, $\mathsf{N_R}$ and the newly introduced predicates. It is clear that p' is injective, because of the way we choose $p'(e)$ and by inductive hypothesis on p. It is also clear that, by this choice, $e \in A^{\mathcal{I}_\star^{\mathsf{FOL}}}$ iff $p'(e) \in A^{\mathcal{N}_\star^{\mathsf{FOL}}}$ holds for all $A \in \mathsf{N_C}$. Since p is a partial isomorphism w.r.t. $\mathsf{N_C}$ by inductive hypothesis this is sufficient to conclude that p' is a partial isomorphism w.r.t. $\mathsf{N_C}$. Next, we show that p' is a partial isomorphism w.r.t. $\mathsf{N_R}$. We notice that for all $e', e'' \in \mathcal{I}_\star^{\mathsf{FOL}}$ the fact that $(e, e') \in r^{\mathcal{I}_\star^{\mathsf{FOL}}}$ iff

$(p'(e), p'(e')) \in r^{\mathcal{N}_\star^{\mathsf{FOL}}}$ holds follows from the fact that in this case, e' and e'' must have distance $1 \leqslant \ell_{i+1}$ in $\mathcal{I}_\star^{\mathsf{FOL}}$, which means that the corresponding values of $p'(e')$ and $p'(e'')$ also have distance 1 and moreover are respectively equal to e' and e'' up to renaming. Finally, we show that p' is a partial isomorphism w.r.t. the newly introduced predicates. By construction of $\mathcal{N}$, $\mathcal{N}_\diamond$ and $\mathcal{I}_\star$ we know that $f^{\mathcal{N}}(e') = f^{\mathcal{I}}(e')$ for all $e' \in \Delta^{\mathcal{N}}$ and all $f \in \mathsf{N_F}$. Using the definition of disjoint union, we then obtain that $f^{\mathcal{I}_\star}((e', \mathcal{N}_j)_\star) = f^{\mathcal{I}}(e')$ and $f^{\mathcal{N}_\diamond}((e', \mathcal{N}_k)_\diamond) = f^{\mathcal{I}}(e')$ for all $j = 1, \dots, q$ and $k = 0, \dots, q$. Clearly, for all $e' \in \Delta^{\mathcal{I}}$ and all $f \in \mathsf{N_F}$ it also holds that $f^{\mathcal{I}_\star}((e', \mathcal{I}_k)_\star) = f^{\mathcal{I}}(e')$ and $f^{\mathcal{N}_\diamond}((e', \mathcal{N}_j)_\diamond) = f^{\mathcal{I}}(e')$ for $j = 1, \dots, q$ and $k = 0, \dots, q$. In other words, the feature values of each individual in $\mathcal{I}$ and $\mathcal{N}$ are duplicated over all copies. This clearly implies that $\mathcal{I}_\star^{\mathsf{FOL}} \models \mathrm{Def}_f(e)$ iff $\mathcal{N}_\diamond^{\mathsf{FOL}} \models \mathrm{Def}_f(p'(e))$, and this property is already satisfied for all other elements for which p' is defined by inductive hypothesis on p. Finally, we show that

$$\mathcal{I}_\star^{\mathsf{FOL}} \models P^{f_1,\dots,f_k}(e_1, \dots, e_k) \text{ iff } \mathcal{N}_\diamond^{\mathsf{FOL}} \models P^{f_1,\dots,f_k}(p'(e_1), \dots, p'(e_k)) \qquad (\dagger)$$

Assuming that $e_j = (e'_j, \mathcal{K}_j)$ for $j = 1, \dots, k$, we have that $p'(e_j) = (e'_j, \mathcal{K}'_j)$ by construction of p' and the inductive hypothesis on p. Combined with the above, we obtain that

$$\mathcal{I}_\star^{\mathsf{FOL}} \models P^{f_1,\dots,f_k}(e_1, \dots, e_k) \text{ iff } \mathcal{I} \models P(f_1, \dots, f_k)(e'_1, \dots, e'_k)$$

and

$$\mathcal{N}_\diamond^{\mathsf{FOL}} \models P^{f_1,\dots,f_k}(p'(e_1), \dots, p'(e_k)) \text{ iff } \mathcal{I} \models P(f_1, \dots, f_k)(e'_1, \dots, e'_k)$$

so we conclude that (†) holds.

We thus showed that p' is a partial isomorphism, and that $I_{q-(i+1)}$ satisfies the i-forth condition. Similarly, we show how to use $p \in I_{q-i}$ and $e' \in \mathcal{N}_\diamond^{\mathsf{FOL}}$ to add a partial isomorphism p' to $I_{q-(i+1)}$ such that $p'(e) = e'$ for some $e \in \mathcal{I}_\star^{\mathsf{FOL}}$, and therefore prove that I_{q-i} satisfies the i-back condition. Overall, we conclude that $I_0, \dots, I_q$ is a q-isomorphism.

As mentioned in the first part of the proof, this implies that $\mathcal{I} \models \phi(d)$ iff $\mathcal{N} \models \phi(c)$ and thus that $\phi(x)$ is ℓ-local w.r.t. $\mathbb{C}$. □

In the following, we assume that the concrete domain $\mathfrak{D}$ is WCUN and has finitely many relations; both conditions are always satisfied by ω-admissible concrete domains [79, 24]. Following the approach employed in the previous section, we introduce a bounded version of $\mathfrak{D}$ bisimulation, where now only the depth is bounded since there are no cardinality constraints.

Definition 8.6. *Let $\mathcal{I}$, $\mathcal{J}$ be interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ and $\ell \in \mathbb{N}$. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a $\mathfrak{D}$* 0-bisimulation *if ρ satisfies the atomic condition of Definition 8.1 and for all k-ary relations P of $\mathfrak{D}$ and $f_1, \dots, f_k \in \mathsf{N_F}$:*

values *if $(d, e) \in \rho$ then $(f_1^{\mathcal{I}}(d), \dots, f_k^{\mathcal{I}}(d)) \in P^D$ iff $(f_1^{\mathcal{J}}(e), \dots, f_k^{\mathcal{J}}(e)) \in P^D$.*

The relation ρ is a $\mathfrak{D}$ $(\ell+1)$-bisimulation *if it is a $\mathfrak{D}$ ℓ-bisimulation that additionally satisfies the features conditions of Definition 8.1, and for all $r \in \mathsf{N_R}$ the following are satisfied:*

*ℓ-**forth** if $(d, e) \in \rho$ and d' is a r-successor of d, then there exist a r-successor e' of e and a $\mathfrak{D}$ ℓ-bisimulation ρ' such that $(d', e') \in \rho'$;*

*ℓ-**back** if $(d, e) \in \rho$ and e' is a r-successor of e, then there exist a r-successor d' of d and a $\mathfrak{D}$ ℓ-bisimulation ρ' such that $(d', e') \in \rho'$.*

The notions of bisimilarity and $\mathbb{C}$-invariance w.r.t. $\mathfrak{D}$ ℓ-bisimulation are defined similarly to how it was done in Definition 4.2.

We show that, under the assumption that the concrete domain $\mathfrak{D}$ is WCUN and has finitely many relations, results analogous to Proposition 4.11, Corollary 4.13, Theorem 4.14, Proposition 4.16 and Theorem 4.18 also hold for $\mathsf{FOL}(\mathfrak{D})$ and $\mathcal{ALC}(\mathfrak{D})$, where $\mathcal{ALC}(\mathfrak{D})$ plays both the role of $\mathcal{ALCSCC}$ and of $\mathcal{ALCQ}t$. Since we can prove that every $\mathsf{FOL}(\mathfrak{D})$ formula that is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation is $\mathbb{C}$-invariant under finite disjoint unions similarly to what is done in Proposition 4.11 for FOL formulae w.r.t. Pr bisimulation, we obtain the following corollary which is analogous to Corollary 4.13.

Corollary 8.7. *If $\mathbb{C}$ is localizable, a $\mathsf{FOL}(\mathfrak{D})$ formula $\phi(x)$ of quantifier depth q that is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation is ℓ-local w.r.t. $\mathbb{C}$ for $\ell := 2^q - 1$.*

We obtain the notions of *tree* and *partial unravelling* to interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ by extending Definition 4.15 to feature names in the obvious way. For trees of depth ℓ, the corresponding version of Theorem 4.14 for $\mathfrak{D}$ ℓ-bisimulation is simplified to the following, where q-isomorphism is replaced by $\mathfrak{D}$ bisimilarity.

Lemma 8.8. *If $\mathcal{I}$, $\mathcal{J}$ are trees of depth ℓ with roots d, e that are $\mathfrak{D}$ ℓ-bisimilar, then these roots are $\mathfrak{D}$ bisimilar.*

Proof. We show that a $\mathfrak{D}$ ℓ-bisimulation ρ between d and e induces a $\mathfrak{D}$ bisimulation ρ' such that if $(d', e') \in \rho'$ then d' and e' have the same distance $0 \leqslant \ell' \leqslant \ell$ from d and e and are $\mathfrak{D}$ $(\ell - \ell')$-bisimilar.

We begin by setting $\rho' := \{(d, e)\}$. Clearly, the tuple (d, e) satisfies the property above with $\ell' = 0$. Assuming that $(d', e') \in \rho'$ are $\mathfrak{D}$ $(\ell - \ell')$-bisimilar, for every r-successor d'' of d' we add to ρ' a tuple (d'', e'') where e'' is an r-successor of e' that is $\mathfrak{D}$ $(\ell - (\ell' + 1))$-bisimilar to d''. This is always possible: if $\ell' < \ell$ then this is guaranteed by the $(\ell - \ell')$-forth condition, and if $\ell' = \ell$ then d' has no r-successors in $\mathcal{I}$ and so the above is vacuously true. In the first, both d'' and e'' have distance $\ell' + 1 \leqslant \ell$ from d and e. Similarly, for every r-successor e'' of e' we add to ρ' a tuple (d'', e'') where d'' is an r-successor of d' that is $\mathfrak{D}$ $(\ell - (\ell' + 1))$-bisimilar to e''.

We show that the relation ρ' obtained by exhaustively repeating the process above for $\ell' = 0, \dots, \ell$ is a $\mathfrak{D}$ bisimulation. Since $(d', e') \in \rho'$ implies that d' and e' are $\mathfrak{D}$ ℓ'-bisimilar with $\ell' \geqslant 0$, it clearly holds that ρ' satisfies the atomic condition. By construction of ρ', the forth and back conditions are also clearly satisfied. To see that features is satisfied by ρ', let $p_1, \dots, p_k$ be feature paths over $\mathsf{N_R}$ and $\mathsf{N_F}$. If $p_i = f_i$ holds for $i = 1, \dots, k$, then the values condition of $\mathfrak{D}$ ℓ-bisimulations applied to d' and e' implies that features is satisfied for p_1, ..., p_k. Otherwise, $p_i = r_i f_i$ holds for some $1 \leqslant i \leqslant k$. If $(v_1, \dots, v_k) \in P^D$ with $v_1 \in p_1^{\mathcal{I}}(d')$, ..., $v_k \in p_k^{\mathcal{I}}(d')$ then d' has some role successors, which means d' and e' are $\mathfrak{D}$ ℓ'-bisimilar with $\ell' > 0$ and so we use the features property of $\mathfrak{D}$ ℓ'-bisimulation to derive that there are $w_1 \in p_1^{\mathcal{J}}(e')$, ..., $w_k \in p_k^{\mathcal{J}}(e')$ such that $(w_1, \dots, w_k) \in P^D$. The other implication is proved similarly, and we conclude that ρ' satisfies the features property. Therefore, ρ' is a $\mathfrak{D}$ bisimulation. □

Similar to the case of Pr (q,ℓ)-bisimulation and q-isomorphism in Corollary 4.17, we obtain the following for $\mathfrak{D}$ ℓ-bisimulation on trees of depth ℓ.

Corollary 8.9. *If $\mathbb{C}$ is closed under partial unravelling and $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ contain $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$ that are $\mathfrak{D}$ ℓ-bisimilar, then $\langle d \rangle \in \Delta^{\mathcal{I}_\ell}$ and $\langle e \rangle \in \Delta^{\mathcal{J}_\ell}$ satisfy the same ℓ-local $\mathsf{FOL}(\mathfrak{D})$ formulae $\phi(x)$ that are $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation.*

By adapting the proof of Theorem 4.18 to use Corollary 8.9 instead of Corollary 4.17 and Corollary 8.7 instead of Corollary 4.13, we obtain the following analogous of Theorem 4.18 for $\mathfrak{D}$ bisimulation.

Theorem 8.10. *Let $\mathbb{C}$ be localizable and closed under partial unravelling. Then, a $\mathsf{FOL}(\mathfrak{D})$ formula $\phi(x)$ is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation iff it is $\mathbb{C}$-invariant under $\mathfrak{D}$ ℓ-bisimulation for some value of ℓ.*

To conclude, we will show that for $\mathsf{FOL}(\mathfrak{D})$ formulae $\phi(x)$ $\mathbb{C}$-invariance under $\mathfrak{D}$ ℓ-bisimulation implies $\mathbb{C}$-equivalence to some $\mathcal{ALC}(\mathfrak{D})$ concept C, where C is in particular a concept of depth ℓ. Similarly to what was done earlier for $\mathcal{ALCQ}t$, we define $\mathcal{ALC}(\mathfrak{D})_\ell$ as the subset of $\mathcal{ALC}(\mathfrak{D})$ whose concepts have nesting level at most ℓ, where the depth of a CD-restriction $\exists p_1, \dots, p_k.P$ is 1 if $p_i = r_i f_i$ for some $i = 1, \dots, k$ and 0 otherwise. As in the case of $\mathcal{ALCQ}t_{q,\ell}$ we observe that $\mathcal{ALC}(\mathfrak{D})_\ell$ is finite, up to $\mathbb{C}$-equivalence.

Proposition 8.11. *If $\mathfrak{D}$ has finitely many relations and $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ are finite, then $\mathcal{ALC}(\mathfrak{D})_\ell$ has finitely many concepts (up to $\mathbb{C}$-equivalence) for all $\ell \in \mathbb{N}$.*

Proof. If $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ are finite then there are only finitely many k-tuples of feature paths over $\mathsf{N_R}$ and $\mathsf{N_F}$ for all values of k; since $\mathfrak{D}$ has finitely many relations, this means that there are only finitely many CD-restrictions in $\mathcal{ALC}(\mathfrak{D})_\ell$.

We prove that our claim holds by induction over ℓ. For $\ell = 0$, this trivially holds because $\mathsf{N_C}$ is finite and by our observation regarding CD-restrictions. For the inductive step, we assume that the claim holds for ℓ and show that the same applies for $\ell + 1$. Every concept in $\mathcal{ALC}(\mathfrak{D})_{\ell+1}$ is a Boolean combination of CD-restrictions, $\mathcal{ALC}(\mathfrak{D})_\ell$ concepts and role restrictions $\exists r.C$ with $r \in \mathsf{N_R}$ and C a $\mathcal{ALC}(\mathfrak{D})_\ell$ concept. By inductive hypothesis, there can be only finitely many role restrictions of this form and $\mathcal{ALC}(\mathfrak{D})_\ell$ concepts (up to $\mathbb{C}$-equivalence). Together with our observation above on the number of CD-restrictions, we deduce that there can only be finitely many non-equivalent Boolean combinations of the described form. Therefore, we conclude that the claim holds for $\mathcal{ALC}(\mathfrak{D})_{\ell+1}$. □

Moreover, under our assumptions on $\mathfrak{D}$, we obtain the invariance of $\mathcal{ALC}(\mathfrak{D})_\ell$ under $\mathfrak{D}$ ℓ-bisimulation.

Proposition 8.12. *If $\mathfrak{D}$ is WCUN and has finitely many relations, then $\mathcal{ALC}(\mathfrak{D})_\ell$ concepts are invariant under $\mathfrak{D}$ ℓ-bisimulation.*

It is again possible, as in the case of Pr (q,ℓ)-bisimulation and $\mathcal{ALCQ}t_{q,\ell}$, to define a *characteristic ℓ-concept* in $\mathcal{ALC}(\mathfrak{D})_\ell$ that describes all individuals that are $\mathfrak{D}$ ℓ-bisimilar to $d \in \Delta^\mathcal{I}$. Assuming that $\mathbf{f}^\mathcal{I}(d) := (f_1^\mathcal{I}(d), \dots, f_k^\mathcal{I}(d))$ with $f_1, \dots, f_k \in \mathsf{N_F}$ and that $p_1, \dots, p_k$ are feature paths over $\mathsf{N_R}$, $\mathsf{N_F}$ we define

$$\begin{aligned}
\mathsf{Values}_\exists[d] &:= \textstyle\bigsqcap\{\exists f_1,\dots,f_k.P \mid \mathbf{f}^\mathcal{I}(d) \in P^D\} \\
\mathsf{Values}_\forall[d] &:= \textstyle\bigsqcap\{\forall f_1,\dots,f_k.P \mid \text{if } f_1^\mathcal{I}(d),\dots,f_k^\mathcal{I}(d) \text{ are defined then } \mathbf{f}^\mathcal{I}(d) \in P^D\} \\
\mathsf{Features}_\exists[d] &:= \textstyle\bigsqcap\{\exists p_1,\dots,p_k.P \mid \textit{some} \text{ tuple in } p_1^\mathcal{I}(d) \times \dots \times p_k^\mathcal{I}(d) \text{ is in } P^D\} \\
\mathsf{Features}_\forall[d] &:= \textstyle\bigsqcap\{\forall p_1,\dots,p_k.P \mid \textit{every} \text{ tuple in } p_1^\mathcal{I}(d) \times \dots \times p_k^\mathcal{I}(d) \text{ is in } P^D\}
\end{aligned}$$

By Proposition 8.11, these concepts are well-defined. Next, we define

$$\begin{aligned}
\mathsf{Forth}_\ell[d] &:= \textstyle\bigsqcap_{r\in\mathsf{N_R}}\bigsqcap_{e\in r^\mathcal{I}(d)}\exists r.\mathsf{Bisim}_\ell[e] \\
\mathsf{Back}_\ell[d] &:= \textstyle\bigsqcap_{r\in\mathsf{N_R}}\forall r.(\bigsqcup_{e\in r^\mathcal{I}(d)}\mathsf{Bisim}_\ell[e])
\end{aligned}$$

and finally introduce the concepts

$$\mathsf{Bisim}_0[d] := \mathsf{Values}_\exists[d] \sqcap \mathsf{Values}_\forall[d] \sqcap \mathsf{Atomic}[d]$$
$$\mathsf{Bisim}_{\ell+1}[d] := \mathsf{Bisim}_\ell[d] \sqcap \mathsf{Features}_\exists[d] \sqcap \mathsf{Features}_\forall[d] \sqcap \mathsf{Forth}_\ell[d] \sqcap \mathsf{Back}_\ell[d]$$

where the concept $\mathsf{Atomic}[d]$ is defined as in Definition 4.24.

Theorem 8.13. *If $\mathfrak{D}$ is WCUN and has finitely many relations and $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ are finite then $d \in \Delta^\mathcal{I}$, $e \in \Delta^\mathcal{J}$ are $\mathfrak{D}$ ℓ-bisimilar iff they satisfy the same $\mathcal{ALC}(\mathfrak{D})_\ell$ concepts.*

Proof. The proof is similar to that of Theorem 4.25, where we additionally need to test that $\rho_\ell := \{(d, e) \in \Delta^\mathcal{I} \times \Delta^\mathcal{J} \mid e \in (\mathsf{Bisim}_\ell[d])^\mathcal{J}\}$ satisfies the values condition and additionally the features condition if $\ell > 0$. To prove that ρ_ℓ satisfies values, assume that for $f_1, \dots, f_k \in \mathsf{N_F}$ and a k-ary relation P of $\mathfrak{D}$ it holds that $(f_1^\mathcal{I}(d), \dots, f_k^\mathcal{I}(d)) \in P^D$. Then, $\exists f_1, \dots, f_k.P$ is a conjunct of $\mathsf{Values}_\exists[d]$, and since $e \in \mathsf{Bisim}_\ell[d]^\mathcal{J}$ we derive that $(f_1^\mathcal{J}(e), \dots, f_k^\mathcal{J}(e)) \in P^D$. Vice versa, if $(f_1^\mathcal{I}(d), \dots, f_k^\mathcal{I}(d)) \notin P^D$, then by WCUN we find k-ary predicates P_1, ..., P_{n_P} such that if $f_j^\mathcal{I}(d)$ is defined for $j = 1, \dots, k$ then $(f_1^\mathcal{I}(d), \dots, f_k^\mathcal{I}(d)) \in P_i^D$ holds for some $1 \leqslant i \leqslant n_P$. This means that $\forall f_1, \dots, f_k.P_i$ is a conjunct of $\mathsf{Values}_\forall[d]$. Then, either $f_j^\mathcal{J}(e)$ is undefined for some $1 \leqslant j \leqslant k$ or $(f_1^\mathcal{J}(e), \dots, f_k^\mathcal{J}(e)) \in P_i^D$ holds, and by WCUN this implies that $(f_1^\mathcal{J}(e), \dots, f_k^\mathcal{J}(e)) \notin P^D$. We conclude that ρ_ℓ satisfies values. Similarly, we verify that ρ_ℓ satisfies features if $\ell > 0$, taking care of replacing tuples $f_1, \dots, f_k \in \mathsf{N_F}$ with tuples $p_1, \dots, p_k$ of feature paths and replacing $\mathsf{Values}_\exists[d]$, $\mathsf{Values}_\forall[d]$ with $\mathsf{Features}_\exists[d]$, $\mathsf{Features}_\forall[d]$. □

Similarly to the proof of Theorem 4.18, these results can be combined to show the following characterization of $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is invariant under $\mathfrak{D}$ bisimulation.

Theorem 8.14. *Let $\mathbb{C}$ be localizable and closed under partial unravelling, $\mathfrak{D}$ be WCUN and have finitely many relations, and $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ be finite. Then the following are equivalent for all $\mathsf{FOL}(\mathfrak{D})$ formulae $\phi(x)$:*

1. *$\phi(x)$ is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation;*
2. *$\phi(x)$ is $\mathbb{C}$-invariant under $\mathfrak{D}$ ℓ-bisimulation for some $\ell \in \mathbb{N}$;*
3. *$\phi(x)$ is equivalent to an $\mathcal{ALC}(\mathfrak{D})$ concept.*

Recall that, in contrast to the case of $\mathcal{ALCSCC}$, where there are concepts that are not FOL-definable, every $\mathcal{ALC}(\mathfrak{D})$ concept is $\mathsf{FOL}(\mathfrak{D})$-definable.

8.2 The Expressive Power of $\mathcal{ALC}_{\mathsf{V+}}(\mathfrak{D})$ and $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{D})$

The notion of $\mathfrak{D}$ bisimulation we introduced is tailored to the expressive power of $\mathcal{ALC}(\mathfrak{D})$, where CD-restrictions can only be the form $\exists p_1, \dots, p_k.P$ or $\forall p_1, \dots, p_k.P$. Here, we consider several variants of the notion of $\mathfrak{D}$ bisimulation, and show how they reflect the different expressivity of the constraints that are allowed in CD-restrictions in the DLs $\mathcal{ALC}(\mathfrak{D})$, $\mathcal{ALC}_{\mathsf{V+}}(\mathfrak{D})$ and $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{D})$. They are defined as follows.

Definition 8.15. *Let $\mathcal{I}$, $\mathcal{J}$ be interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ and $q \in \mathbb{N}$. If p_1, ..., p_k are feature paths over $\mathsf{N_R}$, $\mathsf{N_F}$ and $P_1, \dots, P_n$ are k-ary relations of $\mathfrak{D}$, a $\mathfrak{D}$ bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$ (cf. Definition 8.1) is*

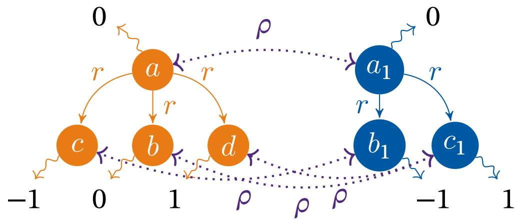


Figure 8.2: A concrete bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$.

combined *if* $(d, e) \in \rho$ *implies that there is* $(v_1, \dots, v_k) \in P_j^D$ *with* $v_1 \in p_1^{\mathcal{I}}(d)$, *...*, $v_k \in p_k^{\mathcal{I}}(d)$ *and* $j = 1, \dots, n$ *iff there is* $(w_1, \dots, w_k) \in P_j^D$ *with* $w_1 \in p_1^{\mathcal{J}}(e)$, *...*, $w_k \in p_k^{\mathcal{J}}(e)$ *and* $j = 1, \dots, n$*;*

q-***isomorphic*** *if* $(d, e) \in \rho$ *implies that for all* $k \leqslant q$ *and for all* $v_1 \in p_1^{\mathcal{I}}(d)$, *...*, $v_k \in p_k^{\mathcal{I}}(d)$ *there are* $w_1 \in p_1^{\mathcal{J}}(e)$, *...*, $w_k \in p_k^{\mathcal{J}}(e)$ *such that the mapping* $v_i \mapsto w_i$ *for* $i = 1, \dots, k$ *is a partial isomorphism (and vice versa).*

Combined $\mathfrak{D}$ bisimulation and $\mathcal{ALC}_{\vee+}(\mathfrak{D})$

We recall that $\mathcal{ALC}_{\vee+}(\mathfrak{D})$ is the extension of $\mathcal{ALC}$ introduced in [79] with CD-restrictions of the form $\exists p_1, \dots, p_k.\phi$ or $\forall p_1, \dots, p_k.\phi$ where ϕ is a disjunction of k-ary predicates $P_i(x_1, \dots, x_k)$ with $i = 1, \dots, n$. In the setting of that paper, where $\mathfrak{D}$ is assumed to ω-admissible and thus JEPD, these restrictions do not increase the expressive power compared to $\mathcal{ALC}(\mathfrak{D})$. Indeed, $\exists p_1, \dots, p_k.\phi$ is equivalent to the disjunction of all restrictions $\exists p_1, \dots, p_k.P_i$ with $i = 1, \dots, n$, while $\forall p_1, \dots, p_k.\phi$ is equivalent to $\neg\exists p_1, \dots, p_k.\phi'$ where ϕ' is the disjunction of all k-ary predicates that do not occur in ϕ.

On the other hand, we can find concrete domains $\mathfrak{D}$ that are WCUN and not JEPD and such that $\mathcal{ALC}_{\vee+}(\mathfrak{D})$ is strictly more expressive than $\mathcal{ALC}(\mathfrak{D})$.

Here, we consider the concrete domain $\mathfrak{Q}_{\mathsf{ord}} := (\mathbb{Q}, <, \leqslant, \geqslant, >)$ with the set of rational numbers as domain and standard binary ordering relations, which is WCUN and not JEPD.

Proposition 8.16. *There is an* $\mathcal{ALC}_{\vee+}(\mathfrak{Q}_{\mathsf{ord}})$ *concept that is not* $\mathbb{C}_{\mathsf{fin}}$*-equivalent to any* $\mathcal{ALC}(\mathfrak{Q}_{\mathsf{ord}})$ *concept.*

Proof. Consider the $\mathcal{ALC}_{\vee+}(\mathfrak{Q}_{\mathsf{ord}})$ concept $C := \forall f, rf.(<(x, y) \vee >(x, y))$ and assume that C is $\mathbb{C}_{\mathsf{fin}}$-equivalent to some $\mathcal{ALC}(\mathfrak{Q}_{\mathsf{ord}})$ concept C'. Using the finite interpretations $\mathcal{I}$ and $\mathcal{J}$ in Figure 8.2, which are related by a $\mathfrak{D}$ bisimulation ρ, and the fact that $(a, a_1) \in \rho$, we deduce by Theorem 8.2 that $a \in C'^{\mathcal{I}}$ iff $a_1 \in C'^{\mathcal{J}}$. This contradicts our assumption that C and C' are equivalent, because $a \notin C^{\mathcal{I}}$, while $a_1 \in C^{\mathcal{J}}$. □

In $\mathcal{ALC}_{\vee+}(\mathfrak{D})$ we gain additional expressivity thanks to the presence of disjunctions of atomic predicates of the same arity over the same tuple. If $\mathfrak{D}$ is WCUN, then combined $\mathfrak{D}$ bisimulations address the fact that unsatisfied universal CD-restrictions in this DL are witnessed by a tuple of values, associated to feature paths, that belongs to the intersection of one or more relations.

Proposition 8.17. *If* $\mathfrak{D}$ *is WCUN, then every* $\mathcal{ALC}_{\vee+}(\mathfrak{D})$ *concept is* $\mathbb{C}$*-invariant under combined* $\mathfrak{D}$ *bisimulation.*

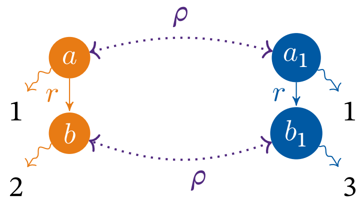


Figure 8.3: A combined $\mathfrak{D}$ bisimulation ρ between $\mathcal{I}$ and $\mathcal{J}$ where the concrete domain is $\mathfrak{N}$.

Proof. The cases for concepts names, concept negation and conjunction as well as existential CD-restrictions $\exists p_1, \dots, p_k.\phi$ are equal to the ones in Theorem 8.2. This last case is already covered, because we can distribute the disjunction of relations over the existential CD-restriction $\exists p_1, \dots, p_k.\phi$ and obtain a $\mathbb{C}$-equivalent concept that is a disjunction of simple CD-restrictions $\exists p_1, \dots, p_k.P$. Thus, we only need to show that universal CD-restrictions of the form $\forall p_1, \dots, p_k.\phi$ with $\phi := \bigvee_{i=1}^{n} P_i(x_1, \dots, x_k)$ are $\mathbb{C}$-invariant under combined $\mathfrak{D}$ bisimulation. Let ρ be a combined $\mathfrak{D}$ bisimulation with $(d, e) \in \rho$.

If $d \notin (\forall p_1, \dots, p_k.\phi)^{\mathcal{I}}$ then there are $v_1 \in p_1^{\mathcal{I}}(d)$, …, $v_k \in p_k^{\mathcal{I}}(d)$ such that $(v_1, \dots, v_k) \notin P_i^D$ for $i = 1, \dots, n$. Since $\mathfrak{D}$ is WCUN, we can find k-ary relations P_1', …, P_n' such that $(v_1, \dots, v_k) \in (P_i')^D$ for $i = 1, \dots, n$. Given that ρ is a concrete $\vee^+$-bisimulation, we can find $w_1 \in p_1^{\mathcal{J}}(e)$, …, $w_k \in p_k^{\mathcal{J}}(e)$ such that $(w_1, \dots, w_k) \in {P_i'}^D$ and thus $(w_1, \dots, w_k) \notin P_i^D$ for $i = 1, \dots, n$, hence $e \notin (\forall p_1, \dots, p_k.\phi)^{\mathcal{J}}$. Likewise, we show that $e \notin (\forall p_1, \dots, p_k.\phi)^{\mathcal{J}}$ implies $d \notin (\forall p_1, \dots, p_k.\phi)^{\mathcal{I}}$. □

q-isomorphic bisimulation and $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{D})$

The DL $\mathcal{ALC}_{\mathsf{fo}}^q(\mathfrak{D})$ extends $\mathcal{ALC}$ by allowing CD-restrictions $\exists p_1, \dots, p_k.\phi$ and $\forall p_1, \dots, p_k.\phi$ where ϕ is a FOL formula over the relations of $\mathfrak{D}$ with $k \leqslant q$ free variables. Then, $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{D})$ [24] is the union of all DLs $\mathcal{ALC}_{\mathsf{fo}}^q(\mathfrak{D})$ with $q \in \mathbb{N}$. Earlier, we showed that for JEPD structures $\mathfrak{D}$ the DLs $\mathcal{ALC}(\mathfrak{D})$ and $\mathcal{ALC}_{\vee+}(\mathfrak{D})$ have the same expressive power. Here, we show that this is not the case for $\mathcal{ALC}_{\vee+}(\mathfrak{D})$ and $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{D})$, by looking at the JEPD concrete domain $\mathfrak{N} := (\mathbb{N}, <, =, >)$.

Proposition 8.18. *There exists a $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{N})$ concept that is not $\mathbb{C}_{\mathsf{fin}}$-equivalent to any $\mathcal{ALC}_{\vee+}(\mathfrak{N})$ concept.*

Proof. Consider the $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{N})$ concept (we use infix notation to ease the reading experience)

$$C := \forall f, rf.(x < y \to \exists z.(x < y \wedge y < z))$$

and assume that C is $\mathbb{C}_{\mathsf{fin}}$-equivalent to some $\mathcal{ALC}_{\vee+}(\mathfrak{N})$ concept C'. Let $\mathcal{I}$ and $\mathcal{J}$ be the two interpretations depicted in Figure 8.3. Given that the relation ρ relating $\mathcal{I}$ and $\mathcal{J}$ is a combined $\mathfrak{D}$ bisimulation that relates a and a_1, we infer from Proposition 8.17 that $a \in (C')^{\mathcal{I}}$ iff $a_1 \in (C')^{\mathcal{J}}$. This contradicts our assumption that C and C' are $\mathbb{C}_{\mathsf{fin}}$-equivalent, because $a \notin C^{\mathcal{I}}$ but $a_1 \in C^{\mathcal{J}}$. □

If we consider two finite and isomorphic substructures $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of a concrete domain $\mathfrak{D}$, then we know that they satisfy the same quantifier-free first-order formulae over the relations of $\mathfrak{D}$. This implies that the partial isomorphism enforced by a isomorphic $\mathfrak{D}$ bisimulation ρ with $(d, e) \in \rho$ between the tuple $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ is enough to ensure that d and e satisfy the same CD-restrictions $\exists p_1, \dots, p_k.\psi(x_1, \dots, x_k)$ and $\forall p_1, \dots, p_k.\psi(x_1, \dots, x_k)$ where

$\psi(x_1, \dots, x_k)$ is quantifier-free. However, this condition alone is not sufficient to cover CD-restrictions of the form above where $\psi(x_1, \dots, x_k)$ contains quantified variables. If we assume $\mathfrak{D}$ to be homogeneous (cf. Chapter 6), though, this turns out to be all we need to capture the full expressive power of $\mathcal{ALC}^q_{\mathsf{fo}}(\mathfrak{D})$.

Theorem 8.19. *If $\mathfrak{D}$ is homogeneous, then every $\mathcal{ALC}^q_{\mathsf{fo}}(\mathfrak{D})$ concept is $\mathbb{C}$-invariant under q-isomorphic $\mathcal{ALC}(\mathfrak{D})$ bisimulation.*

Proof. Similarly to the proof of Theorem 8.2, we only need to show that if $\mathfrak{D}$ is homogeneous and ρ is a isomorphic $\mathfrak{D}$ bisimulation between $\mathcal{I}, \mathcal{J} \in \mathbb{C}$ with $(d, e) \in \rho$ then d and e satisfy the same existential and universal CD-restrictions in $\mathcal{ALC}^q_{\mathsf{fo}}(\mathfrak{D})$. Clearly, $\forall p_1, \dots, p_k.\phi$ and $\neg\exists p_1, \dots, p_k.\, \neg\phi$ are $\mathbb{C}$-equivalent, and thus we only need to show that d and e satisfy the same existential CD-restrictions.

If $d \in (\exists p_1, \dots, p_k.\, \phi)^{\mathcal{I}}$ with $k \leqslant q$, then there are $v_1 \in p_1^{\mathcal{I}}(d)$, ..., $v_k \in p_k^{\mathcal{I}}(d)$ with $k \leqslant q$ such that $(v_1, \dots, v_k) \in \phi^D$. Since ρ is a q-isomorphic $\mathfrak{D}$ bisimulation, there are $w_1 \in p_1^{\mathcal{J}}(e)$, ..., $w_k \in p_k^{\mathcal{J}}(e)$ such that the mapping $h\colon v_i \mapsto w_i$ for $i = 1, \dots, k$ acts as a partial isomorphism. Since $\mathfrak{D}$ is homogeneous, this implies that h can be extended to an isomorphism from $\mathfrak{D}$ to itself. Every first-order formula is invariant under isomorphisms from a structure to itself: therefore, $(w_1, \dots, w_k) \in \phi^D$ holds, and we obtain that $e \in (\exists p_1, \dots, p_k.\, \phi)^{\mathcal{J}}$. Likewise, we show that $d \in (\exists p_1, \dots, p_k.\, \phi)^{\mathcal{I}}$ if $e \in (\exists p_1, \dots, p_k.\, \phi)^{\mathcal{J}}$ and conclude that d and e satisfy the same existential CD-restrictions in $\mathcal{ALC}^q_{\mathsf{fo}}(\mathfrak{D})$. We conclude that every $\mathcal{ALC}^q_{\mathsf{fo}}(\mathfrak{D})$ concept is $\mathbb{C}$-invariant under q-isomorphic $\mathfrak{D}$ bisimulation. □

Relating variants of concrete bisimulation

In Theorem 8.14 we characterized $\mathcal{ALC}(\mathfrak{D})$ as the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is $\mathbb{C}$-invariant under $\mathfrak{D}$ bisimulation, under some assumptions on $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$, on the class $\mathbb{C}$ of interpretations and on $\mathfrak{D}$. In particular, we assumed $\mathfrak{D}$ to be WCUN abd have only finitely many relations.

Instead of repeating the same procedure for $\mathcal{ALC}_{\mathsf{V+}}(\mathfrak{D})$ and $\mathcal{ALC}^q_{\mathsf{fo}}(\mathfrak{D})$, we provide results that we can use to lift Theorem 8.14 to the more expressive settings by choosing an appropriate concrete domain $\mathfrak{D}'$, derived from $\mathfrak{D}$ and with the same domain set, and characterize $\mathbb{C}$-invariance under combined or isomorphic $\mathfrak{D}$ in terms of $\mathbb{C}$-invariance under $\mathfrak{D}'$ bisimulation. Then, the fact that $\mathcal{ALC}(\mathfrak{D}')$ is the fragment of $\mathsf{FOL}(\mathfrak{D}')$ that is $\mathbb{C}$-invariant under $\mathfrak{D}'$ bisimulation implies that $\mathcal{ALC}_{\mathsf{V+}}(\mathfrak{D})$ is the fragment of $\mathsf{FOL}(\mathfrak{D})$ that is $\mathbb{C}$-invariant under combined $\mathfrak{D}$ bisimulation and that $\mathcal{ALC}^q_{\mathsf{fo}}(\mathfrak{D})$ is the fragment that is $\mathbb{C}$-invariant under q-isomorphic $\mathfrak{D}$ bisimulation.

Let $\mathfrak{D}$ be a homogeneous concrete domain whose domain set is countable and that has finitely many relations. We recall that a k-ary relation P is FOL-*definable over* $\mathfrak{D}$ iff there is a first-order formula ψ over the relations of $\mathfrak{D}$ and with k free variables such that $P^D = \psi^D$. Clearly, if $\mathfrak{D}$ has finitely many relations, then for every $k \in \mathbb{N}$ there are only finitely many k-ary relations that we can define with a quantifier-free FOL formula over $\mathfrak{D}$. If $\mathfrak{D}$ is in addition homogeneous, the following result guarantees that the same holds even for relations that are defined by formulae with quantified variables. This is a combination of well-known results from model theory as described in [24] (Theorem 3 and 4).

Theorem 8.20 ([24]). *A countable relational structure $\mathfrak{D}$ with a finite signature is homogeneous iff it admits quantifier elimination and for all $k \in \mathbb{N}$ there are only finitely many k-ary relations that are first-order definable over $\mathfrak{D}$.*

Motivated by these results, we introduce the following concrete domain extensions by definable relations.

Definition 8.21. *A* partial k-orbit *over a concrete domain $\mathfrak{D}$ is a non-empty set O of k-ary relations of $\mathfrak{D}$. Every partial k-orbit induces a k-ary relation R_O over $\mathfrak{D}$ defined as $R_O^D := \bigcap_{R \in O} R^D$. If $\mathfrak{D}$ has finitely many relations, the* orbital decomposition *of $\mathfrak{D}$ is the structure $\mathfrak{D}^o$ obtained by replacing all relations over $\mathfrak{D}$ with all relations R_O that are induced by a partial k-orbit O over $\mathfrak{D}$ with $k \in \mathbb{N}$. Given $q \in \mathbb{N}$, the* first-order q-expansion *of $\mathfrak{D}$ is the structure $\mathfrak{D}^q_{\mathsf{FOL}}$ obtained by expanding $\mathfrak{D}$ with all first-order definable k-ary relations over $\mathfrak{D}$ with $k \leqslant q$.*

Thanks to Theorem 8.20, we obtain that these transformations preserve finiteness of the set of concrete domain relations and WCUN.

Corollary 8.22. *Let $\mathfrak{D}$ be WCUN and have finitely many relations. Then, the orbital decomposition of $\mathfrak{D}$ is WCUN and has finitely many relations. If $\mathfrak{D}$ is additionally homogeneous, then the* FOL *q-expansion of $\mathfrak{D}$ is WCUN and has finitely many relations for all $q \in \mathbb{N}$.*

Proof. In the first case, we must prove that $\mathfrak{D}^o$ is WCUN. Let R be a k-ary relation of $\mathfrak{D}^o$, induced by some partial k-orbit $O = \{R_1, \dots, R_m\}$ over $\mathfrak{D}$. Then, $(d_1, \dots, d_k) \notin R^{D^o}$ iff $(d_1, \dots, d_k) \notin R_i^D$ for $i = 1, \dots, m$. Since $\mathfrak{D}$ is WCUN, $(d_1, \dots, d_k) \notin R_i^D$ happens iff there are k-ary relations R_i^1,...,$R_i^{n_i}$ of $\mathfrak{D}$ such that $(d_1, \dots, d_k) \in \bigcup_{j=1}^{n_i} (R_i^j)^D$. Thus, $(d_1, \dots, d_k) \notin R^{D^o}$ iff $(d_1, \dots, d_k) \in \bigcap_{i=1}^m \bigcup_{j=1}^{n_i} (R_i^j)^D$. By distributing set intersection over set union, we obtain that $(d_1, \dots, d_k)$ is in the union of sets of the form $(R'_1)^D \cap \dots \cap (R'_n)^D$, each of which corresponds to the orbit $O' := \{R'_1, \dots, R'_n\}$. If O_1, ..., O_n are these k-ary orbits, then $(d_1, \dots, d_k) \in (R_{O_1} \vee \dots \vee R_{O_n})^D$ holds, which means that R_{O_1}, ..., R_{O_n} represent the complement of R_O.

In the second case, it is clear that every k-ary relation R_ϕ of $\mathfrak{D}^q_{\mathsf{FOL}}$ induced by a FOL formula $\phi(x_1, \dots, x_k)$ has a complementary k-ary relation R^c_ϕ defined by $\neg\phi(x_1, \dots, x_k)$.

Finally, the finiteness of the relation sets for the two concrete domains has been discussed above. □

We are now ready to prove the relation between $\mathbb{C}$-invariance under variants of concrete bisimulation w.r.t. to $\mathfrak{D}$, its orbit decomposition and its first-order q-expansion.

Theorem 8.23. *Let $\mathbb{C}$ be a class of interpretations of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$ and $\mathfrak{D}$ be WCUN. The following hold for every* FOL($\mathfrak{D}$) *formula $\phi(x)$:*

1. *$\phi(x)$ is $\mathbb{C}$-invariant under combined $\mathfrak{D}$ bisimulation iff it is $\mathbb{C}$-invariant under $\mathfrak{D}^o$ bisimulation;*
2. *if $\mathfrak{D}$ is homogeneous, then $\phi(x)$ is $\mathbb{C}$-invariant under q-isomorphic $\mathfrak{D}$ bisimulation iff it is $\mathbb{C}$-invariant under $\mathfrak{D}^{\mathsf{fo}}$ bisimulation.*

Proof. First, we show that a binary relation ρ between interpretations $\mathcal{I}$ and $\mathcal{J}$ is a combined $\mathfrak{D}$ bisimulation iff it is a $\mathfrak{D}^o$ bisimulation. Let ρ be a $\mathfrak{D}^o$ bisimulation and $(d, e) \in \rho$. We consider the k-ary relations P_1, ..., P_n over $\mathfrak{D}$. Then, there are $v_1 \in p_1^{\mathcal{I}}(d)$, ..., $v_k \in p_k^{\mathcal{I}}(d)$ such that $(v_1, \dots, v_k) \in P_i^D$ for $i = 1, \dots, n$ iff $(v_1, \dots, v_k) \in R_O^{D^o}$, where R_O is the k-ary relation induced by the partial k-orbit $O := \{P_1, \dots, P_n\}$. Since ρ is a $\mathfrak{D}^o$ bisimulation, this happens iff there are values $w_1 \in p_1^{\mathcal{J}}(e)$, ..., $w_k \in p_k^{\mathcal{J}}(e)$ such that $(w_1, \dots, w_k) \in R_O^{D^o}$, which holds iff $(w_1, \dots, w_k) \in P_i^D$ for $i = 1, \dots, n$. Therefore, ρ is a combined $\mathfrak{D}$ bisimulation. Similarly, we prove that if ρ is a combined $\mathfrak{D}$ bisimulation then it is a $\mathfrak{D}^o$ bisimulation.

Next, we show that ρ is a q-isomorphic $\mathfrak{D}$ bisimulation iff it is a $\mathfrak{D}^q_{\mathsf{FOL}}$ bisimulation. Let ρ be a $\mathfrak{D}^q_{\mathsf{FOL}}$ bisimulation and $(d, e) \in \rho$. Then,for all $v_1 \in p_1^{\mathcal{I}}(d)$, ..., $v_k \in p_k^{\mathcal{I}}(d)$ with $k \leqslant q$ there is a FOL formula $\psi(x_1, \dots, x_k)$ over the relations of $\mathfrak{D}$ that describes the isomorphism type of $(v_1, \dots, v_k)$. This means in particular that $(v_1, \dots, v_k) \in R_\psi^{D^q_{\mathsf{FOL}}}$ where R_ψ is the k-ary relation of $\mathfrak{D}^q_{\mathsf{FOL}}$ defined by ψ. Since ρ is a $\mathfrak{D}^q_{\mathsf{FOL}}$ bisimulation, this means that there are $w_1 \in p_1^{\mathcal{J}}(e)$, ..., $w_k \in p_k^{\mathcal{J}}(e)$ such that $(w_1, \dots, w_k) \in R_\psi^{D^q_{\mathsf{FOL}}}$, which in turn implies that the mapping $v_i \mapsto w_i$ for $i = 1, \dots, k$ is a partial isomorphism. Similarly, we show the "vice versa" direction and conclude that ρ is a q-isomorphic $\mathfrak{D}$ bisimulation. On the other hand, let ρ be a q-isomorphic $\mathfrak{D}$ bisimulation and $(d, e) \in \rho$. Every relation R of $\mathfrak{D}^{\mathsf{fo}}$ is first-order definable over $\mathfrak{D}$ and thus preserved by automorphisms over this structure; since $\mathfrak{D}$ is homogeneous, every partial isomorphism $v_i \mapsto w_i$ with $v_i, w_i \in D$ for $i = 1, \dots, k$ can be extended to an automorphism over $\mathfrak{D}$, which means that $(v_1, \dots, v_k) \in R^{D^{\mathsf{fo}}}$ implies $(w_1, \dots, w_k) \in R^{D^{\mathsf{fo}}}$. □

By combining Theorems 8.14 and 8.23 we obtain the characterizations of the extensions of $\mathcal{ALC}(\mathfrak{D})$ in terms of the newly introduced variants of $\mathfrak{D}$ bisimulation.

Corollary 8.24. *Let $\mathbb{C}$ be localizable and closed under partial unravelling,* $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ *be finite and $\mathfrak{D}$ be WCUN and with finitely many relations. Then, the following are equivalent:*

- *$\phi(x)$ is $\mathbb{C}$-invariant under combined $\mathfrak{D}$ bisimulation;*
- *$\phi(x)$ is $\mathbb{C}$-equivalent to a $\mathcal{ALC}_{\mathsf{V+}}(\mathfrak{D})$ concept.*

Corollary 8.25. *Let $\mathbb{C}$ be localizable and closed under partial unravelling,* $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_F}$ *be finite and $\mathfrak{D}$ be homogeneous, WCUN and with finitely many relations. Then, the following are equivalent:*

- *$\phi(x)$ is $\mathbb{C}$-invariant under q-isomorphic $\mathfrak{D}$ bisimulation;*
- *$\phi(x)$ is $\mathbb{C}$-equivalent to a $\mathcal{ALC}^q_{\mathsf{fo}}(\mathfrak{D})$ concept.*

Summary

We introduced the notion of $\mathfrak{D}$ bisimulation, which we used to characterize the expressive power of $\mathcal{ALC}(\mathfrak{D})$ w.r.t. FOL($\mathfrak{D}$), as well as to show that several of its extensions are strictly more expressive, similarly to what was done in Chapter 4. We further compared different, more restrictive notions of $\mathfrak{D}$ bisimulation, and showed how to relate them to obtain characterizations for $\mathcal{ALC}_{\mathsf{V+}}(\mathfrak{D})$ and $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{D})$.

9 Concrete Domains Meet Cardinality Constraints

In this chapter we investigate the DL $\mathcal{ALCOSCC}(\mathfrak{D})$, extending both $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$ with nominals and by further allowing within succ-restrictions the usage of *feature roles* that relate the feature values of an individual and of one of its successors. We show that the consistency problem for TBoxes in this DL is ExpTime-complete if $\mathfrak{D}$ is ExpTime-ω-admissible. Then, we show under which conditions on $\mathfrak{D}$ we can enrich ontologies using *feature assertions* $f(a, c)$ with f a feature name, a an individual and c a constant value, without increasing the complexity of reasoning. Finally, we look at several seemingly harmless extensions of $\mathcal{ALCOSCC}(\mathfrak{D})$ and prove their undecidability.

The work contained in this chapter is based on the paper:

[11] Baader, F. *et al.*: Concrete Domains Meet Expressive Cardinality Restrictions in Description Logics. In: Barrett, C., Waldmann, U. (eds.) Automated Deduction – CADE 30. LNAI,. Vol. 15943, pp. 676–695. Springer, Heidelberg (2025). `https://doi.org/10.1007/978-3-031-99984-0_35`

9.1 Syntax and Semantics of $\mathcal{ALCOSCC}(\mathfrak{D})$

Given finite, disjoint sets $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ of *concept-*, *role-* and *individual names*, $\mathcal{ALCOSCC}$ extends the DL $\mathcal{ALCSCC}$ defined in Chapter 3 with *nominals* $\{a\}$ for $a \in \mathsf{N_I}$ that can be also used within succ-restrictions. An interpretation $\mathcal{I}$ of $\mathsf{N_C}$ and $\mathsf{N_R}$ is additionally defined over $\mathsf{N_I}$ by mapping each $a \in \mathsf{N_I}$ to an individual $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

In this DL, the concept of all individuals that are human and have a child who is not Sam can be written as $\mathsf{Human} \sqcap \exists \mathsf{child}.\neg\{\mathtt{Sam}\}$.

A naive extension of $\mathcal{ALCOSCC}$ with concrete domain reasoning that simply combines succ- and CD-restrictions offers limited expressive power. To improve that, we introduce *feature pointers* α of the form f or $\mathtt{next}\, f$ with $f \in \mathsf{N_F}$ and define *feature roles* $\gamma := P(\alpha_1, \dots, \alpha_k)$, where each α_i is a feature pointer and P is a k-ary predicate of $\mathfrak{D}$. For example, salary is a pointer to the salary of a given individual d, while $\mathtt{next}$ salary is a pointer to the salary of an individual e that we want to compare to d; the feature role ($\mathsf{salary} < \mathtt{next}\, \mathsf{salary}$) describes a binary relation that contains (d, e) iff the salary of d is smaller than that of e.

Then, $\mathcal{ALCOSCC}(\mathfrak{D})$ is the extension of $\mathcal{ALCOSCC}$ with CD-restrictions and succ-restrictions $\mathsf{succ}(\mathfrak{con})$ where $\mathfrak{con}$ can contain feature roles as set variables. We can now describe individuals that earn less than the majority of their children by

$$C_{\mathsf{ex}} := \mathsf{succ}(|\mathsf{child} \cap (\mathsf{salary} < \mathtt{next}\,\mathsf{salary})| > |\mathsf{child} \cap (\mathsf{salary} < \mathtt{next}\,\mathsf{salary})^c|).$$

Feature roles $\gamma := P(\alpha_1, \dots, \alpha_k)$ are mapped by interpretations $\mathcal{I}$ to relations $\gamma^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ such that $(d, e) \in \gamma^{\mathcal{I}}$ iff $(c_1, \dots, c_k) \in P^D$, where $c_i := f_i^{\mathcal{I}}(d)$ if $\alpha_i = f_i$ and $c_i := f_i^{\mathcal{I}}(e)$ if $\alpha_i = \mathtt{next}\, f_i$. The QFBAPA assignment σ_d is extended to map feature roles γ to $\gamma^{\mathcal{I}} \cap \mathsf{ars}^{\mathcal{I}}(d)$, and $\mathsf{succ}(\mathfrak{con})^{\mathcal{I}}$ is defined as before.

An $\mathcal{ALCOSCC}(\mathfrak{D})$ *TBox* $\mathcal{T}$ is a finite set of *concept inclusions (CIs)* $C \sqsubseteq D$ between concepts C, D. For example, we can describe an individual $\mathtt{Jane}$ that earns more than $\mathtt{Sam}$, where the role $\mathsf{ref}_{\mathtt{Sam}}$ always points to $\mathtt{Sam}$:

$$\mathcal{T}_{\mathsf{ex}} := \left\{ \top \sqsubseteq \mathsf{succ}(\mathsf{ref}_{\mathtt{Sam}} = \{\mathtt{Sam}\}),\ \{\mathtt{Jane}\} \sqsubseteq \exists \mathsf{salary}, \mathsf{ref}_{\mathtt{Sam}}\, \mathsf{salary}.> \right\}.$$

A finitely branching interpretation $\mathcal{I}$ is a *model* of $\mathcal{T}$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for every CI $C \sqsubseteq D$ in $\mathcal{T}$. A TBox $\mathcal{T}$ is *consistent* if it has a model.

The Expressive Power of Feature Roles

We claimed above that feature roles are added to $\mathcal{ALCOSCC}(\mathfrak{D})$ to provide enough expressivity to e.g. define the concept C_{ex}. Here, we consider the DL $\mathcal{ALCSCC}(\mathfrak{D})$ as the subset of $\mathcal{ALCOSCC}(\mathfrak{D})$ without nominals occurring in concepts, and the DL $\mathcal{ALCSCC} \oplus \mathcal{ALC}(\mathfrak{D})$ that is obtained by disallowing feature roles in $\mathcal{ALCSCC}(\mathfrak{D})$ concepts. Combining the notions of Pr and $\mathfrak{D}$ bisimulations defined in Chapters 4 and 8, we can formally show that $\mathcal{ALCSCC}(\mathfrak{D})$ and thus $\mathcal{ALCOSCC}(\mathfrak{D})$ is more expressive than $\mathcal{ALCSCC} \oplus \mathcal{ALC}(\mathfrak{D})$.

Definition 9.1. *Given interpretations $\mathcal{I}$ and $\mathcal{J}$ of $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_F}$, the relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a* concrete Presburger (CPr) bisimulation *between $\mathcal{I}$ and $\mathcal{J}$ if it is a Pr bisimulation (cf. Definition 4.2) that satisfies the features condition of Definition 8.1. The notions of bisimilarity and $\mathbb{C}$-invariance w.r.t. CPr bisimulation are defined similarly to what was done in Definition 4.2.*

Similar to $\mathcal{ALCSCC}$, the DL $\mathcal{ALCSCC} \oplus \mathcal{ALC}(\mathfrak{D})$ satisfies Proposition 4.1, i.e. every concept in this DL of the form $\mathsf{succ}(\mathfrak{con})$ is $\mathbb{C}_{\mathsf{fb}}$-equivalent to a concept of the form $\mathsf{succ}(\mathfrak{con}')$ where $\mathfrak{con}'$ only contains set terms of the form $\tau \cap C$, where τ is a safe role type and C an $\mathcal{ALCSCC} \oplus \mathcal{ALC}(\mathfrak{D})$ concept. Using this property and by adapting and combining the proofs of Theorems 4.3 and 8.2 we then obtain the following result.

Theorem 9.2. *If $\mathfrak{D}$ is WCUN then every $\mathcal{ALCSCC} \oplus \mathcal{ALC}(\mathfrak{D})$ concept is $\mathbb{C}_{\mathsf{fb}}$-invariant under CPr bisimulation.*

Similarly to what was done in previous chapters, we use the invariance of $\mathcal{ALCSCC} \oplus \mathcal{ALC}(\mathfrak{D})$ concepts w.r.t. CPr bisimulation to prove that feature roles cannot be expressed in this DL.

Theorem 9.3. *There is no $\mathcal{ALCSCC} \oplus \mathcal{ALC}(\mathfrak{Q})$ concept without feature roles that is $\mathbb{C}_{\mathsf{fb}}$-equivalent to $C := \mathsf{succ}(|r \cap (f < \mathtt{next}\, f)| > |r \cap (f < \mathtt{next}\, f)^c|)$.*

Proof. Assume that there is an $\mathcal{ALCSCC} \oplus \mathcal{ALC}(\mathfrak{Q})$ concept D that is $\mathbb{C}_{\mathsf{fb}}$-equivalent to C. Consider the interpretations $\mathcal{I}$ and $\mathcal{J}$ depicted in Figure 9.1. Then, $a \in C^{\mathcal{I}} = D^{\mathcal{I}}$ and $a_1 \notin C^{\mathcal{J}} = D^{\mathcal{J}}$. However, the relation $\rho := \{(a, a_1), (b, b_1), (c, c_1), (d, d_1)\}$ is a CPr bisimulation that relates a and a_1. This leads to a contradiction, since $a \in D^{\mathcal{I}}$ iff $a_1 \in D^{\mathcal{J}}$ is supposed to hold. We conclude that C and D cannot be $\mathbb{C}_{\mathsf{fb}}$-equivalent. $\square$

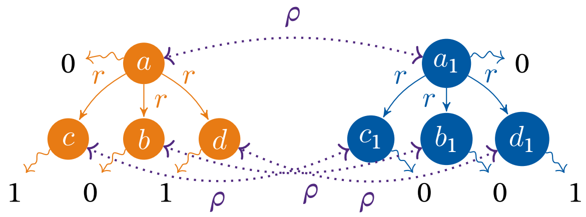


Figure 9.1: A CPr bisimulation ρ between $\mathcal{I}$ (left) and $\mathcal{J}$ (right) with $\mathfrak{Q}$ as concrete domain.

9.2 Deciding Consistency

Let $\mathfrak{D}$ be an ExpTime-ω-admissible concrete domain and $\mathcal{T}$ an $\mathcal{ALCOSCC}(\mathfrak{D})$ TBox. We assume w.l.o.g. that $\mathsf{N_C}$, $\mathsf{N_R}$, $\mathsf{N_I}$ and $\mathsf{N_F}$ contain exactly the names occurring in $\mathcal{T}$ and that there is at least one individual name; indeed, $\mathcal{T}$ is consistent iff $\mathcal{T} \cup \{\{a\} \sqsubseteq \{a\}\}$ is consistent, where a is a fresh individual name. We define the notion of *individual types*, describing sets of equivalent individual names in an interpretation.

Definition 9.4. *An* individual type $\mathfrak{a}$ *w.r.t.* $\mathsf{N_I}$ *is a non-empty subset of* $\mathsf{N_I}$*, and a set of individual types* $\mathbb{I}$ *is an* individual type system *for* $\mathsf{N_I}$ *if* $\mathbb{I}$ *partitions* $\mathsf{N_I}$*. Given an interpretation* $\mathcal{I}$*, an individual* $d \in \Delta^{\mathcal{I}}$ has individual type $\mathfrak{a}_{\mathcal{I}}(d) := \{a \in \mathsf{N_I} \mid a^{\mathcal{I}} = d\}$ *if this set is non-empty, and* d *is* anonymous *otherwise.*

We now fix an individual type system $\mathbb{I}$. Let $\mathcal{M}$ be the set of all subconcepts appearing in $\mathcal{T}$, as well as their negations. Employing the same notion of *type* w.r.t. $\mathcal{M}$ introduced in Definition 6.2, we say that a type t is *named with* an individual type $\mathfrak{a}_t$ if for all $a \in \mathsf{N_I}$, $a \in \mathfrak{a}_t$ iff $\{a\} \in t$, and is *anonymous* if it is not named with any individual type.

Following the approach used in [7], we construct a QFBAPA formula ϕ_t that is induced by the succ-restrictions $\mathsf{succ}(\mathfrak{con})$ in a type t and enriched with constraints derived from the individual type system $\mathbb{I}$ and the set of role names $\mathsf{N_R}$. Formally, ϕ_t is defined as the conjunction of

- $\phi_{\mathfrak{con}}$ if $\mathsf{succ}(\mathfrak{con}) \in t$ and $\neg\phi_{\mathfrak{con}}$ otherwise, where $\phi_{\mathfrak{con}}$ is derived from $\mathfrak{con}$ by replacing role names r, feature roles γ and concepts C with set variables X_r, X_γ and X_C, respectively;
- $|\bigcap_{a\in\mathfrak{a}} X_{\{a\}}| \leqslant 1$ for every $\mathfrak{a} \in \mathbb{I}$; and
- $\mathcal{U} = \bigcup_{r\in\mathsf{N_R}} X_r$.

All formulae ϕ_t contain exactly the same set variables and thus have the same Venn regions (cf. Section 3.1), called the *Venn regions of* $\mathcal{T}$. A Venn region v of $\mathcal{T}$ *has individual type* $\mathfrak{a}_v = \{a \in \mathsf{N_I} \mid X_{\{a\}} \in v\}$ if this set is non-empty, and v is *anonymous* otherwise. The following example shows that ϕ_t does not yet account for the cardinality constraints induced by the CD-restrictions in t.

Example 9.5. *Let* $\mathcal{T} = \{\top \sqsubseteq (\exists \mathsf{salary}, \mathsf{child}\ \mathsf{salary}.<) \sqcap (\mathsf{succ}(|\mathsf{child}| \leqslant 0))\}$*. For every model* $\mathcal{I}$ *of* $\mathcal{T}$ *and* $d \in \Delta^{\mathcal{I}}$*, the type* $t := t_{\mathcal{I}}(d)$ *contains both conjuncts appearing in this CI. The QFBAPA formula* $\phi_t := |X_{\mathsf{child}}| \leqslant 0 \wedge \mathcal{U} = X_{\mathsf{child}}$ *is satisfied by any solution assigning the empty set to* $\mathcal{U}$*. However,* t *cannot be realized: the first conjunct implies that* d *has a* child*-successor* $e \neq d$ *such that* $\mathsf{salary}^{\mathcal{I}}(d) < \mathsf{salary}^{\mathcal{I}}(e)$*, while the last conjunct forces* d *to have no* child*-successor.*

To realize the CD-restrictions in t, we may need up to $M_\mathcal{T} := R_\mathcal{T} \cdot P_\mathcal{T}$ distinct role successors, where $R_\mathcal{T}$ is the number of CD-restrictions in $\mathcal{M}$ and $P_\mathcal{T}$ is the maximal arity of predicates of $\mathfrak{D}$ occurring in $\mathcal{M}$. We add this information to the QFBAPA formula ϕ_t with additional constraints over a set of pre-selected Venn regions, representing sets of role successors whose existence is implied by the CD-restrictions in t. Let S be a set of at most $M_\mathcal{T}$ Venn regions v, each associated with a natural number $0 \leqslant n_v \leqslant M_\mathcal{T}$. By Lemma 3.1, the QFBAPA formula $\phi_{t,S}$, which extends ϕ_t with a conjunct $|v| \geqslant n_v$ for each $v \in S$, is satisfiable iff there is a natural number $n_\mathcal{T}$ of polynomial size w.r.t. the size of ϕ_t and $M_\mathcal{T}$ s.t. $\phi_{t,S}$ has a solution in which at most $n_\mathcal{T}$ Venn regions are non-empty. Moreover, since all formulae ϕ_t are nearly of the same size (except for the difference between ϕ_{con} and $\neg\phi_{\mathsf{con}}$) and $|S|$ and the numbers n_v are bounded by $M_\mathcal{T}$, we can assume that the bound $n_\mathcal{T}$ is independent of the choice of S and t, is polynomial w.r.t. the size of $\mathcal{T}$ and can be computed in polynomial time.

To formalize these additional restrictions, we consider *bags*, i.e. functions V assigning to every Venn region v of $\mathcal{T}$ a *multiplicity* $V(v) \in \mathbb{N}$, whose *support* $\mathsf{supp}(V)$ is the set of Venn regions of $\mathcal{T}$ with multiplicity $V(v) \geqslant 1$. The associated QFBAPA formula ϕ_V is the conjunction of the constraint $\mathcal{U} = \bigcup \mathsf{supp}(V)$ and all constraints $|v| \geqslant c$ where $v \in \mathsf{supp}(V)$ and $c = V(v)$.

Definition 9.6. *A* Venn bag *for a type t w.r.t. $\mathcal{T}$ is a bag V of Venn regions of $\mathcal{T}$ s.t. $|\mathsf{supp}(V)| \leqslant n_\mathcal{T}$, $V(v) \leqslant M_\mathcal{T} + 1$ holds for all $v \in \mathsf{supp}(V)$ and the QFBAPA formula $\phi_{t,V} := \phi_t \wedge \phi_V$ is satisfiable.*

By Lemma 3.1, $\phi_{t,S}$ is satisfiable iff there is a Venn bag V for t such that $\phi_{t,V}$ includes all constraints from $\phi_{t,S}$.

Finally, we take care of actually satisfying the CD-restrictions occurring in a type by using complete constraint systems to describe all relevant feature values. Feature values that are not represented in these systems correspond to undefined values. To ensure that all types agree on the feature values of individual names, we fix an *individual constraint system* $\mathfrak{C}_\mathbb{I}$ w.r.t. $\mathbb{I}$, i.e. a complete constraint system over variables of the form $f^\mathfrak{a}$, where $f \in \mathsf{N_F}$ and $\mathfrak{a} \in \mathbb{I}$, that refer to the feature values of named individuals. Then, we define constraint systems $\mathfrak{C}_{t,V}$ representing the relations between the feature values associated with an individual of type t and those of its role successors as specified by a Venn bag V for t. The system $\mathfrak{C}_{t,V}$ extends $\mathfrak{C}_\mathbb{I}$ by adding variables of the form

- $f^\star$, representing the value of the feature $f \in \mathsf{N_F}$ at the current individual;
- $f^{(v,j)}$ with $v \in \mathsf{supp}(V)$ and $1 \leqslant j \leqslant V(v)$ for the f-values at the successors, in order to express the relevant CD-restrictions.

Again, not all these variables actually need to occur in the constraint system, only the ones whose associated feature values should be defined. To handle named types and named Venn regions, we define the indexing functions

$$\iota(t) := \begin{cases} \star & \text{if } t \text{ is anonymous} \\ \mathfrak{a}_t & \text{otherwise} \end{cases} \qquad \text{and } \iota((v,j)) := \begin{cases} (v,j) & \text{if } v \text{ is anonymous} \\ \mathfrak{a}_v & \text{otherwise} \end{cases}$$

for all $v \in \mathsf{supp}(V)$ and $1 \leqslant j \leqslant V(v)$. Additionally, we do not allow more variables of the form $f^\mathfrak{a}$ than those already contained in $\mathfrak{C}_\mathbb{I}$.

Definition 9.7. *Let t be a type w.r.t. $\mathcal{T}$ and V a Venn bag for t. A* local system *for t, V is a complete constraint system $\mathfrak{C}_{t,V}$ that includes $\mathfrak{C}_\mathbb{I}$ and no additional variables of the form $f^\mathfrak{a}$, $\mathfrak{a} \in \mathbb{I}$, such that:*

Algorithm 2 Type elimination algorithm for $\mathcal{ALCOSCC}(\mathfrak{D})$

Input: An $\mathcal{ALCOSCC}(\mathfrak{D})$ TBox $\mathcal{T}$
Output: consistent if $\mathcal{T}$ is consistent, and inconsistent otherwise

```
guess an individual type system 𝕀 and an individual constraint system ℭ_𝕀
guess augmented types t_a = (t_a, V_a, ℭ_a) for a ∈ 𝕀 s.t. t_a is named with a
𝕋 ← {t = (t, V, ℭ) augmented type | t is anonymous} ∪ {t_a | a ∈ 𝕀}
while there is t ∈ 𝕋 that is not patched by 𝕋 do 𝕋 ← 𝕋 \ {t}
if t_a ∈ 𝕋 for all a ∈ 𝕀 then return consistent
else return inconsistent
```

1. *if* $C := \exists p_1, \dots, p_k.P \in \mathcal{M}$, *then* $C \in t$ *iff* $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}_{t,V}$ *such that*

$$x_i = \begin{cases} \iota(t) & \text{if } p_i = f_i, \text{ or} \\ \iota((v, j)) & \text{if } p_i = r f_i, \text{ for some } 1 \leqslant j \leqslant V(v) \text{ and } X_r \in v; \end{cases}$$

2. *for all set variables* $X_{P(\alpha_1,\dots,\alpha_k)}$, *all* $v \in \mathsf{supp}(V)$, *and* $1 \leqslant j \leqslant V(v)$ *it holds that* $X_{P(\alpha_1,\dots,\alpha_k)} \in v$ *iff* $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}_{t,V}$, *where*

$$x_i = \begin{cases} \iota(t) & \text{if } \alpha_i = f_i, \text{ and} \\ \iota((v, j)) & \text{if } \alpha_i = \mathtt{next}\ f_i. \end{cases}$$

In the following definition, we denote with S_v the subset of $\mathcal{M}$ that contains $C \in \mathcal{M}$ if $X_C \in v$ and $\neg C \in \mathcal{M}$ if $X_C \notin v$ (cf. Section 3.1).

Definition 9.8. *An* augmented type *for* $\mathcal{T}$ *is a tuple* $\mathfrak{t} := (t, V, \mathfrak{C}_\mathfrak{t})$, *where* t *is a type w.r.t.* $\mathcal{T}$, V *is a Venn bag for* t, *and* $\mathfrak{C}_\mathfrak{t}$ *is a satisfiable local system for* t, V. *The* root *of* $\mathfrak{t}$ *is* $\mathsf{root}(\mathfrak{t}) := t$.

An augmented type $\mathfrak{t}' = (t', V', \mathfrak{C}_{\mathfrak{t}'})$ patches $\mathfrak{t}$ *at* (v, i), *where* $v \in \mathsf{supp}(V)$ *and* $1 \leqslant i \leqslant V(v)$, *if* $S_v \subseteq t'$ *and the* merged system $\mathfrak{C}_\mathfrak{t} \triangleleft_{(v,i)} \mathfrak{C}_{\mathfrak{t}'}$ *has a solution, where* $\mathfrak{C}_\mathfrak{t} \triangleleft_{(v,i)} \mathfrak{C}_{\mathfrak{t}'}$ *is obtained as the union of* $\mathfrak{C}_\mathfrak{t}$ *and the result of replacing all variables in* $\mathfrak{C}_{\mathfrak{t}'}$ *as follows:*

$$\begin{array}{lll} f^\star & \mapsto & f^{(v,i)} \quad \text{if } t' \text{ is anonymous}; \\ f^{(w,j)} & \mapsto & f^{(w,j)'} \quad \text{for all anonymous } w \in \mathsf{supp}(V') \text{ and } 1 \leqslant j \leqslant V'(w); \\ f^\mathfrak{a} & \mapsto & f^\mathfrak{a} \quad \text{for all } \mathfrak{a} \in \mathbb{I}. \end{array}$$

A set of augmented types $\mathbb{T}$ patches $\mathfrak{t}$ *if, for all* $v \in \mathsf{supp}(V)$ *and* $1 \leqslant i \leqslant V(v)$, *there is a* $\mathfrak{t}' \in \mathbb{T}$ *that patches* $\mathfrak{t}$ *at* (v, i).

The merging operation identifies all features associated to (v, i) in $\mathfrak{C}_\mathfrak{t}$ with those associated to t' in $\mathfrak{C}_{\mathfrak{t}'}$, while keeping the remaining variables associated to anonymous individuals separate. If t' is not anonymous (and thus $\mathfrak{C}_{\mathfrak{t}'}$ contains no variable of the form $f^\star$) then the condition $S_v \subseteq t'$ ensures that $\mathfrak{a}_v = \mathfrak{a}_{t'}$, and thus the variable $f^{\iota((v,i))} = f^{\mathfrak{a}_v} = f^{\mathfrak{a}_{t'}}$ in $\mathfrak{C}_\mathfrak{t}$ is already identical to $f^{\iota(t')} = f^{\mathfrak{a}_{t'}}$ in $\mathfrak{C}_{\mathfrak{t}'}$.

The augmented types are now used by Algorithm 2 to decide consistency of an $\mathcal{ALCOSCC}(\mathfrak{D})$ TBox via a type elimination approach. We show that Algorithm 2 is indeed sound and complete.

Soundness

We observe that for every patchwork $\mathfrak{D}$, the fact that $\mathfrak{D}$ satisfies AP (see Chapter 2) implies the following property:

AP^{+} if $\mathbb{C}$ is a finite set of constraint systems and V a set of variables such that $V(\mathfrak{B}) \cap V(\mathfrak{C}) = V$ and $\mathfrak{B}$ and $\mathfrak{C}$ agree on V for all $\mathfrak{B}, \mathfrak{C} \in \mathbb{C}$, then the constraint system $\bigcup \mathbb{C}$ is satisfiable iff each $\mathfrak{C} \in \mathbb{C}$ is satisfiable.

With this property in mind, we show that Algorithm 2 is sound.

In particular, we assume that there is a run of the type elimination algorithm that returns consistent, and use the individual type system $\mathbb{I}$ and the set $\mathbb{T}$ of augmented types constructed in this run to define a model $\mathcal{I}$ of $\mathcal{T}$. The domain $\Delta^{\mathcal{I}}$ consists of tuples $(\mathfrak{a}, w)$, where $\mathfrak{a} \in \mathbb{I}$ and w is a word over the alphabet Σ of all tuples $(\mathfrak{t}, v, i)$ with $\mathfrak{t} \in \mathbb{T}$, v a Venn region of $\mathcal{T}$ and $i \geqslant 1$ a natural number. We associate to each tuple $(\mathfrak{a}, w)$ the augmented type $\mathsf{end}(\mathfrak{a}, w) \in \mathbb{T}$ defined as $\mathsf{end}(\mathfrak{a}, \varepsilon) := \mathfrak{t}_{\mathfrak{a}}$ and $\mathsf{end}(\mathfrak{a}, w' \cdot (\mathfrak{t}, v, i)) := \mathfrak{t}$ for $w' \in \Sigma^*$.

We define $\Delta^{\mathcal{I}}$ as the union of sets Δ^m with $m \in \mathbb{N}$, where Δ^0 contains $(\mathfrak{a}, \varepsilon)$ for every $\mathfrak{a} \in \mathbb{I}$ and Δ^{m+1} is defined inductively in the following. Given $(\mathfrak{a}, w) \in \Delta^m$ with $\mathsf{end}(\mathfrak{a}, w) = \mathfrak{t} = (t, V, \mathfrak{C}_{t,V}) \in \mathbb{T}$ we observe that

- the QFBAPA formula ϕ_t has a solution $\sigma_{\mathfrak{a},w}$ such that $\sigma_{\mathfrak{a},w}(|v|) \geqslant V(v)$ if $v \in \mathsf{supp}(V)$ and $\sigma_{\mathfrak{a},w}(|v|) = 0$ otherwise for all Venn regions v of $\mathcal{T}$,
- for $v \in \mathsf{supp}(V)$ and $i = 1, \dots, V(v)$ there exists an augmented type $\mathfrak{t}_{(v,i)} \in \mathbb{T}$ patching $\mathfrak{t}$ at (v, i), as otherwise $\mathfrak{t}$ would have been eliminated from $\mathbb{T}$.

Using these augmented types, we define for $r \in \mathsf{N_R}$ the set $\Delta_r^{m+1}[\mathfrak{a}, w]$ containing $(\mathfrak{a}, w \cdot (\mathfrak{t}_{(v,i)}, v, j))$ iff $X_r \in v$, $\mathsf{root}(\mathfrak{t}_{(v,i)})$ is anonymous, $j = 1, \dots, \sigma_{\mathfrak{a},w}(|v|)$ and $i = \max(j, V(v))$. We now define Δ^{m+1} as the extension of Δ^m by all sets $\Delta_r^{m+1}[\mathfrak{a}, w]$ for which $r \in \mathsf{N_R}$ and $(\mathfrak{a}, w) \in \Delta^m$.

The extensions of $a \in \mathsf{N_I}$, $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$ are defined as:

$$
\begin{aligned}
a^{\mathcal{I}} &:= (\mathfrak{a}, \varepsilon), \text{ where } \mathfrak{a} \text{ is the unique individual type in } \mathbb{I} \text{ containing } a;\\
A^{\mathcal{I}} &:= \{(\mathfrak{a}, w) \in \Delta^{\mathcal{I}} \mid \mathsf{end}(\mathfrak{a}, w) = \mathfrak{t} \text{ and } A \in \mathsf{root}(\mathfrak{t})\};\\
r^{\mathcal{I}} &:= \{((\mathfrak{a}, w), (\mathfrak{b}, \varepsilon)) \mid \mathsf{end}(\mathfrak{a}, w) = \mathfrak{t} \text{ and } \mathfrak{t}_{\mathfrak{b}} \text{ patches } \mathfrak{t} \text{ at } (v, i) \text{ with } X_r \in v\} \cup\\
&\quad\ \{((\mathfrak{a}, w), (\mathfrak{a}, w')) \mid (\mathfrak{a}, w) \in \Delta^m \text{ and } (\mathfrak{a}, w') \in \Delta_r^{m+1}[\mathfrak{a}, w] \text{ with } m \in \mathbb{N}\}.
\end{aligned}
$$

For $f \in \mathsf{N_F}$, $f^{\mathcal{I}}$ is defined as follows. If $(\mathfrak{a}, w) \in \Delta^{\mathcal{I}}$ and $\mathsf{end}(\mathfrak{a}, w) = (t, V, \mathfrak{C}_{t,V})$, we extend $\mathfrak{C}_{t,V}$ with all variables $f^{(v,i)}$ where $i > V(v)$ and $f \in \mathsf{N_F}$ such that $f^{(v,V(v))}$ occurs in $\mathfrak{C}_{t,V}$ and $(\mathfrak{a}, w \cdot (\mathfrak{t}', v, i))$ occurs in $\Delta^{\mathcal{I}}$. Then, we add all constraints $P((f_1^{x_1})', \dots, (f_k^{x_k})')$ obtained by replacing every occurrence of $f^{(v,V(v))}$ in a constraint $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}_{t,V}$ with some variable $f^{(v,i)}$ among those occurring in the extended system with $i \geqslant V(v)$. In this way, the feature values of all role successors of $(\mathfrak{a}, w)$ are handled correctly w.r.t. one another and w.r.t. those of $(\mathfrak{a}, w)$. Next, we replace all variables $f^{\mathfrak{a}}$ with $f^{\mathfrak{a},\varepsilon}$, all variables $f^{\star}$ with $f^{\mathfrak{a},w}$ and all variables $f^{(v,i)}$ with $f^{\mathfrak{a},u}$ where u is the unique word of the form $w \cdot (\mathfrak{t}', v, i)$ for which $(\mathfrak{a}, u) \in \Delta^{\mathcal{I}}$.

Let $\mathfrak{C}_{\mathfrak{a},w}$ be the resulting complete constraint system and $\mathfrak{C}^m$ with $m \in \mathbb{N}$ be the union of all $\mathfrak{C}_{\mathfrak{a},w}$ with $(\mathfrak{a}, w) \in \Delta^m$.

Lemma 9.9. *For every $m \in \mathbb{N}$, the constraint system $\mathfrak{C}^m$ has a solution.*

Proof. Since $\mathfrak{D}$ is in particular a patchwork, we can extend every constraint system $\mathfrak{C}$ over $\mathfrak{D}$ with a solution h to a complete and satisfiable constraint system. Indeed, the fact that $\mathfrak{D}$ is JEPD implies that, for all $k \in \mathbb{N}$, either $\mathfrak{D}$ has no k-ary relations or for all variables $v_1, \ldots, v_k$ the system $\mathfrak{C}$ contains at most one constraint $P(v_1, \ldots, v_k)$ with a k-ary relation P of $\mathfrak{D}$. If $\mathfrak{C}$ contains no such constraint, then we can add to it the unique constraint $P(v_1, \ldots, v_k)$ for which $(h(v_1), \ldots, h(v_k)) \in P^D$ holds. In this case, h is still a solution of the extended system. Hereafter, we call a *completion* of a satisfiable constraint system $\mathfrak{C}$ a complete constraint system obtained using this procedure.

Our second observation is that for all $(\mathfrak{a}, w) \in \Delta^{\mathcal{I}}$ the complete constraint system $\mathfrak{C}_{\mathfrak{a},w}$ is satisfiable. Assuming that $\mathsf{end}(\mathfrak{a}, w) = (t, V, \mathfrak{C}_{t,V}) \in \mathbb{T}$, we know from Definition 9.8 that $\mathfrak{C}_{t,V}$ has a solution h'. We extend h' to a solution h of $\mathfrak{C}_{\mathfrak{a},w}$ by first setting $h'(f^{(v,i)}) := h'(f^{(v,V(v))})$ for $i = V(v) + 1, \ldots, \sigma_{\mathfrak{a},w}(|v|)$ if $f^{(v,V(v))}$ occurs in $\mathfrak{C}_{t,V}$ and then renaming the variables in the domain of h' using the renaming used to construct $\mathfrak{C}_{\mathfrak{a},w}$, thus obtaining h.

We prove the claim by induction over $m \in \mathbb{N}$. For $m = 0$, the system $\mathfrak{C}^0$ corresponds to the union of all systems $\mathfrak{C}_{\mathfrak{a},\varepsilon}$ with $\mathfrak{a} \in \mathbb{I}$. Every two systems of this form share exactly the variables $f^{\mathfrak{b},\varepsilon}$ for which $f^{\mathfrak{b}}$ occurs in $\mathfrak{C}_{\mathbb{I}}$; moreover, they agree on these variables, since they all include (up to renaming) a copy of the complete constraint system $\mathfrak{C}_{\mathbb{I}}$. By AP^+, we conclude that $\mathfrak{C}^0$ is satisfiable.

Now, we inductively assume that the constraint system $\mathfrak{C}^m$ has a solution for $m \in \mathbb{N}$ and show that this implies that $\mathfrak{C}^{m+1}$ is satisfiable, too. We notice that $\mathfrak{C}^{m+1}$ is the union of $\mathfrak{C}^m$ and all systems $\mathfrak{C}_{\mathfrak{a},w}$ with $(\mathfrak{a}, w) \in \Delta^{m+1} \setminus \Delta^m$. By construction, for every $(\mathfrak{a}, w) \in \Delta^{m+1} \setminus \Delta^m$ there is a unique $(\mathfrak{a}, w') \in \Delta^m$ such that $w = w' \cdot (\mathfrak{t}, v, i)$ for some augmented type $\mathfrak{t} \in \mathbb{T}$, some Venn region v of $\mathcal{T}$ and some natural number i. In particular, $\mathfrak{t}$ patches $\mathsf{end}(\mathfrak{a}, w')$ at (v, i) by definition of Δ^{m+1}. Let $\mathfrak{C}'$ be a completion of $\mathfrak{C}^m$ and consider the complete constraint system $\mathfrak{C}_{\mathfrak{a},w}$. The variables shared by $\mathfrak{C}'$ and $\mathfrak{C}_{\mathfrak{a},w}$ are exactly (up to renaming) those occurring in $\mathfrak{C}_{\mathbb{I}}$ and those of the form $f^{\mathfrak{a},w}$ for $f \in \mathsf{N_F}$ that occur in the complete constraint systems $\mathfrak{C}_{\mathfrak{a},w'}$ and $\mathfrak{C}_{\mathfrak{a},w}$. Due to the fact that $\mathfrak{t}$ patches $\mathsf{end}(\mathfrak{a}, w')$ at (v, i), we deduce that these two systems agree on their shared variables. Given that $\mathfrak{C}_{\mathfrak{a},w'}$ is a subsystem of $\mathfrak{C}'$ and that both $\mathfrak{C}'$ and $\mathfrak{C}_{\mathfrak{a},w}$ are complete, we deduce that these two systems agree on their shared variables, and since both are satisfiable we conclude that $\mathfrak{C}' \cup \mathfrak{C}^w_{\mathfrak{a}}$ is satisfiable. By taking a completion of this constraint system and iteratively repeating this process for every other element in $(\mathfrak{a}, w) \in \Delta^{m+1} \setminus \Delta^m$, we obtain a satisfiable constraint system that includes $\mathfrak{C}^{m+1}$ as a subsystem, which is then satisfiable. □

Let $\mathfrak{C}^{\mathcal{I}}$ be the union of all systems $\mathfrak{C}^m$ with $m \in \mathbb{N}$. Every finite subsystem of $\mathfrak{C}^{\mathcal{I}}$ is a subsystem of $\mathfrak{C}^m$ for some $m \in \mathbb{N}$ and is thus satisfiable. Thus, by homomorphism ω-compactness of $\mathfrak{D}$, we can infer that $\mathfrak{C}^{\mathcal{I}}$ has a solution $h^{\mathcal{I}}$. For every $f \in \mathsf{N_F}$ and $(\mathfrak{a}, w) \in \Delta^{\mathcal{I}}$, we now define $f^{\mathcal{I}}((\mathfrak{a}, w)) := h^{\mathcal{I}}(f^{\mathfrak{a},w})$ if $f^{\mathfrak{a},w}$ occurs in $\mathfrak{C}^{\mathcal{I}}$ and leave it undefined otherwise.

Lemma 9.10. *For all $d = (\mathfrak{a}, w) \in \Delta^{\mathcal{I}}$ and $C \in \mathcal{M}$, $C \in \mathsf{root}(\mathsf{end}(d))$ iff $d \in C^{\mathcal{I}}$.*

Proof. We prove this claim by structural induction over $C \in \mathcal{M}$. We assume that $\mathsf{end}(\mathfrak{a}, w) = \mathfrak{t} = (t, V, \mathfrak{C}_{t,V})$ and first prove the base cases where C is either a concept name, a nominal or an existential CD-restriction.

- The case $C = A \in \mathsf{N_C}$ is trivially covered by the definition of $A^{\mathcal{I}}$.
- If $C = \{a\}$, we notice that $\{a\} \in \mathsf{root}(\mathfrak{t})$ iff $\mathfrak{t} = \mathfrak{t}_{\mathfrak{a}}$ with $a \in \mathfrak{a}$ and $\mathfrak{a}_{\mathfrak{t}} = \mathfrak{a}$; by construction of $\Delta^{\mathcal{I}}$, this holds iff $(\mathfrak{a}, w) = (\mathfrak{a}, \varepsilon) \in \{a\}^{\mathcal{I}}$.

- Let $C = \exists p_1, \dots, p_k.P \in \mathcal{M}$. If $C \in \mathsf{root}(\mathfrak{t})$, then Definition 9.7 together with $\mathfrak{t} \in \mathbb{T}$ implies that there is a constraint $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}_{t,V}$ such that for $i = 1, \dots, k$ either $x_i = \mathfrak{a}_i$ with $\mathfrak{a}_i \in \mathbb{I}$ or $x_i = \star$ or $x_i = (v_i, j_i)$ for some $v_i \in V$ with $1 \leqslant j_i \leqslant V(v_i)$. In the last case, we know that there is $\mathfrak{t}_{(v_i,j_i)} \in \mathbb{T}$ that patches $\mathfrak{t}$ at (v_i, j_i). Using these indices and augmented types, we select for $i = 1, \dots, k$ the domain elements

$$d_i = (\mathfrak{a}_i, w^i) := \begin{cases} (\mathfrak{a}_i, \varepsilon) & \text{if } x_i = \mathfrak{a}_i \in \mathbb{I} \\ (\mathfrak{a}, w) & \text{if } x_i = \star \\ (\mathfrak{a}, w \cdot (\mathfrak{t}_{(v_i,j_i)}, v_i, j_i)) & \text{if } x_i = (v_i, j_i). \end{cases}$$

  Then, $P(f_1^{\mathfrak{a}_1,w^1}, \dots, f_k^{\mathfrak{a}_k,w^k}) \in \mathfrak{C}^{\mathcal{I}}$ holds and thus $(f_1^{\mathcal{I}}(d_1), \dots, f_k^{\mathcal{I}}(d_k)) \in P^D$ by definition of $\mathcal{I}$. We prove that $(\mathfrak{a}, w) \in C^{\mathcal{I}}$ by showing that $f_i^{\mathcal{I}}(d_i) \in p_i^{\mathcal{I}}(\mathfrak{a}, w)$ holds for $i = 1, \dots, k$:

  - If $p_i = f_i$, then $x_i = \iota(t)$ is either $\star$ if t is anonymous or $x_i = \mathfrak{a}_i = \mathfrak{a}_t = \mathfrak{a}$ and $w^i = w = \varepsilon$. In the former case, $f_i^{\mathcal{I}}(d_i) = f_i^{\mathcal{I}}(\mathfrak{a}, w) \in p_i^{\mathcal{I}}(\mathfrak{a}, w)$. In the latter case, we have $f_i^{\mathcal{I}}(d_i) = f_i^{\mathcal{I}}(\mathfrak{a}_i, \varepsilon) \in p_i^{\mathcal{I}}(\mathfrak{a}, w)$.
  - If $p_i = r_i f_i$, then $x_i = \iota((v_i, j_i))$ is either (v_j, j_i) if v_j is anonymous or $x_i = \mathfrak{a}_i = \mathfrak{a}_{v_i}$ and $w^i = \varepsilon$. Moreover, in both cases, we have $X_{r_i} \in v_i$, and thus $((\mathfrak{a}, w), (\mathfrak{a}, w^i)) \in r_i^{\mathcal{I}}$ or $((\mathfrak{a}, w), (\mathfrak{a}_{v_i}, \varepsilon)) \in r_i^{\mathcal{I}}$, respectively, Therefore, $f_i^{\mathcal{I}}(d_i) \in p_i^{\mathcal{I}}(\mathfrak{a}, w)$ holds in both cases.

  Vice versa, assume that $C \notin \mathsf{root}(\mathfrak{t})$. We show that for all $(c_1, \dots, c_k)$ where $c_i \in p_i^{\mathcal{I}}(w)$ for $i = 1, \dots, k$ it holds that $(c_1, \dots, c_k) \notin P^D$. We choose individuals $d_i = (\mathfrak{a}_i, w^i)$ for $i = 1, \dots, k$ such that $d_i = (\mathfrak{a}, w)$ and $f_i^{\mathcal{I}}(d_i) = c_i$ if $p_i = f_i$ and $d_i = (\mathfrak{a}', w')$ is an r_i-successor of $(\mathfrak{a}, w)$ with $f_i^{\mathcal{I}}(d_i) = c_i$ if $p_i = r_i f_i$. The construction of $\mathfrak{C}_{\mathfrak{a},w}$ from $\mathfrak{C}_{t,V}$, together with $C \notin \mathsf{root}(\mathfrak{t})$ and thus $\neg C \in \mathsf{root}(\mathfrak{t})$, implies that $P(f_1^{\mathfrak{a}_1,w^1}, \dots, f_k^{\mathfrak{a}_k,w^k}) \notin \mathfrak{C}_{\mathfrak{a},w}$ according to Definition 9.7. Since $f^{\mathcal{I}}(d_i)$ is defined, however, $f_i^{\mathfrak{a}_i,w^i}$ must occur in $\mathfrak{C}_{\mathfrak{a},w}$ for $i = 1, \dots, k$. Given that $\mathfrak{C}_{\mathfrak{a},w}$ is complete, there must be $P'(f_1^{\mathfrak{a}_1,w^1}, \dots, f_k^{\mathfrak{a}_k,w^k}) \in \mathfrak{C}_{\mathfrak{a},w}$ for some $P' \neq P$ k-ary. The interpretation of features of $\mathcal{I}$ is a solution of $\mathfrak{C}^{\mathcal{I}}$ and thus of $\mathfrak{C}_{\mathfrak{a},w}$, so we deduce that $(c_1, \dots, c_k) = (f_1^{\mathcal{I}}(d_1), \dots, f_k^{\mathcal{I}}(d_k)) \in (P')^D$, hence $(c_1, \dots, c_k) \notin P^D$ (by JEPD).

Assume that the claim holds for $C_1, C_2, D \in \mathcal{M}$ to prove the inductive cases.

- If $C = \neg D \in \mathcal{M}$, then $C \in \mathsf{root}(\mathfrak{t})$ iff $D \notin \mathsf{root}(\mathfrak{t})$ iff $(\mathfrak{a}, w) \notin D^{\mathcal{I}}$ iff $(\mathfrak{a}, w) \in C^{\mathcal{I}}$, where the equivalences hold due to Definition 6.2, the inductive hypothesis, and the semantics of negation, respectively.

- If $C = C_1 \sqcap C_2 \in \mathcal{M}$, then $C \in \mathsf{root}(\mathfrak{t})$ iff $C_i \in \mathsf{root}(\mathfrak{t})$ for $i = 1, 2$ iff $(\mathfrak{a}, w) \in C_i^{\mathcal{I}}$ for $i = 1, 2$ iff $(\mathfrak{a}, w) \in C^{\mathcal{I}}$, with equivalences justified as before.

- If $C = \mathsf{succ}(\mathfrak{con}) \in \mathcal{M}$, we show that the solution $\sigma_{\mathfrak{a},w}$ of $\phi_{t,V}$ used to determine the role successors of $(\mathfrak{a}, w)$ satisfies the formula $\phi_{\mathfrak{con}}$ iff the QFBAPA assignment $\zeta_{\mathfrak{a},w}$ induced by $(\mathfrak{a}, w)$ in $\mathcal{I}$ is a solution of $\mathfrak{con}$ (recall that $\zeta_{\mathfrak{a},w}(\mathcal{U}) = \mathsf{ars}^{\mathcal{I}}(\mathfrak{a}, w)$). Since $C \in \mathsf{root}(\mathfrak{t})$ iff $\phi_{\mathfrak{con}}$ occurs in $\phi_{t,V}$, this allows us to conclude that $C \in \mathsf{root}(\mathfrak{t})$ iff $\zeta_{\mathfrak{a},w}$ is a solution of $\mathfrak{con}$, which, by the semantics of $\mathcal{ALCOSCC}(\mathfrak{D})$, happens iff $(\mathfrak{a}, w) \in C^{\mathcal{I}}$.

  To show that $\sigma_{\mathfrak{a},w}$ satisfies $\phi_{\mathfrak{con}}$ iff $\zeta_{\mathfrak{a},w}$ satisfies $\mathfrak{con}$, we construct a bijection $\pi \colon \zeta_{\mathfrak{a},w}(\mathcal{U}) \to \sigma_{\mathfrak{a},w}(\mathcal{U})$ such that $(\mathfrak{a}', w') \in \zeta_{\mathfrak{a},w}(\delta)$ iff $\pi(\mathfrak{a}', w') \in \sigma_{\mathfrak{a},w}(X_\delta)$ for each role name, concept or feature role δ. For every named element $(\mathfrak{a}', \varepsilon) \in \zeta_{\mathfrak{a},w}(\mathcal{U})$, we define $\pi(\mathfrak{a}', \varepsilon)$ to be the unique element in $\sigma_{\mathfrak{a},w}(\bigcap_{a\in\mathfrak{a}'} X_{\{a\}})$: since $(\mathfrak{a}', \varepsilon)$ is a role successor of $(\mathfrak{a}, w)$, we know that $\mathfrak{t}_{\mathfrak{a}'}$ patches $\mathfrak{t}$, which means that a Venn region v with $S_v \subseteq \mathsf{root}(\mathfrak{t}_{\mathfrak{a}'})$ and thus $\mathfrak{a}_v = \mathfrak{a}'$ must occur in $\mathsf{supp}(V)$, which means that $\sigma_{\mathfrak{a},w}(\mathcal{U})$ contains at least one element satisfying $\bigcap_{a\in\mathfrak{a}'} X_{\{a\}}$; moreover, there can be at most one such element since $\sigma_{\mathfrak{a},w}$ satisfies ϕ_t. For each anonymous element $(\mathfrak{a}, w')$ with $w' = w \cdot (\mathfrak{t}_{(v,i)}, v, j)$, we can assign a unique element $\pi(\mathfrak{a}', w) \in \sigma_{\mathfrak{a},w}(v)$, since we created exactly $\sigma_{\mathfrak{a},w}(|v|)$ many such elements w'. This mapping π is a bijection since named Venn regions must have cardinality 1 w.r.t. $\sigma_{\mathfrak{a},w}$, we explicitly created enough elements for each anonymous Venn region $v \in \mathsf{supp}(V)$, each such element is a role successor of $(\mathfrak{a}, w)$ due to the constraint $X_{r_1} \cup \dots \cup X_{r_n} = \mathcal{U}$, and there are no other role successors of $(\mathfrak{a}, w)$ in $\mathcal{I}$.

It remains to show that $(\mathfrak{a}', w') \in \zeta_{\mathfrak{a},w}(\delta)$ iff $\pi(\mathfrak{a}', w') \in \sigma_{\mathfrak{a},w}(X_\delta)$. For anonymous role successors $(\mathfrak{a}, w')$ with $w' = w\cdot(\mathfrak{t}_{(v,i)}, v, j)$ of $(\mathfrak{a}, w)$, this amounts to showing the following:

- $X_r \in v$ iff $(\mathfrak{a}, w') \in r^{\mathcal{I}}((\mathfrak{a}, w))$ for all role names $r \in \mathsf{N_R}$;
- $X_D \in v$ iff $(\mathfrak{a}, w') \in D^{\mathcal{I}}$ for all concepts $D \in \mathcal{M}$; and
- $X_\gamma \in v$ iff $(\mathfrak{a}, w') \in \gamma^{\mathcal{I}}((\mathfrak{a}, w))$ for all feature roles γ.

The first point is a consequence of the definition of $\mathcal{I}$. Using again this definition, we observe that $\mathfrak{t}'$ patches $\mathfrak{t}$ at (v, i), and thus $S_v \subseteq \mathsf{root}(\mathfrak{t}_{(v,i)})$ holds. Since for $D \in \mathcal{M}$ we have that $D \in S_v$ iff $X_D \in v$ and by inductive hypothesis we have that $D \in \mathsf{root}(\mathfrak{t}_{(v,i)})$ iff $(\mathfrak{a}, w') \in D^{\mathcal{I}}$, we conclude that $(\mathfrak{a}, w') \in D^{\mathcal{I}}$ iff $X_D \in v$. Finally, by Definition 9.7 and using the renaming we adopted before, for $\gamma := P(\alpha_1, \dots, \alpha_k)$ we have that $X_\gamma \in v$ iff $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}_{\mathfrak{a},w}$ where $x_i = (\mathfrak{a}, w)$ if $\alpha_i = f_i$ and $x_i = (\mathfrak{a}, w')$ if $\alpha_i = \texttt{next}\ f_i$. By construction of $\mathcal{I}$ and the definition of x_i, we have that $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}^{\mathcal{I}}$ iff $(f_1^{\mathcal{I}}(x_1), \dots, f_k^{\mathcal{I}}(x_k)) \in P^D$, and by the semantics of $\mathcal{ALCOSCC}(\mathfrak{D})$ this happens iff $(\alpha_1^{\mathcal{I}}((\mathfrak{a}, w), (\mathfrak{a}, w')), \dots, \alpha_k^{\mathcal{I}}((\mathfrak{a}, w), (\mathfrak{a}, w'))) \in P^D$, i.e. iff $(\mathfrak{a}, w') \in \gamma^{\mathcal{I}}(\mathfrak{a}, w)$.

We can show that a similar characterization for named role successors $(\mathfrak{a}', \varepsilon)$ of $(\mathfrak{a}, w)$, by considering the unique singleton Venn region v of $\sigma_{\mathfrak{a},w}$ that satisfies $\bigcap_{a\in\mathfrak{a}'} X_{\{a\}}$.

□

It is a direct consequence that $\mathcal{I}$ satisfies all CIs in $\mathcal{T}$ and is thus a model of $\mathcal{T}$. Hence, $\mathcal{T}$ is consistent. Therefore, we conclude that Algorithm 2 is sound.

Theorem 9.11. *If there is a run of Algorithm 2 that returns consistent, then $\mathcal{T}$ is consistent.*

Completeness

Next, we show that Algorithm 2 is complete. Let $\mathcal{I}$ be a model of $\mathcal{T}$ and $\mathbb{I}$ be the individual type system that contains an individual type $\mathfrak{a}$ iff $\mathfrak{a} = \mathfrak{a}_{\mathcal{I}}(d)$ for some $d \in \Delta^{\mathcal{I}}$. Then $\mathbb{I}$ is well-defined, as every $a \in \mathsf{N_I}$ is uniquely assigned to $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. For $\mathfrak{a} \in \mathbb{I}$, we denote by $\mathfrak{a}^{\mathcal{I}}$ the unique $d \in \Delta^{\mathcal{I}}$ with $\mathfrak{a} = \mathfrak{a}_{\mathcal{I}}(d)$. Further, let $T_{\mathcal{I}} := \{t_{\mathcal{I}}(d) \mid d \in \Delta^{\mathcal{I}}\}$ be the set of all types that are realized in $\mathcal{I}$.

For each individual type $\mathfrak{a} \in \mathbb{I}$, we define $t_{\mathfrak{a}} := t_{\mathcal{I}}(\mathfrak{a}^{\mathcal{I}})$. Using $\mathbb{I}$ and $T_{\mathcal{I}}$, we build a constraint system $\mathfrak{C}_{\mathbb{I}}$ and a set $\mathbb{T}_{\mathcal{I}} := \{\mathfrak{t}_{\mathcal{I}}(d) \mid d \in \Delta^{\mathcal{I}}\}$ of augmented types, containing unique types $\mathfrak{t}_{\mathfrak{a}}$ whose roots are named with $\mathfrak{a} \in \mathbb{I}$. We define the individual constraint system $\mathfrak{C}_{\mathbb{I}}$ over all variables $f^{\mathfrak{a}}$ with $f \in \mathsf{N_F}$ and $\mathfrak{a} \in \mathbb{I}$ for which $f^{\mathcal{I}}(\mathfrak{a}^{\mathcal{I}})$ is defined, such that $P(f_1^{\mathfrak{a}_1}, \dots, f_k^{\mathfrak{a}_k}) \in \mathfrak{C}_{\mathbb{I}}$ iff $\big(f_1^{\mathcal{I}}(\mathfrak{a}_1^{\mathcal{I}}), \dots, f_k^{\mathcal{I}}(\mathfrak{a}_k^{\mathcal{I}})\big) \in P^D$. Clearly, $\mathfrak{C}_{\mathbb{I}}$ is complete.

Next, we associate to each $d \in \Delta^{\mathcal{I}}$ an augmented type $\mathfrak{t}_{\mathcal{I}}(d) := (t_{\mathcal{I}}(d), V_d, \mathfrak{C}_d)$. If e is a role successor of d, let v_e be the Venn region of $\mathcal{T}$ whose variables X_r, X_C, X_γ for role names r, concepts C and feature roles γ satisfy the following:

- $X_r \in v_e$ iff e is an r-successor of d;
- $X_C \in v_e$ iff $C \in t_{\mathcal{I}}(e)$;
- $X_\gamma \in v_e$ iff $e \in \gamma^{\mathcal{I}}(d)$.

For every non-negated CD-restriction $\exists p_1, \dots, p_k.P \in t_{\mathcal{I}}(d)$, we can find values $c_i \in p_i^{\mathcal{I}}(d)$ for $i = 1, \dots, k$ such that $(c_1, \dots, c_k) \in P^D$. If $p_i = r_i f_i$, this implies that there is $e_i \in r_i^{\mathcal{I}}(d)$ such that $f_i^{\mathcal{I}}(e_i) = c_i$. We collect all these successors of d, which are at most $M_{\mathcal{T}}$ many distinct elements, in the set S_{cd}. For $e \in \mathsf{S}_{\mathsf{cd}}$, let n_e be the number of elements $e' \in \mathsf{S}_{\mathsf{cd}}$ such that $v_e = v_{e'}$.

Lemma 9.12. *There is a Venn bag V_d for $t_{\mathcal{I}}(d)$ w.r.t. $\mathcal{T}$ such that $V_d(v_e) = n_e$ for all $e \in \mathsf{S}_{\mathsf{cd}}$, and for all other $v \in \mathsf{supp}(V_d)$ we have $V_d(v) = 1$ and there is a role successor $e \in \Delta^{\mathcal{I}} \setminus \mathsf{S}_{\mathsf{cd}}$ of d such that $v = v_e$.*

Proof. Given that S_{cd} contains at most $M_{\mathcal{T}}$ elements, there can be at most $M_{\mathcal{T}}$ different constraints $|v_e| \geqslant n_e$ generated by elements e of S_{cd} and $1 \leqslant n_e \leqslant M_{\mathcal{T}} + 1$ holds for all these constraints. Consider the QFBAPA formula ϕ_d that is the conjunction of $\phi_{t_{\mathcal{I}}(d)}$ and the constraints $|v_e| \geqslant n_e$ for $e \in \mathsf{S}_{\mathsf{cd}}$. By construction, the QFBAPA assignment σ_d induced by $\mathcal{I}$ (see Section 9.1) is a solution of $\phi_{t_{\mathcal{I}}(d)}$ with $\sigma_d(\mathcal{U}) = \mathsf{ars}^{\mathcal{I}}(d)$, which satisfies the additional constraints $|v_e| \geqslant n_e$ with $e \in \mathsf{S}_{\mathsf{cd}}$ since $\mathsf{S}_{\mathsf{cd}} \subseteq \mathsf{ars}^{\mathcal{I}}(d)$, and is thus a solution of ϕ_d. By Lemma 3.1, there is another solution σ'_d of ϕ_d for which there are at most $n_{\mathcal{T}}$ Venn regions v with $\sigma'_d(v) \neq \emptyset$, and, whenever $\sigma'_d(v) \neq \emptyset$, then also $\sigma_d(v) \neq \emptyset$; that is, every non-empty Venn region v according to σ'_d still corresponds to an element $e \in \mathsf{ars}^{\mathcal{I}}(d)$ such that $v_e = v$. We define the bag V_d as follows:

$$V_d(v) := \begin{cases} n_e & \text{if } v = v_e \text{ and } e \in \mathsf{S}_{\mathsf{cd}}, \\ 1 & \text{if } v \neq v_e \text{ for } e \in \mathsf{S}_{\mathsf{cd}} \text{ and } \sigma'_d(v) \neq \emptyset, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathsf{supp}(V_d)$ contains at most $M_{\mathcal{T}}$ Venn regions v, all satisfying $V_d(v) \leqslant M_{\mathcal{T}} + 1$. Since σ'_d is also a solution of $\phi_{t_{\mathcal{I}}(d), V_d}$ and $\mathsf{supp}(V_d)$ contains exactly the Venn regions that are assigned non-empty sets by σ'_d, we conclude that V_d satisfies all conditions of Definition 9.6, hence that V_d is a Venn bag for $t_{\mathcal{I}}(d)$. □

It remains to define the local system $\mathfrak{C}_d$. Consider the set X_d that contains $\iota(t_{\mathcal{I}}(d))$ (either ★ or $\mathfrak{a}_{t_{\mathcal{I}}(d)}$), all $\mathfrak{a} \in \mathbb{I}$, and all pairs (v, j) with anonymous Venn bags $v \in \mathsf{supp}(V_d)$ and $1 \leqslant j \leqslant V_d(v)$ (cf. Definition 9.7). Let λ_d be a bijection mapping every such (v, j) to an anonymous successor e of d satisfying $v_e = v$, such that $\lambda_d((v_e, j)) \in \mathsf{S}_{\mathsf{cd}}$ for all $e \in \mathsf{S}_{\mathsf{cd}}$. Such a bijection exists due to Lemma 9.12. We extend this bijection to $\mathbb{I}$ by setting $\lambda_d(\mathfrak{a}) := \mathfrak{a}^{\mathcal{I}}$. Furthermore, we extend λ_d to ξ_d with $\xi_d(\iota(t_{\mathcal{I}}(d))) := d$. Then, ξ_d is injective except for ★: if d is its own role successor, it can happen that $\xi_d(\star) = d = \xi_d((v, j))$. We define the complete constraint system $\mathfrak{C}_d$ over variables f^x with $f \in \mathsf{N_F}$, $x \in X_d$, such that $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}_d$ iff $\big(f_1^{\mathcal{I}}(\xi_d(x_1)), \dots, f_k^{\mathcal{I}}(\xi_d(x_k))\big) \in P^D$ holds for all $x_1, \dots, x_k \in X_d$. If $f^{\mathcal{I}}(\xi_d(x))$ is undefined, then f^x does not occur in $\mathfrak{C}_d$.

Lemma 9.13. *$\mathfrak{C}_d$ is a satisfiable local system for $t_{\mathcal{I}}(d)$ and V_d.*

Proof. Clearly, $h_d(f^x) := f^{\mathcal{I}}(\xi_d(x))$ for $f \in \mathsf{N_F}$, $x \in X_d$ defines a solution of $\mathfrak{C}_d$. We show that $\mathfrak{C}_d$ is a local system for $t_{\mathcal{I}}(d)$ and V_d, according to Definition 9.7:

1. We have $\exists p_1, \dots, p_k.P \in t_{\mathcal{I}}(d)$ iff $(c_1, \dots, c_k) \in P^D$ for some values $c_i \in p_i^{\mathcal{I}}(d)$ with $i = 1, \dots, k$; by construction of V_d, if $p_i = r_i f_i$ we can find $e_i \in \mathsf{S_{cd}}$ such that $(d, e_i) \in r_i^{\mathcal{I}}$ and $f_i^{\mathcal{I}}(e_i) = c_i$, and set $x_i := \lambda_d^{-1}(e_i)$; if $p_i = f_i$ (and $f_i^{\mathcal{I}}(d) = c_i$), we set $x_i := \iota(t_{\mathcal{I}}(d))$. In both cases, we have $f_i^{\mathcal{I}}(\xi_d(x_i)) = c_i$. By definition of $\mathfrak{C}_d$, then, $(c_1, \dots, c_k) \in P^D$ iff $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}_d$.

2. Consider any variable of the form $X_{P(\alpha_1,\dots,\alpha_k)}$, a Venn region $v \in \mathsf{supp}(V_d)$ and $1 \leqslant j \leqslant V_d(v)$. We know that $v = v_e$ for some individual $e = \lambda_d(x)$ with $x \in X_d$, and thus $X_{P(\alpha_1,\dots,\alpha_k)} \in v = v_e$ iff $\big(\alpha_1^{\mathcal{I}}(d,e), \dots, \alpha_k^{\mathcal{I}}(d,e)\big) \in P^D$. Setting $x_i := \iota(t_{\mathcal{I}}(d))$ if $\alpha_i = f_i$ and $x_i := x$ if $\alpha_i = \texttt{next}\, f_i$, we obtain that $\big(\alpha_1^{\mathcal{I}}(d,e), \dots, \alpha_k^{\mathcal{I}}(d,e)\big) \in P^D$ iff $\big(f_1^{\mathcal{I}}(\xi_d(x_1)), \dots, f_k^{\mathcal{I}}(\xi_d(x_k))\big) \in P^D$ iff $P(f_1^{x_1}, \dots, f_k^{x_k}) \in \mathfrak{C}_d$.

□

Thus, $\mathfrak{t}_{\mathcal{I}}(d) = \big(t_{\mathcal{I}}(d), V_d, \mathfrak{C}_d\big)$ is an augmented type. Furthermore, we can show that every augmented type in $\mathbb{T}_{\mathcal{I}}$ is patched in $\mathbb{T}_{\mathcal{I}}$.

Lemma 9.14. *Every augmented type in $\mathbb{T}_{\mathcal{I}}$ is patched in $\mathbb{T}_{\mathcal{I}}$.*

Proof. Consider an augmented type $\mathfrak{t}_{\mathcal{I}}(d) = \big(t_{\mathcal{I}}(d), V_d, \mathfrak{C}_d\big) \in \mathbb{T}_{\mathcal{I}}$ as defined above, $v \in \mathsf{supp}(V_d)$, and $1 \leqslant i \leqslant V_d(v)$. We consider $e := \xi_d((v,i))$ if v is anonymous, and $e := \mathfrak{a}_v^{\mathcal{I}} = \xi_d(\mathfrak{a}_v)$ otherwise. In both cases, we have $v_e = v$ by construction of ξ_d. We show that $\mathfrak{t}_{\mathcal{I}}(e)$ patches $\mathfrak{t}_{\mathcal{I}}(d)$ at (v,i). First,

$$
\begin{aligned}
S_v &= \{C \in \mathcal{M} \mid X_C \in v\} \cup \{\neg C \in \mathcal{M} \mid X_C^c \in v\} \\
&\subseteq \{C \in \mathcal{M} \mid C \in t_{\mathcal{I}}(e)\} \cup \{\neg C \in \mathcal{M} \mid C \notin t_{\mathcal{I}}(e)\} \\
&= t_{\mathcal{I}}(e).
\end{aligned}
$$

Second, we consider the system $\mathfrak{C}_d \lhd_{(v,i)} \mathfrak{C}_e$ obtained by renaming the variables in $\mathfrak{C}_e$ as in Definition 9.8. As discussed above, $h_d(f^x) := f^{\mathcal{I}}(\xi_d(x))$ defines a solution of $\mathfrak{C}_d$, and similarly $h_e(f^x) := f^{\mathcal{I}}(\xi_e(x))$ is a solution of $\mathfrak{C}_e$. In particular, $h_d(f^{\mathfrak{a}}) = f^{\mathcal{I}}(\mathfrak{a}^{\mathcal{I}}) = h_e(f^{\mathfrak{a}})$ for all $\mathfrak{a} \in \mathbb{I}$ and, if e is anonymous, $h_d(f^{(v,i)}) = f^{\mathcal{I}}(\xi_d((v,i))) = f^{\mathcal{I}}(e) = f^{\mathcal{I}}(\xi_e(\star)) = h_e(f^{\star})$. Thus, the mapping h defined as the union of h_d and h_e (after renaming) is a solution of $\mathfrak{C}_d \lhd_{(v,i)} \mathfrak{C}_e$. □

If Algorithm 2 guesses $\mathbb{I}$, $\mathfrak{C}_{\mathbb{I}}$, and $\mathfrak{t}_{\mathfrak{a}}$, where $\mathfrak{a} \in \mathbb{I}$, then the initial set $\mathbb{T}$ must contain $\mathbb{T}_{\mathcal{I}}$, and no augmented type from $\mathbb{T}_{\mathcal{I}}$ can ever be removed from $\mathbb{T}$. This shows that the augmented types $\{\mathfrak{t}_{\mathfrak{a}} \mid \mathfrak{a} \in \mathsf{N_I}\} \subseteq \mathbb{T}_{\mathcal{I}}$ remain in $\mathbb{T}$ throughout the execution of the algorithm. Since the algorithm terminates, it thus returns consistent. We conclude that Algorithm 2 is compelte.

Theorem 9.15. *If $\mathcal{T}$ is consistent, then there is a run of Algorithm 2 that returns consistent.*

Termination and Complexity

Because Algorithm 2 runs in exponential time, we obtain a matching upper bound to the ExpTime-hardness inherited from $\mathcal{ALC}$. Indeed, as there are at most exponentially many individual type systems and polynomially many individual types in such a type system, all guesses can be implemented by enumerating all choices in exponential time. The main elimination procedure also runs in exponential time as the number of augmented types is exponentially bounded, and all required tests can be performed in exponential time, provided that $\mathfrak{D}$ is ExpTime-ω-admissible. We thus obtain the following result.

Theorem 9.16. *Let $\mathfrak{D}$ be an ExpTime-ω-admissible concrete domain. Then, consistency checking in $\mathcal{ALCOSCC}(\mathfrak{D})$ is ExpTime-complete.*

Proof. ExpTime-hardness follows from ExpTime-hardness for $\mathcal{ALC}$ [94]. It remains to show that Algorithm 2 can be executed in deterministic exponential time by enumerating all choices in Lines 1 and 2 instead of guessing them. First, there are only exponentially many individual type systems $\mathbb{I}$ and individual constraint systems for Line 1 of Algorithm 2. Moreover, there are only exponentially many augmented types $(t, V, \mathfrak{C}_{t,V})$ since the size of Venn bags V is bounded polynomially and thus also $\mathfrak{C}_{t,V}$ can contain only polynomially many variables and constraints. In addition, satisfiability of the polynomially large $\phi_{t,V}$ and $\mathfrak{C}_{t,V}$ can be checked in exponential time since satisfiability of QFBAPA formulae is NP-complete [72] and $\mathfrak{D}$ is ExpTime-ω-admissible, respectively. Therefore, the initial set $\mathbb{T}$ in Line 3 can be constructed in exponential time and there are also only exponentially many possibilities to assign augmented types $\mathfrak{t}_{\mathsf{a}}$ to the individual types in $\mathbb{I}$ in Line 2. Since each iteration of the loop in Line 4 removes one augmented type from $\mathbb{T}$, there can be at most exponentially many iterations. Each iteration can be performed in exponential time, as each check for patching involves a polynomial test to check whether $S_v \subseteq t'$ and an exponential check for satisfiability of a constraint system of polynomial size, and at most exponentially many patching checks occur. By Theorems 9.11 and 9.15, we conclude that consistency of an $\mathcal{ALCOSCC}(\mathfrak{D})$ TBox is decidable in exponential time. □

9.3 Reasoning with ABoxes

In Chapter 6 we complemented $\mathcal{ALC}(\mathfrak{D})$ TBoxes $\mathcal{T}$ with *ABoxes* containing *concept assertions* $C(a)$ and *role assertions* $r(a, b)$, where $a, b \in \mathsf{N_I}$, $r \in \mathsf{N_R}$, and C is a concept, with the obvious semantics. In $\mathcal{ALCOSCC}(\mathfrak{D})$, those assertions can be expressed in the TBox using nominals [22]. In the presence of a concrete domain, however, we may want to use additional kinds of assertions: *predicate assertions* $P(f_1(a_1), \dots, f_k(a_k))$ with $f_i \in \mathsf{N_F}$, $a_i \in \mathsf{N_I}$, $i = 1, \dots, k$, and a k-ary predicate P of $\mathfrak{D}$, and *feature assertions* $f(a, c)$ with $f \in \mathsf{N_F}$, $a \in \mathsf{N_I}$, and a constant $c \in D$. The former requires every model $\mathcal{I}$ to satisfy $(f_1^{\mathcal{I}}(a_1^{\mathcal{I}}), \dots, f_k^{\mathcal{I}}(a_k^{\mathcal{I}})) \in P^D$, and the latter states that $f^{\mathcal{I}}(a^{\mathcal{I}}) = c$.

Using predicate assertions, we can rewrite the TBox $\mathcal{T}_{\mathsf{ex}}$ in Section 9.1 into a single, intuitive assertion salary(`Sam`) < salary(`Jane`). This also demonstrates how predicate assertions can be simulated by CIs: instead of $P(f_1(a_1), \dots, f_k(a_k))$, we can use $\top \sqsubseteq \mathsf{succ}(\mathsf{ref}_{a_i} = \{a_i\})$ for $i = 1, \dots, k$ and $\top \sqsubseteq \exists \mathsf{ref}_{a_1} f_1, \dots, \mathsf{ref}_{a_k} f_k.P$.

On the other hand, with feature assertions, we can give specific values and state, for instance, that `Sam`'s salary is 100,001 € with salary(`Sam`, 100,001). Feature assertions seemingly increase the expressivity, since we can use them to refer to constant values. If $\mathfrak{D}$ has *singleton predicates* $=_c$ with $(=_c)^D = \{c\}$, then one can express $f(a, c)$ by $\{a\} \sqsubseteq \exists f.{=_c}$. Since an ω-admissible concrete domain $\mathfrak{D}$ has a finite signature, however, this only works for a fixed, finite set of values $c \in D$. Due to the JD and JEPD conditions, it turns out that feature assertions are actually equivalent to *additional singleton predicates* $=_c$ that are not part of $\mathfrak{D}$, but can be used in concepts with the same semantics as defined above.

We notice that the results of this section also hold for $\mathcal{ALC}(\mathfrak{D})$, as they can be shown without resorting to succ-restrictions or nominals [34] and without relying on the fact that interpretations are finitely branching. Even if we were to impose such a restriction, though, the fact that $\mathcal{ALC}(\mathfrak{D})$ ontologies have the *finitely branching model property* (cf. Corollary 6.14) implies that the results of this section still apply.

Referring to feature values of named individuals We can use the roles ref_a introduced above in arbitrary CD-restrictions to refer to the feature values of named individuals.

We introduce a related construction here, variants of which will be used in several of the following proofs. The idea is to introduce features like $\mathsf{salary}_{\mathtt{Sam}}$ that can be used to express feature roles like $\mathtt{next}\,\mathsf{salary} < \mathsf{salary}_{\mathtt{Sam}}$ within succ-restrictions, in order to quantify the number of successors with a salary smaller than Sam's. For this purpose, the interpretation of the feature f_a needs to be such that $f_a^{\mathcal{I}}(d)$ is equal to $f^{\mathcal{I}}(a^{\mathcal{I}})$ at every individual $d \in \Delta^{\mathcal{I}}$. The idea is to use the role ref_a to enforce this, using a CI like $\top \sqsubseteq \forall \mathsf{ref}_a f, f_a.{=}$. However, $\mathfrak{D}$ does not necessarily contain the equality predicate $=$, which means that this may not be a valid CI in $\mathcal{ALCOSCC}(\mathfrak{D})$. Nevertheless, by JD, we know that there is a quantifier-free, equality-free first-order formula $\phi_=(x, y)$ over the signature of $\mathfrak{D}$ that is equivalent to $x = y$. Moreover, by JEPD and finiteness of the signature, we can express negated atoms as disjunctions of positive atoms, so that we may assume $\phi_=(x, y)$ to be a disjunction of conjunctions of positive atoms.

We can use this to construct the concept $C_{\mathsf{ref}_a f = f_a}$ that is obtained from $\phi_=(x, y)$ by replacing $\wedge$ with $\sqcap$, $\vee$ with $\sqcup$ and every atom $P(t_1, \dots, t_n)$ with $\forall p_1, \dots, p_n.P$, where $p_i = \mathsf{ref}_a f$ whenever $t_i = x$ and $p_i = f_a$ whenever $t_i = y$. This concept is equivalent to the intended CD-restriction $\forall \mathsf{ref}_a f, f_a.{=}$ since ref_a is functional, i.e. every individual has exactly one ref_a-successor, namely a. Thus, the CI $\top \sqsubseteq C_{\mathsf{ref}_a f = f_a}$ enforces that, whenever both $f^{\mathcal{I}}(a^{\mathcal{I}})$ and $f_a^{\mathcal{I}}(d)$ are defined, then they must be equal. Finally, we can complement this CI by similar ones to express that these feature values are either both defined or both undefined:

$$\exists f_a, f_a.{=} \sqsubseteq \exists \mathsf{ref}_a f, \mathsf{ref}_a f.{=} \text{ and} \exists \mathsf{ref}_a f, \mathsf{ref}_a f.{=} \sqsubseteq \exists f_a, f_a.{=}$$

(we can construct concepts $C_{f_a = f_a}$ and $C_{\mathsf{ref}_a f = \mathsf{ref}_a f}$ equivalent to $\exists f_a, f_a.{=}$ and $\exists \mathsf{ref}_a f, \mathsf{ref}_a f.{=}$, respectively, similarly to $C_{\mathsf{ref}_a f = f_a}$ above).

Concrete domains with constants. In the following proofs, we need to check the satisfiability of constraints that contain one or more constants. For this purpose, we assume $\mathfrak{D}$ to be *ExpTime-ω-admissible with constants*, i.e. ExpTime-ω-admissible and such that given $c_1, \dots, c_k \in \mathfrak{D}$ we can find an encoding of these elements such that CSP($\mathfrak{D}$) remains decidable in exponential time if constraints are allowed to contain c_1, ..., c_k. Making this assumption is not unreasonable, since it holds for the three main examples of ExpTime-ω-admissible concrete domains. For $\mathfrak{Q}$ and and Allen's interval algebra, it is required that satisfiability of $P(c, d)$ for constants c and d can be decided in polynomial time [65], which is the case if all involved numbers are given as integer fractions with the integers represented in binary. For RCC8, constants can only denote polygonal regions in the 2D plane, with their finitely many vertices specified by rational coordinates [75].

Lemma 9.17. *For $\mathcal{ALCOSCC}(\mathfrak{D})$ with a concrete domain $\mathfrak{D}$ that is ExpTime-ω-admissible with constants, the following hold:*

1. *we can reduce consistency of a TBox $\mathcal{T}$ with additional singleton predicates to consistency of an ontology $\mathcal{O}$ with feature assertions in exponential time, where $\mathcal{O}$ has polynomial size w.r.t. $\mathcal{T}$;*
2. *we can reduce consistency of an ontology $\mathcal{O}$ with feature assertions to consistency of a TBox $\mathcal{T}$ with additional singleton predicates in polynomial time.*

Proof. We can express every feature assertion $f(a,c)$ by $\{a\} \sqsubseteq \exists f.{=_c}$. For the other direction, consider an $\mathcal{ALCOSCC}(\mathfrak{D})$ TBox $\mathcal{T}$ that uses additional singleton predicates. We show how to construct a TBox $\mathcal{T}'$ and an ABox $\mathcal{A}'$ that simulate all additional singleton predicates $=_c$ in $\mathcal{T}$ by using feature assertions. Since $=_c$ is unary, it can occur only in CD-restrictions of the form $\exists f.{=_c}$ or $\exists rf.{=_c}$ and feature roles $=_c(f)$ or $=_c(\texttt{next}\, f)$. CD-restrictions $\exists rf.{=_c}$ can be equivalently expressed as $\mathsf{succ}(|r \cap \exists f.{=_c}| \geqslant 1)$, and $=_c(\texttt{next}\, f)$ can directly be replaced by $\exists f.{=_c}$. This means that we can assume that $=_c$ occurs only in expressions of the form $\exists f.{=_c}$ or $=_c(f)$.

The main idea is to store the value c in a special feature f_c by using feature assertions, and make sure that the value of f_c is equal to c at every element reachable from a named individual by a role chain. We can then express $\exists f.{=_c}$ and $=_c(f)$ by making f equal to f_c, for which we exploit JD.

First, we ensure that f_c is a total function by adding the axiom $\top \sqsubseteq \exists f_c.\top_{\mathfrak{D}}$ to $\mathcal{T}'$, where $\top_{\mathfrak{D}}$ is interpreted as D. Although $\top_{\mathfrak{D}}$ may not be a predicate of $\mathfrak{D}$, by JEPD and the fact that the signature of $\mathfrak{D}$ is non-empty and finite, $\top_{\mathfrak{D}}$ can be expressed as the disjunction of some k-ary predicates $P_1, \dots, P_m$. This implies that for every $d \in D$ there is exactly one k-ary predicate P_i such that $(d, \dots, d) \in P_i^D$. Thus, we can write $\exists f.\top_{\mathfrak{D}}$ equivalently as $\exists f, \dots, f.P_1 \sqcup \dots \sqcup \exists f, \dots, f.P_m$, where each restriction $\exists f, \dots, f.P_i$ repeats f for k times.

We next give f_c the value c for all elements reachable from a named individual. We start by adding all feature assertions $f_c(a,c)$ to $\mathcal{A}'$, for every individual name a occurring in $\mathcal{T}$. If $\mathcal{T}$ does not contain any individual names, we instead add only $f_c(a^*, c)$ for a fresh individual name a^*. It remains to transfer this value to all reachable elements.

Since $\mathfrak{D}$ is ω-admissible, equality between two variables x, y can be expressed using a formula $\phi_=(x,y)$ that is a disjunction of conjunctions of positive atoms over the signature of $\mathfrak{D}$ (i.e., not including the additional singleton predicates). Now consider the formula $\phi_=(c,y)$, where x is replaced by the constant c. Since $\phi_=(c,y)$ is equivalent to $c = y$, we can find a single disjunct $\beta(c,y)$ of $\phi_=(c,y)$ such that $\beta(c,y)$ is satisfiable and equivalent to $c = y$; otherwise, $\phi_=(c,y)$ would be satisfied also for values of y other than c. Overall, we can compute $\beta(c,y)$ in exponential time, using the fact that $\mathfrak{D}$ is ExpTime-ω-admissible with constants. For every $r \in \mathsf{N_R}(\mathcal{O})$, we now obtain the concept $C_{r,c}$ from $\beta(c,y)$ by replacing $\wedge$ with $\sqcap$ and every atom $P(t_1, \dots, t_n)$ with $\forall p_1, \dots, p_n.P$, where $p_i = f_c$ if $t_i = c$ and $p_i = rf_c$ if $t_i = y$, and add the axiom $\top \sqsubseteq C_{r,c}$ to $\mathcal{T}'$. This ensures that, in every model $\mathcal{I}$ of $\mathcal{A}'$ and $\mathcal{T}'$, for all elements d that are reachable from a named individual by a sequence of role connections, we have that $f_c^{\mathcal{I}}(d) = c$.

We can now replace every concept of the form $\exists f.{=_c}$ in $\mathcal{T}$ with a concept $C_{f=c}$ that is obtained from $\beta(c,y)$ by replacing $\wedge$ with $\sqcap$ and atoms $P(t_1, \dots, t_n)$ with $\exists f_1, \dots, f_n.P$, where $f_i = f_c$ if $t_i = c$ and $f_i = f$ if $t_i = y$. Similarly, we can replace feature roles $=_c(f)$ by $\gamma_{f=c}$ obtained from $\beta(c,y)$ by replacing $\wedge$ with $\cap$ and atoms $P(t_1, \dots, t_n)$ with $P(\alpha_1, \dots, \alpha_n)$, where $\alpha_i = f_c$ if $t_i = c$ and $\alpha_i = f$ if $t_i = y$. The constructed TBox $\mathcal{T}'$ and ABox $\mathcal{A}'$ are of polynomial size w.r.t. the size of $\mathcal{T}$ since each assertion, concept, or concrete role $f_c(a,c)$, $\exists f_c.\top_{\mathfrak{D}}$, $C_{r,c}$, $C_{f=c}$, $\gamma_{f=c}$ is of linear size, the replacements of $\exists f.{=_c}$ by $C_{f=c}$ are independent of each other since $\alpha(c,y)$ cannot contain the additional singleton predicates, and similarly for $\gamma_{f=c}$.

Let now $\mathcal{I}$ be a model of $\mathcal{T}$. By interpreting f_c as the total function with $f_c(d) = c$ for all $d \in \Delta^{\mathcal{I}}$ and, optionally, a^* as an arbitrary element from $\Delta^{\mathcal{I}}$, we obtain a model of $\mathcal{A}'$ and $\mathcal{T}'$. Conversely, let $\mathcal{I}'$ be a model of $\mathcal{A}'$ and $\mathcal{T}'$. We restrict $\mathcal{I}'$ to the subdomain of all elements reachable from a named element $a^{\mathcal{I}'}$ by a chain of role connections $r^{\mathcal{I}'}$ for $r \in \mathsf{N_R}$. The resulting interpretation $\mathcal{I}''$ is still a model of $\mathcal{A}'$ and $\mathcal{T}'$ since the evaluation of concepts on $\Delta^{\mathcal{I}''}$ does not depend on unconnected elements from $\Delta^{\mathcal{I}''} \setminus \Delta^{\mathcal{I}'}$ (see Section 9.1). The new axioms in $\mathcal{A}'$ and $\mathcal{T}'$ ensure that $f_c(d) = c$ holds for all $d \in \Delta^{\mathcal{I}''}$, and therefore $\mathcal{I}''$ is also a model of $\mathcal{T}$. □

Additionally, feature assertions can be expressed by predicate assertions if $\mathfrak{D}$ is *homogeneous*, i.e. such that every isomorphism between finite substructures of $\mathfrak{D}$ can be extended to an isomorphism from $\mathfrak{D}$ to itself [24]. All known ω-admissible concrete domains are homogeneous [24].

Lemma 9.18. *For $\mathcal{ALCOSCC}(\mathfrak{D})$ with a concrete domain $\mathfrak{D}$ that is ExpTime-ω-admissible with constants and homogeneous, consistency of an ontology $\mathcal{O}$ with feature assertions can be reduced to consistency of an ontology $\mathcal{O}'$ without feature assertions in exponential time, where $\mathcal{O}'$ is of polynomial size w.r.t. $\mathcal{O}$.*

Proof. Let $\mathcal{T}$ be an $\mathcal{ALCOSCC}(\mathfrak{D})$ TBox and $\mathcal{A}$ an ABox containing feature assertions. Since predicate assertions can be expressed by TBox axioms, it suffices to show how to simulate the feature assertions by using predicate assertions. Let $\mathcal{A}'$ result from $\mathcal{A}$ by removing all feature assertions and adding the predicate assertions $P(f_1(a_1), \dots, f_k(a_k)) \in \mathcal{A}'$ for all combinations of feature assertions $f_i(a_i, c_i) \in \mathcal{A}$, $i = 1, \dots, k$, with $(c_1, \dots, c_k) \in P^D$. The size of $\mathcal{A}'$ is polynomial in the input, since the signature of $\mathfrak{D}$ is fixed, and we can compute $\mathcal{A}'$ in exponential time since $\mathfrak{D}$ is ExpTime-ω-admissible with constants.

It is easy to see that every model of $\mathcal{T}$ and $\mathcal{A}$ is also a model of $\mathcal{A}'$. Conversely, let $\mathcal{I}$ be a model of $\mathcal{T}$ and $\mathcal{A}'$ and let $\mathfrak{D}_{\mathcal{A}}, \mathfrak{D}_{\mathcal{I}}$ be the finite substructures of $\mathfrak{D}$ over the domains

$$D_{\mathcal{A}} := \{c \mid f(a, c) \in \mathcal{A}\} \text{ and } D_{\mathcal{I}} := \{f^{\mathcal{I}}(a) \mid f(a, c) \in \mathcal{A}\},$$

respectively. By definition of $\mathcal{A}'$ and JEPD, we have $\big(f_1^{\mathcal{I}}(a_1^{\mathcal{I}}), \dots, f_k^{\mathcal{I}}(a_k^{\mathcal{I}})\big) \in P^D$ iff $(c_1, \dots, c_k) \in P^D$, for all combinations of feature assertions $f_i(a_i, c_i)$ in $\mathcal{A}$. By JD, this in particular implies that $f_1^{\mathcal{I}}(a_1^{\mathcal{I}}) = f_2^{\mathcal{I}}(a_2^{\mathcal{I}})$ iff $f_1(a_1, c), f_2(a_2, c) \in \mathcal{A}$ for some value $c \in D$, which means that the two substructures have the same number of elements. Moreover, by the first equivalence, the mapping $f^{\mathcal{I}}(a^{\mathcal{I}}) \mapsto c$ for all $f(a, c) \in \mathcal{A}$ is an isomorphism between $\mathfrak{D}_{\mathcal{I}}$ and $\mathfrak{D}_{\mathcal{A}}$. Since $\mathfrak{D}$ is homogeneous, there exists an isomorphism $h\colon D \to D$ such that $h(f^{\mathcal{I}}(a^{\mathcal{I}})) = c$ if $f(a, c) \in \mathcal{A}$. Now, we obtain $\mathcal{I}'$ from $\mathcal{I}$ by changing the interpretation of feature names to $f^{\mathcal{I}'}(d) := h(f^{\mathcal{I}}(d))$ iff this value is defined for $f \in \mathsf{N_F}$ and $d \in \Delta^{\mathcal{I}}$. Since h is an isomorphism, we have $C^{\mathcal{I}} = C^{\mathcal{I}'}$ for all concepts C, including CD-restrictions and succ-restrictions with feature roles, which shows that $\mathcal{I}'$ is a model of $\mathcal{T}$. Moreover, it also satisfies all feature assertions $f(a, c) \in \mathcal{A}$ since $f^{\mathcal{I}'}(a^{\mathcal{I}'}) = h(f^{\mathcal{I}}(a^{\mathcal{I}})) = c$ by construction. □

Together, Lemmas 9.17 to 9.18 show that, under these conditions, we can use constant values (either in feature assertions or additional singleton predicates) in $\mathcal{ALCOSCC}(\mathfrak{D})$, without increasing the complexity of reasoning. The following result then follows together with Theorem 9.16.

Theorem 9.19. *If $\mathfrak{D}$ is ExpTime-ω-admissible with constants and homogeneous, then consistency in $\mathcal{ALCOSCC}(\mathfrak{D})$ with feature assertions and additional singleton predicates is ExpTime-complete.*

9.4 Undecidable Extensions

To conclude our investigations, we show that several extensions of $\mathcal{ALCOSCC}(\mathfrak{D})$, inspired by existing DLs or obtained by seemingly harmless tweaks to the syntax and semantics, are undecidable. Hereafter, we assume that the domain set of $\mathfrak{D}$ is infinite and that $\mathfrak{D}$ is JD (cf. Chapter 2). If equality over $\mathfrak{D}$ is defined by the quantifier-free, equality-free formula $\phi_=(x, y)$, we write $(f = \texttt{next}\, g)$ to denote the set term obtained by replacing every atom $P(x_1, \dots, x_k)$ in $\phi_=(x, y)$ with the feature role $P(\alpha_1, \dots, \alpha_k)$, where $\alpha_i = f$ if $x_i = x$ and $\alpha_i = \texttt{next}\, g$ if $x_i = y$ for $i = 1, \dots, k$, and every Boolean connective with the corresponding set operation.

Comparing set cardinalities and feature values. If $\mathfrak{D}$ is a numerical concrete domain where D is either $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$, it is natural to consider comparisons between feature values of an individual d and the cardinalities of sets of role successors of d. For example, we could describe individuals whose age is fifteen times as much as the number of their children using the concept $\mathsf{succ}(\mathsf{age} = 15 \cdot |\mathsf{child}|)$. This could be achieved by allowing succ-restrictions to contain *mixed numerical constraints* $f = \ell$, where ℓ is a PA expression (cf. Section 3.1) that is allowed in $\mathcal{ALCOSCC}(\mathfrak{D})$ and $f \in \mathsf{N_F}$; then, we extend $\cdot^\mathcal{I}$ by defining $d \in \mathsf{succ}(f = \ell)^\mathcal{I}$ iff $f^\mathcal{I}(d) = \sigma_d(\ell)$. Unfortunately, for the CDs considered here, this leads to undecidability, which can be shown by a reduction to $\mathcal{ALC}(\mathfrak{D})$ with the concrete domain $\mathfrak{D} = (\mathbb{N}, +_1)$ where $+_1$ is the successor relation. TBox consistency in this DL is known to be undecidable [24].

Theorem 9.20. *If $\mathfrak{D}$ is a numerical concrete domain that is JD, then consistency of $\mathcal{ALCOSCC}(\mathfrak{D})$ TBoxes with mixed numerical constraints is undecidable.*

Proof. We force $r \in \mathsf{N_R}$ to be functional with the CI $\top \sqsubseteq \mathsf{succ}(|r| \leqslant 1)$. We encode the CD-restriction $C := \exists p_0, p_1.+_1$ using $C_0 \sqcap C_1$, where

$$C_i := \begin{cases} \mathsf{succ}(f_i = |S| + i) & \text{if } p_i = f_i \\ \mathsf{succ}(f_i' = |S| + i) \sqcap \mathsf{succ}(|r_i \cap (f_i' = \texttt{next}\, f_i)| \geqslant 1) & \text{if } p_i = r_i f_i. \end{cases}$$

for $i = 0, 1$, with fresh names $S \in \mathsf{N_C}$, $f_i' \in \mathsf{N_F}$. □

Local and global cardinality constraints. It is possible to extend $\mathcal{ALCSCC}$ by replacing succ-restrictions, ranging over sets of role successors, with sat-*restrictions* $\mathsf{sat}(\mathtt{con})$ ranging over the whole domain of an interpretation. For the resulting DL, called $\mathcal{ALCSCC}^{++}$, the consistency problem is NExpTime-complete [10]. In this DL, we can state that an individual likes *all* existing cars using the concept $\mathsf{sat}(\mathsf{likes} \cap \mathsf{Car} = \mathsf{Car})$; in contrast, $\mathsf{succ}(\mathsf{likes} \cap \mathsf{Car} = \mathsf{Car})$ describes an individual that likes all cars to which it is related by some role.

If we consider the DL $\mathcal{ALCSCC}^{++}(\mathfrak{D})$ obtained by adding sat-restrictions in the presence of concrete domains, then these restrictions may additionally contain feature roles. For example, the concept $\mathsf{sat}(\top = (\mathsf{age} \geqslant \texttt{next}\, \mathsf{age}))$ describes the *overall* oldest individuals, by saying that their age is greater or equal to those of all individuals, while $\mathsf{succ}(\top = (\mathsf{age} \geqslant \texttt{next}\, \mathsf{age}))$ describes individuals that are not younger than any individuals related to them by some role name.

Both $\mathcal{ALCSCC}^{++}$ and $\mathcal{ALCSCC}^{++}(\mathfrak{D})$ are evaluated over *finite* interpretations. In [10], this has been used to show that the consistency problem for the extension of $\mathcal{ALCSCC}^{++}$ with inverse roles is undecidable. Similarly, we can use sat-restrictions with feature roles to simulate multiplication of cardinalities of *finite* sets, and thus reduce Hilbert's tenth problem [81] to the consistency of a $\mathcal{ALCSCC}^{++}(\mathfrak{D})$ TBox, provided that $\mathfrak{D}$ is JD. Writing $C \equiv D$ as a shorthand for $C \sqsubseteq D$ and $D \sqsubseteq C$, we can encode the equation $\mathfrak{e} = (x = y \cdot z)$ over integers as a product of cardinalities $|A_x^\mathcal{I}| = |A_y^\mathcal{I}| \cdot |A_z^\mathcal{I}|$, in three steps. First, we enforce $r_\mathfrak{e}^\mathcal{I} = A_y^\mathcal{I} \times A_z^\mathcal{I}$ to hold with $A_y \equiv \mathsf{sat}(|r_\mathfrak{e}| \geqslant 1)$ and $A_y \equiv \mathsf{sat}(r_\mathfrak{e} = A_z)$; then, we enforce $|s_\mathfrak{e}^\mathcal{I}| = |A_x^\mathcal{I}|$ by adding $\top \sqsubseteq \mathsf{sat}(s_\mathfrak{e} = (f_\mathfrak{e} = \texttt{next}\, g_\mathfrak{e}))$ and the CIs

$$\top \sqsubseteq \mathsf{sat}(|(g_\mathfrak{e} = \texttt{next}\, f_\mathfrak{e})| \leqslant 1) \text{ and } A_x \sqsubseteq \mathsf{sat}(|(g_\mathfrak{e} = \texttt{next}\, f_\mathfrak{e})| \geqslant 1).$$

Finally, we add $\top \sqsubseteq \mathsf{sat}(|r_\mathfrak{e}| = |s_\mathfrak{e}|)$, so that, for every finite model $\mathcal{I}$ of all these CIs, $|A_x^\mathcal{I}| = |s_\mathfrak{e}^\mathcal{I}| = |r_\mathfrak{e}^\mathcal{I}| = |A_y^\mathcal{I} \times A_z^\mathcal{I}| = |A_y^\mathcal{I}| \cdot |A_z^\mathcal{I}|$ holds. We reduce the solvability of a system of Diophantine equations $\mathcal{E}$ over the natural numbers to the consistency of a $\mathcal{ALCSCC}^{++}(\mathfrak{D})$ TBox $\mathcal{T}_\mathcal{E}$. Without loss of generality, we assume that every equation in $\mathcal{E}$ is of the form $x = y \cdot z$, $x = y + z$ or $x = n$ with x, y, z variables and n a natural number. The TBox $\mathcal{T}_\mathcal{E}$ contains a concept name A_x for every variable x occurring in $\mathcal{E}$ and a conjunction of CIs $\mathcal{T}_\mathfrak{e}$ for every equation $\mathfrak{e} \in \mathcal{E}$, built as follows:

- if $\mathfrak{e} = (x = n)$, then $\mathcal{T}_\mathfrak{e}$ contains the CI $\top \sqsubseteq \mathsf{sat}(|A_x| = n)$;
- if $\mathfrak{e} = (x = y + z)$, then $\mathcal{T}_\mathfrak{e}$ contains the CI $\top \sqsubseteq \mathsf{sat}(|A_x| = |A_y| + |A_z|)$;
- if $\mathfrak{e} = (x = y \cdot z)$, then $\mathcal{T}_\mathfrak{e}$ contains the following CIs and *concept definitions* $C \equiv D$, which are a shorthand for $C \sqsubseteq D$ and $D \sqsubseteq C$:
  - $A_y \equiv \mathsf{sat}(|r_\mathfrak{e}| \geqslant 1)$ and $A_y \equiv \mathsf{sat}(r_\mathfrak{e} = A_z)$ where $r_\mathfrak{e}$ is a fresh role name;
  - $\top \sqsubseteq \mathsf{sat}(s_\mathfrak{e} = (f_\mathfrak{e} = \texttt{next}\, g_\mathfrak{e}))$ and $\top \sqsubseteq \mathsf{sat}(|(g_\mathfrak{e} = \texttt{next}\, f_\mathfrak{e})| \leqslant 1)$ as well as $A_x \sqsubseteq \mathsf{sat}(|(g_\mathfrak{e} = \texttt{next}\, f_\mathfrak{e})| \geqslant 1)$ with $s_\mathfrak{e} \in \mathsf{N_R}$ and $f_\mathfrak{e}, g_\mathfrak{e} \in \mathsf{N_F}$ fresh names;
  - $\top \sqsubseteq \mathsf{sat}(|r_\mathfrak{e}| = |s_\mathfrak{e}|)$.

The key result for the correctness of the reduction is the following.

Lemma 9.21. *If $\mathfrak{e} = (x = y \cdot z)$ then $|A_x^\mathcal{I}| = |A_y^\mathcal{I}| \cdot |A_z^\mathcal{I}|$ for every model $\mathcal{I}$ of $\mathcal{T}_\mathfrak{e}$.*

Proof. First, we show that $r_\mathfrak{e}^\mathcal{I} = A_y^\mathcal{I} \times A_z^\mathcal{I}$. If $(d, e) \in r_\mathfrak{e}^\mathcal{I}$ holds, then $d \in A_y^\mathcal{I}$ follows from $A_y \equiv \mathsf{sat}(|r_\mathfrak{e}| \geqslant 1)$; in turn, this implies that $e \in A_z^\mathcal{I}$ due to $A_y \equiv \mathsf{sat}(r_\mathfrak{e} = A_z)$. Vice versa, if $d \in A_y^\mathcal{I}$ and $e \in A_z^\mathcal{I}$, then $A_y \equiv \mathsf{sat}(r_\mathfrak{e} = A_z)$ implies that $(d, e) \in r_\mathfrak{e}^\mathcal{I}$. We conclude that $(d, e) \in r_\mathfrak{e}^\mathcal{I}$ iff $d \in A_y^\mathcal{I}$ and $e \in A_z^\mathcal{I}$.

Next, we show that $|s_\mathfrak{e}^\mathcal{I}| = |A_x^\mathcal{I}|$. To show that $|s_\mathfrak{e}^\mathcal{I}| \geqslant |A_x^\mathcal{I}|$ holds, we observe that for every $e \in A_x^\mathcal{I}$ there exists $d \in \Delta^\mathcal{I}$ such that $f_\mathfrak{e}^\mathcal{I}(d) = g_\mathfrak{e}^\mathcal{I}(e)$ by $A_x \sqsubseteq \mathsf{sat}(|(g_\mathfrak{e} = \texttt{next}\, f_\mathfrak{e})| \geqslant 1)$, and this implies that $(d, e) \in s_\mathfrak{e}^\mathcal{I}$ by $\top \sqsubseteq \mathsf{sat}(s_\mathfrak{e} = (f_\mathfrak{e} = \texttt{next}\, g_\mathfrak{e}))$. Thus, $s_\mathfrak{e}^\mathcal{I}$ contains at least as many tuples as elements of $A_x^\mathcal{I}$. To establish $|s_\mathfrak{e}^\mathcal{I}| \leqslant |A_x^\mathcal{I}|$, we notice that the function h mapping $(d, e) \in s_\mathfrak{e}^\mathcal{I}$ to $e \in \Delta^\mathcal{I}$ is an injective function from $s_\mathfrak{e}^\mathcal{I}$ to $A_x^\mathcal{I}$. All the CIs in $\mathcal{T}_\mathfrak{e}$ imply that $e \in A_x^\mathcal{I}$. Assuming that $h((d, e)) = h((d', e'))$ and thus $e = e'$, the fact that

$$f_\mathfrak{e}^\mathcal{I}(d) = g_\mathfrak{e}^\mathcal{I}(e) = g_\mathfrak{e}^\mathcal{I}(e') = f_\mathfrak{e}^\mathcal{I}(d')$$

together with $\top \sqsubseteq \mathsf{sat}(|(g_\mathfrak{e} = \texttt{next}\, f_\mathfrak{e})| \leqslant 1)$ implies $d = d'$, hence that h is injective. Finally, we use the CI $\top \sqsubseteq \mathsf{sat}(|r_\mathfrak{e}| = |s_\mathfrak{e}|)$ to conclude that

$$|A_x^\mathcal{I}| = |s_\mathfrak{e}^\mathcal{I}| = |r_\mathfrak{e}^\mathcal{I}| = |A_y^\mathcal{I} \times A_z^\mathcal{I}| = |A_y^\mathcal{I}| \cdot |A_z^\mathcal{I}|,$$

where the last identity holds because the domain of $\mathcal{I}$ is finite. □

Theorem 9.22. *A system of Diophantine equations $\mathcal{E}$ has a solution over the natural numbers iff the TBox $\mathcal{T}_\mathcal{E}$ is consistent.*

Proof. Assume that $\mathcal{I}$ is a finite model of $\mathcal{T}_\mathcal{E}$. Then, the assignment $x_\star := |A_x^\mathcal{I}|$ to every variable x is a solution of all equations $\mathfrak{e} \in \mathcal{E}$. This is trivial for $\mathfrak{e} = (x = n)$ and $\mathfrak{e} = (x = y + z)$, and Lemma 9.21 shows that this holds for $\mathfrak{e} = (x = y \cdot z)$.

Vice versa, assume that $\mathcal{E}$ has a solution assigning the natural number $x_\star$ to every variable x, and that every value assigned by this solution is smaller or equal than the natural number $n_\mathcal{E}$. We define the finite interpretation $\mathcal{I}$ with domain $\Delta^\mathcal{I} := \{1, \dots, n_\mathcal{E}\}$ with $A_x^\mathcal{I} := \{1, \dots, x_\star\}$ for every variable x. Assuming that $\mathfrak{e} = (x = y \cdot z)$, we define the interpretation of role names

$$r_\mathfrak{e}^\mathcal{I} := A_y^\mathcal{I} \times A_z^\mathcal{I} \text{ and } s_\mathfrak{e}^\mathcal{I} := \{(i, i \cdot j) \mid i \in A_y^\mathcal{I}, 1 \leqslant j \leqslant z_\star\}.$$

To define the interpretation of feature names $f_\mathfrak{e}$, $g_\mathfrak{e}$, we assume that $h_\mathfrak{e}$ is an injective mapping from $A_y^\mathcal{I}$ to D, which always exists since we assumed $\mathfrak{D}$ to be infinite. Then, we define $f_\mathfrak{e}^\mathcal{I}(i) := h_\mathfrak{e}(i)$ iff $i \in A_y^\mathcal{I}$ and $g_\mathfrak{e}^\mathcal{I}(j) := f_\mathfrak{e}^\mathcal{I}(i)$ iff $i \in A_y^\mathcal{I}$ and $(i, j) \in s_\mathfrak{e}^\mathcal{I}$. It is then straightforward to verify that $\mathcal{I}$ is a finite model of $\mathcal{T}_\mathcal{E}$. □

Theorem 9.23. *If the concrete domain $\mathfrak{D}$ is infinite and JD, then the consistency problem for $\mathcal{ALCSCC}^{++}(\mathfrak{D})$ TBoxes is undecidable.*

Transitive roles. Often, we may want a role name to be interpreted as a transitive relation: for instance, trans(ancestor) in the TBox expresses the fact that the ancestor of an ancestor is also an ancestor. The interaction between number restrictions and transitivity axioms in the presence of role inclusions is known to lead to undecidability [64]. It is possible to regain decidability by disallowing transitive roles within number restrictions, even in the presence of inverse roles [64]. Another restriction that leads to decidability is to replace number restrictions with role functionality axioms; in this case, decidability holds even if one additionally allows nominals and inverse roles [57].

In the DL $\mathcal{SSCC}$ that extends $\mathcal{ALCSCC}$ with transitivity axioms, consistency is undecidable even under all syntactic constraints mentioned above. In particular, we require that numerical constraints contain no transitive roles and no constants other than 0 or 1. We show how to reduce the solvability of a tiling problem P (cf. Definition 7.19) to the consistency of a restricted $\mathcal{SSCC}$ TBox $\mathcal{T}_P$. Adapting the reduction introduced in [64], we introduce concept names A_t for $t \in T$ and role names h and v meant to capture the matching conditions of P. If $\mathcal{I}$ is a model of $\mathcal{T}_P$, we ensure that every $d \in \Delta^\mathcal{I}$ has exactly one tile type and enforce the existence of exactly one h- and one v-successor for d using the CI

$$\begin{aligned} \top &\sqsubseteq \textstyle\bigsqcup_{t \in T} A_t \sqcap \bigsqcap_{t \neq t' \in T} \neg(A_t \sqcap A_{t'}) \\ \top &\sqsubseteq \mathsf{succ}(|h| = 1 \wedge |v| = 1). \end{aligned} \qquad \text{(successors)}$$

The matching conditions of P are encoded in $\mathcal{T}_P$ by adding for $t \in T$ the CIs

$$A_t \sqsubseteq \mathsf{succ}(h \subseteq \textstyle\bigsqcup_{(t,t') \in H} A_{t'}) \text{ and } A_t \sqsubseteq \mathsf{succ}(v \subseteq \bigsqcup_{(t,t') \in V} A_{t'}). \qquad \text{(matching)}$$

For every model $\mathcal{I}$ of $\mathcal{T}_P$ we want to ensure that every v-successor of a h-successor of $d \in \Delta^\mathcal{I}$ is also a role successor of d, and similarly for every h-successor of a v-successor. As $\mathcal{I}$ must be finitely branching, though, it is not possible to simply use one transitive role r that includes both

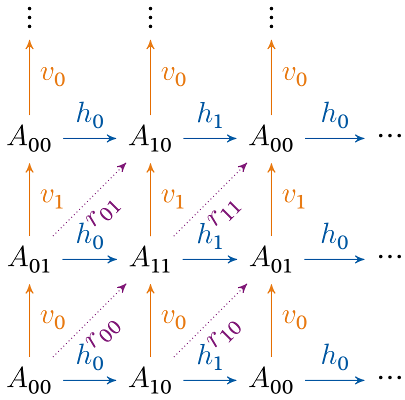


Figure 9.2: A representation of the structure enforced using transitive roles.

h and v, as this would imply that d has infinitely many successors. Instead, we introduce role names h_i, v_j and r_{ij} with $i, j \in \{0, 1\}$, where r_{ij} is a transitive superrole of h_i and v_j thanks to

$$\mathsf{trans}(r_{ij}) \text{ and } h_i \sqsubseteq r_{ij} \text{ and } v_j \sqsubseteq r_{ij} \text{ for } i, j \in \{0, 1\}, \tag{super}$$

where $r \sqsubseteq s$ is an abbreviation for $\top \sqsubseteq \mathsf{succ}(r \subseteq s)$. Then, we partition the domain with four concept names A_{ij} with $i, j \in \{0, 1\}$ using a similar CI as the one used in (successors). If d is labelled with A_{ij}, then none of its successors should be labelled with the same concept A_{ij}; further, for d the roles h_i and v_j act as h and v and connect to other individuals d' labelled by concepts $A_{i'j'}$ following the CIs

$$\begin{aligned} &A_{ij} \sqsubseteq \mathsf{succ}(h \subseteq A_{(1-i)j} \wedge v \subseteq A_{i(1-j)}) \text{ and} \\ &A_{ij} \sqsubseteq \mathsf{succ}(h = h_i \wedge v = v_j) \text{ for } i, j \in \{0, 1\}. \end{aligned} \tag{local}$$

These axioms enforce an alternating pattern of roles that ensures that, if d belongs to A_{ij}, then it has finitely many r_{ij}-successors.

What is left is to ensure for models $\mathcal{I}$ of $\mathcal{T}_P$ is that the v-successor of the h-successor of $d \in \Delta^{\mathcal{I}}$ is equal to the h-successor of the v-successor of d. Since both are also successors of d thanks to the presence of transitive roles, we force them to be equal by introducing the CI

$$\top \sqsubseteq \mathsf{succ}(|h^c \cap v^c| = 1). \tag{$\star$}$$

The effect of (super), (local) and ($\star$) on the models of $\mathcal{T}_P$ is showed in Figure 9.2. We establish the crucial property enjoyed by the models of $\mathcal{T}_P$ below.

Lemma 9.24. *If $\mathcal{I}$ is a model of $\mathcal{T}_P$, then the binary relations $h^{\mathcal{I}} \circ v^{\mathcal{I}}$ and $v^{\mathcal{I}} \circ h^{\mathcal{I}}$ coincide and are functional.*

Proof. If $\mathcal{I}$ be a model of $\mathcal{T}_P$, (successors) guarantees that for every $d \in \Delta^{\mathcal{I}}$ that there are four individuals $d_1, d_2, e_1, e_2 \in \Delta^{\mathcal{I}}$ such that $(d, d_1) \in h^{\mathcal{I}}$, $(d_1, d_2) \in v^{\mathcal{I}}$, $(d, e_1) \in v^{\mathcal{I}}$ and $(e_1, e_2) \in h^{\mathcal{I}}$. By (super) and (local) we deduce that $d \in A_{ij}^{\mathcal{I}}$ iff $(d, d_2) \in r_{ij}^{\mathcal{I}}$ and $(d, e_2) \in r_{ij}^{\mathcal{I}}$ for $i, j \in \{0, 1\}$. Moreover, d_2 and e_2 are both different from d_1 and e_1, since the concepts A_{ij} are disjoint for $i, j \in \{0, 1\}$ and by (local). Together with (successors), this implies that $d_2, e_2 \notin h^{\mathcal{I}}(d)$ and $d_2, e_2 \notin v^{\mathcal{I}}(d)$. Then, we conclude by ($\star$) that $d_2 = e_2$ must hold and that $h^{\mathcal{I}} \circ v^{\mathcal{I}}$ and $v^{\mathcal{I}} \circ h^{\mathcal{I}}$ coincide. □

Lemma 9.25. *The tiling problem P has a solution iff $\mathcal{T}_P$ is consistent.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{T}_P$. We define the mapping $\pi\colon \mathbb{N}\times\mathbb{N}\to\Delta^{\mathcal{I}}$ inductively, as follows. First, let $\pi(0,0)$ be an arbitrary individual in $\mathcal{I}$, which exists since this set must be non-empty. Assuming that for $i,j\in\mathbb{N}$ the value $\pi(i,j) := d$ is defined, we define $\pi(i+1,j)$ as the unique h-successor of d in $\mathcal{I}$ $\pi(i,j+1)$ as the unique v-successor of d in $\mathcal{I}$. Lemma 9.24 guarantees that π is well-defined: indeed, the individual $\pi(i+1,j+1)$ is supposed to be both the unique h-successor of $\pi(i,j+1)$ and the unique v-successor of $\pi(i,j+1)$, and the lemma ensures that these are indeed the same elements. Clearly, for all $i,j\in\mathbb{N}$

$$(\pi(i,j),\pi(i+1,j))\in h^{\mathcal{I}} \text{ and } (\pi(i,j),\pi(i,j+1))\in v^{\mathcal{I}}. \tag{9.1}$$

Using π, we define $\pi_P\colon \mathbb{N}\times\mathbb{N}\to T$ as $\pi_P(i,j) := t$ iff $\pi(i,j)\in A_t^{\mathcal{I}}$. Then, the fact that $\mathcal{I}$ satisfies (successors) and (9.1) ensures that π_P is a solution of P.

Next, let π be a solution of P. We define the interpretation $\mathcal{I}_\pi$ with domain $\mathbb{N}\times\mathbb{N}$ as follows. For each tile type $t\in T$, we set $A_t^{\mathcal{I}_\pi}$ as the set of elements (m,n) for which $\pi(m,n)=t$. For each element (m,n) in the domain, we add $((m,n),(m+1,n))$ to $h^{\mathcal{I}_\pi}$ and $((m,n),(m,n+1))$ to $v^{\mathcal{I}_\pi}$. Then, writing $(m\equiv i \mod 2)$ to denote that the remainder of the division of $m\in\mathbb{N}$ by 2 is i (and similarly for n and j), we set for $i,j\in\{0,1\}$

$$\begin{aligned}
A_{ij}^{\mathcal{I}_\pi} &:= \{(m,n)\in\Delta^{\mathcal{I}_\pi} \mid m\equiv i \mod 2,\ n\equiv j \mod 2\}\\
h_i^{\mathcal{I}_\pi} &:= \{((m,n),(m+1,n)) \mid (m,n)\in A_{ij}^{\mathcal{I}_\pi}\}\\
v_j^{\mathcal{I}_\pi} &:= \{((m,n),(m,n+1)) \mid (m,n)\in A_{ij}^{\mathcal{I}_\pi}\}\\
r_{ij}^{\mathcal{I}_\pi} &:= h_i^{\mathcal{I}_\pi}\cup v_j^{\mathcal{I}_\pi}\cup\{((m,n),(m+1,n+1)) \mid (m,n)\in A_{ij}^{\mathcal{I}_\pi}\}
\end{aligned}$$

It is straightforward to verify that $\mathcal{I}_\pi$ is a model of $\mathcal{T}_P$. □

Theorem 9.26. *Consistency in $\mathcal{SSCC}$ is undecidable, even if numerical constraints contain no transitive roles and no constants other than* 0 *or* 1.

In the above reduction, transitive roles do not explicitly occur within number restrictions. However, the semantics of $\mathcal{SSCC}$ is such that every succ-restriction implicitly ranges over all roles, including the transitive ones. If we disallow the usage of the set complement operator $\cdot^c$ on roles, which is employed in (★), we could still enforce correctness of the reduction by adding $A_{ij}\sqsubseteq \mathsf{succ}(|A_{(1-i)(1-j)}|=1)$ to $\mathcal{T}_P$, which works because of the implicit ranging over role successors induced by the semantics of $\mathcal{SSCC}$.

One may ask if lifting this semantic condition allows us to regain decidability. If we consider the extension $\mathcal{SSCC}^{++}$ of $\mathcal{SSCC}$, defined in the same spirit of $\mathcal{ALCSCC}^{++}$ w.r.t. $\mathcal{ALCSCC}$, we can replace (★) with $A_{ij}\sqsubseteq \mathsf{sat}(|r_{ij}\cap A_{(1-i)(1-j)}|=1)$ to cause undecidability. While only using coefficients 0 or 1, this variant requires an explicit usage of transitive roles in number restrictions. It is unclear if the same can effect can be achieved while disallowing transitive roles within numerical constraints.

Summary

We introduced the very expressive DL $\mathcal{ALCOSCC}(\mathfrak{D})$ that supports concrete domain restrictions and role successor restrictions involving feature values. We showed that consistency in this logic is ExpTime-complete, the same as for the basic DL $\mathcal{ALC}$, if $\mathfrak{D}$ is ExpTime-ω-admissible. Moreover, we have discussed the consequences of adding assertions, transitive roles, unrestricted semantics, or mixed constraints, most of which make the logic undecidable. Additionally to these results, we showed that reasoning in $\mathcal{ALCOSCC}(\mathfrak{D})$ is undecidable if we allow the comparison of feature values and cardinalities of sets of role successors if $\mathfrak{D}$ is a numerical concrete domain.

10 Conclusion

This thesis provides a comprehensive view on Description Logics extended with concept or terminological constructors based on *cardinality constraints* expressed in the logic QFBAPA or on *concrete domains*. We established the complexity of reasoning in the DLs obtained by considering one extension at a time or multiple extensions at once. In particular, we presented results for $\mathcal{ALCSCC}$, $\mathcal{ALCSCC}^\infty$, $\mathcal{ALC}(\mathfrak{D})$, $\mathcal{ALCOSCC}(\mathfrak{D})$ and RCBoxes, ERCBoxes and ECBoxes. Further, we investigated the expressive power of these logics, using for each a suitable notion of bisimulation that allowed us to compare the expressive power of DLs or to characterize it w.r.t. FOL or FOL($\mathfrak{D}$). To enable the comparison of logics with and without concrete domains, we also introduced the notion of *abstract expressive power* and used it to derive interesting first-order properties of logics with concrete domains.

In Chapter 3 we presented existing results on concept satisfiability w.r.t. a TBox written in the DLs $\mathcal{ALCSCC}$ [7] and $\mathcal{ALCSCC}^\infty$ [16], and on generalizations of TBoxes and CBoxes based on *(semi-)restricted* or *extended cardinality constraints* [19, 93, 8, 10]. We derived complexity results for $\mathcal{ALCSCC}$ and arbitrary interpretations that correspond to what was shown in these papers w.r.t. finite or finitely branching models. Additionally, we established the complexity of the entailment problem for Boolean combinations of (semi-)restricted cardinality constraints, where reducibility to consistency depends on the employed coefficients, enriching the existing landscape of complexity results for reasoning with global cardinality constraints in description logics. The main complexity results derived in Chapter 3 are reported in Table 10.1.

In Chapter 4 we analyzed the expressive power of $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ as concept languages using the notion of *local Presburger (Pr) bisimulation*, which strengthens the known notion of *counting bisimulation* [80] by applying the back-and-forth conditions to *safe role types* rather than role names. We showed that both $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ concepts are invariant under Pr bisimulation and used this property to show non-definability results w.r.t. these DLs. In [16] we showed that the first-order definable fragment of $\mathcal{ALCSCC}^\infty$ corresponds to the DL $\mathcal{ALCQ}t$, by showing that this DL is exactly the fragment of first-order logic that is invariant under Pr bisimulation. Here, we generalized this result to both $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ w.r.t. arbitrary, finitely branching and finite interpretations, using an approach based on strong locality properties of FOL that follows the treatment of Otto for graded modal logic (i.e. $\mathcal{ALCQ}$) in terms of counting bisimulation [84]. Using (q,ℓ)-bisimulations, we showed that $\mathcal{ALCQ}t$ and $\mathcal{ALCSCC}$ can be separated, thus showing that the latter DL contains concepts that are not first-order definable and is more expressive than the former DL. These results are summarized in Figure 10.1,

		$\mathcal{ALCSCC}$	$\mathcal{ALCSCC}^\infty$
Concept satisfiability	with no TBox	PSpace-c. [7]	PSpace-c. [16]
	w.r.t. a TBox	ExpTime-c. [7]	ExpTime-c. [16]
Boolean TBox consistency		ExpTime-c.	ExpTime-c.
RCBox consistency		ExpTime-c. [8]	ExpTime-c.
ERCBox	consistency	ExpTime-c. [10]	ExpTime-c.
	entailment	coNExpTime-c.	coNExpTime-c.
CBox consistency		NExpTime-c. [8]	NExpTime-c.
ECBox consistency		NExpTime-c. [8]	NExpTime-c.

Table 10.1: Complexity results discussed for Chapter 3, where we assume binary coding of numbers. Complexity of entailment is only indicated where it differs from consistency. Each entry without citation corresponds to a contribution of this thesis.

where we further relate the expressive power of these logics with other concept languages studied in this thesis.

Chapter 5 has been devoted to the analysis of the expressive power of knowledge bases written using local and global cardinality constraints, which we conducted by means of *global* and *comparative* Pr *bisimulations*. In particular, we showed that the set of Boolean $\mathcal{ALCQ}t$ TBoxes is the fragment of first-order logic that is invariant under global Pr bisimulation, and that Boolean $\mathcal{ALCQ}t$ CBoxes play a similar role w.r.t. comparative Pr bisimulation. Using the 0-1 law of first-order logic, we showed that even simple RCBoxes cannot be defined in first-order logic, both w.r.t. arbitrary and finite interpretations. A classification of the expressive power results obtained in Chapter 5 is depicted in Figure 10.2.

In Chapter 6 we introduced the notion of ExpTime-ω-admissible concrete domain, and proved that the consistency problem for $\mathcal{ALC}(\mathfrak{D})$ ontologies is ExpTime-complete if $\mathfrak{D}$ is ExpTime-ω-admissible, thus proving the conjecture posed in [79] and further showing that concept and role assertions can be added without an increase in complexity.

In Chapter 7 we studied the notion of *abstract expressive power* of logics with concrete domains. We established sufficient conditions on the concrete domain that ensure that the resulting extensions of FOL and $\mathcal{ALC}$ satisfy (countable) compactness or other important first-order properties, and use the notion of abstract (non-)definability to derive (un-)decidability results for several fragments of FOL($\mathfrak{D}$). These results are summarised in Table 10.2 and Table 10.3.

In Chapter 8 we introduced a notion of bisimulation for $\mathcal{ALC}(\mathfrak{D})$ and used it to characterize its expressive power w.r.t. FOL($\mathfrak{D}$), as well as to show that several of its extensions are strictly more expressive, similarly to what was done in Chapter 4. Moreover, we compared different notions of bisimulations with concrete domains and showed how to relate them to obtain characterizations for $\mathcal{ALC}_{\vee+}(\mathfrak{D})$ and $\mathcal{ALC}_{\mathsf{fo}}(\mathfrak{D})$.

Finally, we presented the very expressive DL $\mathcal{ALCOSCC}(\mathfrak{D})$ that supports concrete domain restrictions and role successor restrictions involving feature values. We have shown that consistency in this logic is ExpTime-complete, the same as for the basic DL $\mathcal{ALC}$. Moreover, we have discussed the consequences of adding assertions, transitive roles, unrestricted semantics, or mixed constraints, most of which make the logic undecidable. For reasoning with ExpTime-ω-admissible concrete domains, we report the results obtained in Chapters 6 and 9 in Table 10.4. Additionally to these results, in Chapter 9 we showed that reasoning in $\mathcal{ALCOSCC}(\mathfrak{D})$ is unde-

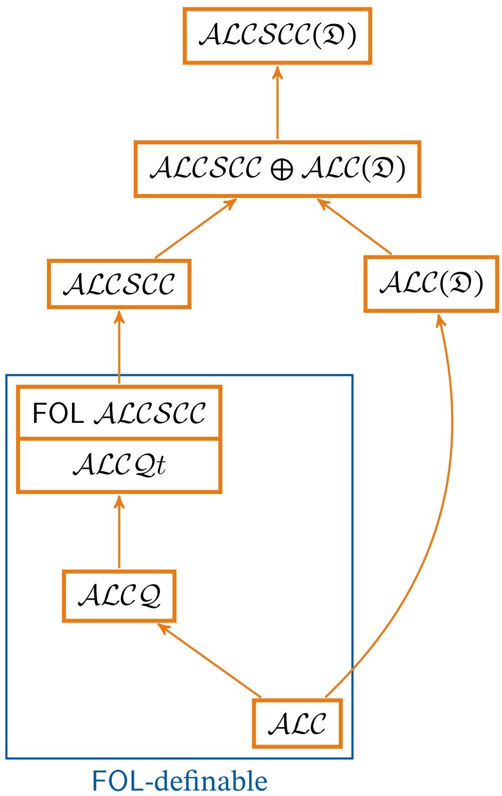


Figure 10.1: The relative expressivity of the concept languages studied in this thesis. An arrow from a node N to a node N′ means that the concept language N can be expressed in N′ and that N is strictly less expressive than N′.

cidable if we allow the comparison of feature values and cardinalities of sets of role successors if $\mathfrak{D}$ is a numerical concrete domain.

Related Work and Open Problems

In addition to what has already been mentioned, we would like to point out other existing work that relates to what analyzed in this thesis, highlighting potential venues for future work.

Expressive Power of ECBoxes and TBoxes over restricted classes of models. The characterizations of the expressive power of box formalisms shown in Chapter 5 only hold w.r.t. arbitrary interpretations. We have seen in Chapter 4 that, by adopting appropriate model transformations, the characterization results for the expressive power of concept languages could be also w.r.t. finitely branching and finite interpretations. We conjecture that this approach can be adopted to generalize the results contained in Chapter 5. In [85], for instance, this locality-based approach has been used to characterize the modal logic K with a global operator, which is analogous to Boolean $\mathcal{ALC}$ TBoxes. While the results shown in Chapter 4 and [85] use a notion of locality based on the Gaifman graph of an interpretation, we could use other notions of locality, such as Hanf locality, and related results from finite model theory (see e.g. [44]).

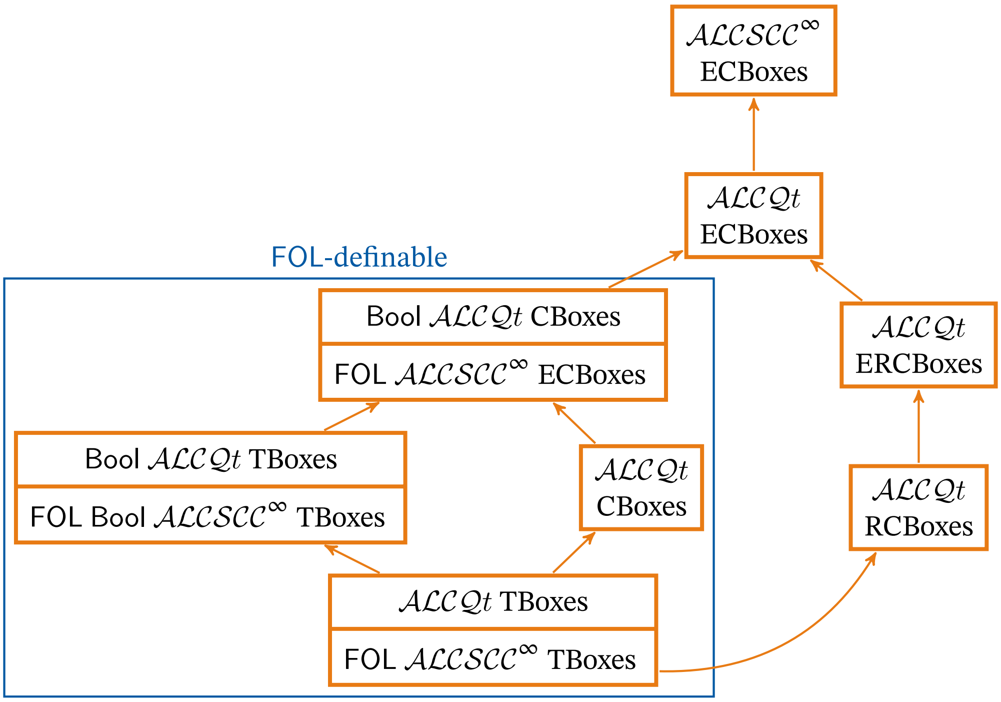


Figure 10.2: A visual representation of the expressivity hierarchy for TBoxes and their extensions. An arrow from a node N to a node N′ means that all the languages in N, which are equivalent, are strictly less expressive than those in N′.

Conservative extensions and rewritability. The notion of expressive power employed in this thesis focuses on the classes of models defined by a concept, knowledge base or formula over a fixed signature. However, there are alternative notions of expressive power that allow the usage of additional predicates in the translation of one of these objects, similarly to what we called *projective abstract definability* in Chapter 7. This alternative view is based on the concept of *conservative extension* [6]. For TBoxes, the problem of deciding if a TBox in a certain DL admits a rewriting in a weaker DL was tackled in [80], and these results have been successively expanded to account for conservative rewritability [69]. For the extended CRs that we investigated in Chapter 5, early results on conservative rewritability or RCBoxes and ERCBoxes appeared in [93], where it is shown that these constructs can be expressed using a number of powerful DL constructors. Recently, some techniques discussed in these papers have been applied to separate counting logics from non-counting ones [70]. It is yet unclear if $\mathcal{ALCSCC}$ concepts can be conservatively rewritten in $\mathcal{ALCQ}$, though we suspect that by adapting the results contained in [80] one could establish the decidability of the rewritability problem of first-order definable $\mathcal{ALCSCC}$ TBoxes into $\mathcal{ALCQ}$ ontologies. In Chapter 7 we found instances of $\mathfrak{D}$ such that $\mathcal{ALC}(\mathfrak{D})$ TBoxes cannot be expressed in $\mathcal{ALC}$ nor in FOL, and instances that instead allowed us to capture the abstract expressive power of these TBoxes in FOL or even $\mathcal{ALC}$. Similarly to what we discussed for cardinality constraints, we think that by combining the techniques in [80, 69] one can investigate the problem of deciding, for a given $\mathcal{ALC}(\mathfrak{D})$ TBox, whether it can be abstractly (projectively) defined in FOL ($\mathcal{ALC}$).

	$\mathsf{FOL}(\mathfrak{D})$	$\mathcal{ALC}(\mathfrak{D})$
$\mathfrak{D}$ is strongly positive and homomorphism ω-compact		
Downward Löwenheim-Skolem	yes	yes
Countable Compactness	yes	yes
Upward Löwenheim-Skolem	—	yes
Craig Interpolation	abstract	—
$\mathfrak{D}$ is strongly positive and unary		
Abstract definability	FOL (projective)	$\mathcal{ALC}$ (projective)
Craig Interpolation	abstract	abstract
$\mathfrak{D}$ is countably infinite and jointly diagonal		
Abstract FOL-definability	no	no
Upward Löwenheim-Skolem	no	—

Table 10.2: Abstract expressive power results shown in Chapter 7.

	$\mathfrak{D}$ strongly positive and unary	$\mathfrak{D}$ infinite and jointly diagonal
Satisfiability of $\mathsf{GF}_2(\mathfrak{D})$	decidable	—
Satisfiability of $\mathsf{FOL}_2(\mathfrak{D})$	decidable	undecidable
Satisfiability of $\mathcal{C}^2(\mathfrak{D})$	decidable	undecidable

Table 10.3: Complexity results derived in Chapter 7.

Adding more to $\mathcal{ALCOSCC}(\mathfrak{D})$. In Chapter 9 we presented $\mathcal{ALCOSCC}(\mathfrak{D})$, a very expressive DL that supports concrete domain restrictions and role successor restrictions involving feature values through *feature roles*. We have shown that consistency in this logic is ExpTime-complete, the same as for the basic DL $\mathcal{ALC}$. While feature roles can already express a restricted form of inverse roles, in the future, we would like to investigate the decidability and complexity of $\mathcal{ALCOISCC}(\mathfrak{D})$ with full inverse roles, for which it is known that they increase the complexity of classical DLs with nominals and number restrictions to NExpTime [96].

Two-variable guarded fragment with Presburger counting. The logic GPres_2 [27] is an extension of GC_2 with local Presburger constraints that generalizes $\mathcal{ALCSCC}$ and $\mathcal{ALCSCC}^\infty$ by allowing e.g. for inverse roles. Its satisfiability problem is ExpTime-complete [26, 77] w.r.t. finitely branching interpretations and decidable in 3NExpTime if only finite interpretations are allowed [27]. In terms of expressive power, one could ask if GC_2 can be characterized as the first-order fragment of GPres_2 w.r.t. the classes of finitely branching and finite interpretations, perhaps using a stronger notion of bisimulation than the one employed in Chapter 4 that is tailored to the guarded setting. On the other hand, adding concrete domain reasoning with definedness and concrete domain predicates as done in Chapter 7 may yield a decidable logic with very expressive cardinality constraints and concrete domains. This logic subsumes $\mathcal{ALCOISCC}(\mathfrak{D})$ and thus, as discussed earlier, reasoning would become at least NExpTime-hard [96].

Extending the two-variable fragment with counting. As mentioned in Chapter 2 the satisfiability problem for the logic $\mathcal{C}^2$ is NExpTime-complete for both unary and binary coding of

	$\mathcal{ALC}(\mathfrak{D})$	$\mathcal{ALCOSCC}(\mathfrak{D})$
Consistency	ExpTime-c.	ExpTime-c.
Consistency with constants (†)	ExpTime-c.	ExpTime-c.
	$\mathcal{SSCC}$	$\mathcal{ALCSCC}^{++}(\mathfrak{D})$
Consistency	undecidable	undecidable

Table 10.4: Complexity results derived in Chapters 6 and 9, where $\mathfrak{D}$ is ExpTime-ω-admissible. Where marked with †, $\mathfrak{D}$ fulfills additional conditions stated in Chapter 9.

numbers [90]. We have seen in Chapter 3 that the consistency problem for $\mathcal{ALCSCC}^\infty$ and $\mathcal{ALCSCC}$ ECBoxes is similarly NExpTime-complete, both for finite and arbitrary models. The next logical step would be to investigate the satisfiability problem for $\mathcal{C}^2$ extended with Presburger counting.

In this context, looking at extensions of $\mathcal{C}^2$ with reasoning over ω-admissible concrete domains is not fruitful, as we already established for FOL_2 that the satisfiability problem of $\mathsf{FOL}_2(\mathfrak{D})$ is undecidable if $\mathfrak{D}$ is infinite and JD (cf. Chapter 7). Additionally, the handling of concrete domain reasoning w.r.t. finite interpretations is not well understood, and even for less expressive concrete domains it may be unclear how to establish decidability results. One exception would be the case of unary concrete domains that are closed under negation: in this setting, we already proved in Chapter 7 that reasoning in the corresponding extension of $\mathcal{C}^2$ is decidable.

A Unifying Theory of Concrete Domains. As mentioned earlier, there are several existing complexity results on satisfiability w.r.t. a TBox for extensions of $\mathcal{ALC}$ by concrete domains that are not ω-admissible [36, 42, 43, 74] that complement our work and previous work on ω-admissible concrete domains [79, 24]. It is unclear whether these lines of research can converge, providing generalized criteria on concrete domains that yield decidability.

Expressive power of Graph Neural Networks (GNNs). Recently, the tools used to investigate the expressive power of counting logics have been employed to characterize the expressive power of GNNs [55]. In particular, it has been shown that $\mathcal{ALCQ}$ corresponds to the set of FOL-definable node classifiers that can be expressed using a class of GNNs called *aggregate-combine GNNs* [25]. This result has been further extended by investigating the relationship of GNNs and the *two-guarded fragment with Presburger counting* GPres_2 [28] or other extensions of $\mathcal{ALCQ}$ with fixpoint operators or Datalog construction [1, 89]. Given that the characterizations of $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$ that we provided in Chapters 4 and 8 holds also when restricted to finite models, it would be interesting to apply them to characterize what properties of node classifiers that are definable in these DLs can be learnt using a GNN.

Craig interpolation and concrete domains. In the concrete domain setting, we defined variants of Craig interpolations, according to whether the feature symbols in concrete domain restrictions are taken into account or not. Our results in Chapter 7 tackle abstract interpolation, where the feature symbols are ignored. It would be interesting to see whether one can also show concrete versions of the Craig interpolation property, maybe depending on whether the concrete domain itself satisfies an appropriate interpolation property [35].

Support for reasoning. The considerable expressive power of $\mathcal{ALCSCC}$ and RCBoxes, paired with their robust decidability, has been used to derive decidability for other extensions of standard DLs. For example, $\mathcal{ALCSCC}^{\infty}$ has been used to show the decidability of $\mathcal{ALC}$ extended with perceptron-like concepts [49], while CBoxes were already used in earlier results on DLs with circumscription reasoning [30] and RCBoxes provide an upper bound for reasoning with probabilistic conditionals on finite models [88]. However, practical support for reasoning with the DLs presented in this paper is currently missing, or only partially supported through limited encodings to OWL (e.g. [93] for RCBoxes). Similarly, reasoners for DLs with non-trivial concrete domains only exist for $\mathcal{ALC}(\mathfrak{D})$ and $\mathcal{EL}(\mathfrak{D})$ with so-called p-admissible concrete domains and without feature paths [3]. A plausible venue of research in this direction would be the design and implementation of reasoners that combine logical and numerical reasoning based on simplex, branch-and-bound, and SAT-based column generation [47] or with existing solvers for Satisfiability Modulo Theory theories that subsume p-admissible or ω-admissible concrete domains. SAT-based column generation has also been used to develop a reasoner for probabilistic conditionals [68] based on a different semantics than those defined in [88], which only consider finite interpretations and thus require additional checks that are not implemented in the reasoner presented in [68].

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