

The Emergence of Global Properties from Local Interactions

Static Properties and One-Dimensional Patterns

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ABSTRACT

Groups of simple objects, interacting under local rules, can produce global properties of great complexity and rich functionality. In this paper, we address of whether a given global property can emerge from purely local interactions. We provide a necessary condition for the case of static properties, and provide complementary sufficient conditions for patterns on the special case of one-dimensional lattices. This work shows that two general considerations, local checkability and symmetry, are the main determinants of “emergability,” at least in these special cases. This work also shows that relatively unified constructions provide solutions to large classes of patterns.

A General Model

Groups of simple objects, interacting under local rules, can produce global properties of great complexity and rich functionality ([1, 2, 3, 4]). In this paper, we address of whether a given global property can emerge from purely local interactions. First, we describe a model of stateless, stigmergy-less agents. The three fundamental objects in the model are configurations, local dynamics, and global properties. Configurations represent agents and the spaces in which they are situated as labeled graphs.

Definition 1 [Configurations] (Fig. 1)

A configuration is a graph $X = (N_X, E_X)$ of the form

$$X = (N_X, E_X) = (N_a \cup N_g, E_a \cup E_g),$$

in which: N_a is a set of finite set of nodes labeled by the symbol a ; N_g is a set of nodes labeled by the symbol g ; E_g is a set of edges (directed or undirected) between nodes in N_g ; and E_a is a set containing, for each $x \in N_a$, one directed edge of the form (x, y) , where y is a node in N_g that is unique for each x .

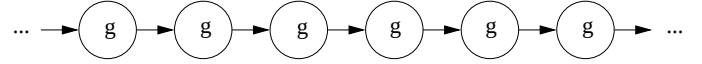
Denote by $\mathcal{C}_G$ the set of configurations for which $(N_g, E_g) = G$. For a set $\mathcal{G}$ of graphs G (called underlying spatial graphs), let $\mathcal{C}_\mathcal{G} = \bigcup_{G \in \mathcal{G}} \mathcal{C}_G$.

For example, the “directed line”, denoted $\overrightarrow{\mathbb{Z}}$, is the one dimensional integer lattice with global orientation given by taking $N_g = \{(g, n) | n \in \mathbb{Z}\}$ and $E_g = \{(i, i+1) | i \in \mathbb{Z}\}$. The graph $\overrightarrow{\mathbb{Z}}$, the “undirected line”, has the same N_g , but $E_g = \{(n, n+1) | n \in \mathbb{Z}\}$. The graph G could also be higher dimensional lattices $\overrightarrow{\mathbb{Z}}^n$, $\overline{\mathbb{Z}}^n$ defined in analogy to their one-dimensional versions, in addition to rings, spheres, and graphs with less regular structure.

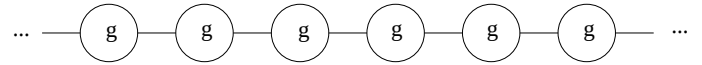
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AAMAS’06 May 8–12 2006, Hakodate, Hokkaido, Japan
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A)



B)



C)

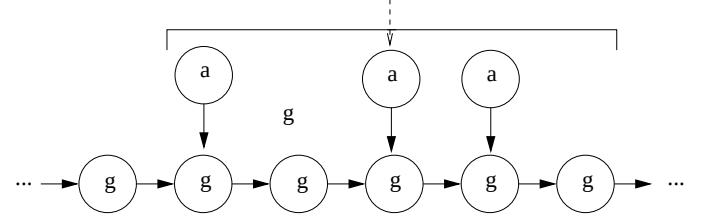


Figure 1: A) The infinite integer lattice with global orientation (“the directed line”). B) The infinite integer lattice without orientation. C) A configuration of three agents on the directed line. The bracket above the middle agent indicates the ball of radius 2 around it.

For a configuration $X = (N_g \cup N_a, E)$, let the *agents* of X be the set $Ag(X) = N_a$, and $|Ag(X)| = |N_a|$ be the number of agents in X . Given $x \in Ag(X)$, let the unique node number m for which $(x, (g, m)) \in E_a$ be denoted by $pos(x, X)$, the *position* of x in X . For $x, y \in N_g$, let $dist(x, y, X)$ be the graph distance in the underlying spatial graph G of X ; this extends to a distance between $x, y \in Ag(X)$ as the graph distance between $pos(x, X)$ and $pos(y, X)$. For $x \in X$, define the *neighborhood of radius r about x in X* , denoted $B_r(x, X)$, as the subgraph $(N_B, E_B) \subset (N_X, E_X)$ where $N_B = \{y \in N_X | dist(x, y, X) \leq r\}$. Let $\mathcal{C}_\mathcal{G}(r)$ denote the set of all r -neighborhoods $B_r(x, X)$ for $x \in Ag(X)$ and $X \in \mathcal{C}_\mathcal{G}$. A configuration X is *r -connected* if $\bigcup_{x \in Ag(x)} B_r(x, X)$ is a connected subgraph of X , and *r -disconnected* if it is not r -connected. The set of r -connected configurations is denoted by $\mathcal{C}_\mathcal{G}[r]$. For two subconfigurations $A = (N_A, E_A)$ and $B = (N_B, E_B)$, define the “graph sum” $A \oplus B = (N_A \cup N_B, E_A \cup E_B)$.

The dynamics of the model are specified by local agent-based rules of motion. For each agent $x \in Ag(X)$, the rule looks at the neighborhood B of some fixed radius r around x and decides whether to move x from its initial position $pos(x, B)$ to some other point g' that is at most one position away. Formally, such a rule can be described by a map f such that for each r -neighborhood $B_r(x, X) = (N_B, E_B)$,

$$f : (N_B, E_B) \mapsto (N_B, E_B \setminus \{(x, pos(x, B))\} \cup \{(x, g')\}) \quad (1)$$

where $dist(x, g', B) \leq 1$.

Definition 2 [Local Rules] A local rule with information radius r is any function $f : \mathcal{C}_G(r) \rightarrow \mathcal{C}_G(r)$ that obeys condition 1 above. Let $\mathcal{D}_r$ denote the set of all such f , and $\mathcal{D} = \cup_r \mathcal{D}_r$. We write $r(f)$ for the information radius of f .

When f fixes an agent $x \in X$, we write $[f(x, X)] = S$ (S stands for “stay”).

Given a sequence $s = (s_1, s_2, \dots) \in \text{Ag}^\infty(X)$ of calls to agents and an initial configuration X_0 , we could first apply f to (x_1, X_0) ; we could then apply f to $(x_2, X_1 = f(x_1, X_0))$, and repeat this iteratively. The resulting sequence represents trajectories generated by totally asynchronous updating, denoted by $f_s^\infty(X_0)$. Any such trajectory has an ω -limit set, denoted by $\Omega(f, s, X_0)$, which is the set of configurations that appear infinitely often in $\{f_s^n(X_0)\}$. $\Omega(f, s, X_0)$ is empty when no well-defined limit actually exists.

A choice of models for the concurrency of agent calls is given by specifying a set of sets $S = \{S_X | X \in \mathcal{C}_G\}$ of valid call sequences that is invariant under the graph-equivalence $\cong$ on $\mathcal{C}_G$. Totally asynchronous update (with no restriction on the relative frequency of calls to different agents) corresponds to setting $S_X^{\text{asynch}} = \text{Ag}(X)^\infty$, whereas totally synchronous update corresponds to the choice $S_X^{\text{synch}} = \{(\text{Ag}(X), \text{Ag}(X), \text{Ag}(X), \dots)\}$.

Global properties are given by specifying a target set $P \subset \mathcal{C}_G$ of ‘desired’ configurations. For example, *patterns*:

Definition 3 [Patterns] Fix a graph G . Let $(q)^{\oplus k}$ denote the set of subconfigurations Y of graphs over G of the form $Y = q_1 \oplus q_2 \oplus \dots \oplus q_k$, where the $q_1, \dots, q_k$ are all disjoint and graph-equivalent to q . A pattern over G is a subset $P \subset \mathcal{C}_G$ for which there exists a finite q such that

$$P = \bigcup_{k=0}^{\infty} \{Y \oplus G \mid Y \in (q)^{\oplus k}\}.$$

In this case, we write $P = \langle q \rangle$. We define the length of P , denoted $|P|$, to be the length of the underlying spatial graph of Y and the size of P , denoted $|\text{Ag}(P)|$, to be the number of agents in q .

Given a property $P \subset \mathcal{C}_G$, our motivating interest is to find (or prove the non-existence of) *solutions* to it, those dynamic rules whose trajectories satisfy the property.

Definition 4 [Solutions] A dynamic rule $f \in \mathcal{D}$ is a P -solution over $(\mathcal{G}, S)$ if, for all $X \in \mathcal{C}_G$, and all call-sequences $s \in S_X$, $\Omega(\{f_s^n(X)\}) \subset P$ holds.

The solution space $\mathcal{F}(\mathcal{G}, S, P)$ is the set of all φ -solutions, that is

$$\mathcal{F}(\mathcal{G}, S, P) = \{f \in \mathcal{D} \mid f \text{ is a } P\text{-solution over } (\mathcal{G}, S)\}.$$

P is locally solvable when $\mathcal{F}(\mathcal{G}, S, \varphi) \neq \emptyset$. Otherwise, P is locally unsolvable.

Local Checkability

Definition 5 A local check scheme with radius r is a function $\Theta : \mathcal{C}_G(r) \rightarrow \{0, 1\}$. The check radius $\Xi(P)$ of a property P is the minimal $r > 0$ such that there exists local check scheme Θ with $r(\Theta) = r$ such that

$$\bigwedge_{a \in \text{Ag}(X)} (\Theta(B_r(a, X)) = 1) \wedge X \in \mathcal{C}_G \Rightarrow X \in P,$$

if such exists. The connected check radius $\Xi^c(P)$ is the minimal $r > 0$ such that there exists Θ with $r(\Theta) = r$ such that

$$\bigwedge_{a \in \text{Ag}(X)} (\Theta(B_r(a, X)) = 1) \wedge X \in \mathcal{C}_G[r] \Rightarrow X \in P$$

if it exists.

If $\Xi(P) < \infty$, we say P is locally checkable.

Proposition 1 For all $\mathcal{G}$, S and P : $f \in \mathcal{F}(\mathcal{G}, S, P)$ only if $r(f) \geq \Xi(P)$.

Hence, $\Xi(P) = \infty \Rightarrow \mathcal{F}(\mathcal{G}, S, P) = \emptyset$; and $\Xi^c(P) = \infty \Rightarrow \mathcal{F}(\mathcal{G}, S, P) = \emptyset$. Thus:

Corollary 1 Static properties P is locally solvable over $\mathcal{G}$ only if it is are locally checkable over $\mathcal{G}$:

Obviously, patterns *do* meet the local checkability condition.

Corollary 1 establishes a necessary condition that must be met for any static property to be emergeable. In the next section, we answer the converse question when restricted to the special case of connected patterns on $\mathcal{G} = \{\vec{\mathbb{Z}}\}, \{\vec{\mathbb{Z}}\}$, and $\mathcal{R}$, the locally one-dimensional lattices. We begin by defining some notation.

Patterns on the Directed Line

Given $X \in \mathcal{C}_{\vec{\mathbb{Z}}}$, and $x, y \in X$, we say that x is *to the right of* y if there exists a directed path from $\text{pos}(x, X)$ to $\text{pos}(y, X)$ in $\vec{\mathbb{Z}}$, and if so, we write $y \leq x$. If such a path does not exist, then we say x is *to the left of* y , and write $x \leq y$. For any connected subconfiguration $Y \subset X$, let $\min(Y)$ be the left-most spatial node in Y , and $\max(Y)$ be that of the right-most. For $a, b \in \vec{\mathbb{Z}}$ with $a \leq b$, let $X[a, b]$ be the maximal connected subconfiguration of $Y \subset X$ with $\min(Y) = a$ and $\max(Y) = b$. For $x \in \text{Ag}(X) \in \mathcal{C}_{\vec{\mathbb{Z}}}$, define x^{+k} to be the k th agent to the right of x , if such exists; and x^{-k} to be the k -th agent to the left of x , if such exists. Define $\text{re}(X)$ to be the unique $x \in \text{Ag}(X)$ for which $x^+ = x^{+1}$ does not exist, and $\text{le}(X)$ to be the unique agent for which $x^- = x^{-1}$ does not exist. (These are the “right-end” and “left-end” agents of X , respectively.)

Any local rule f with radius r operating over the directed line $\vec{\mathbb{Z}}$ has three possible actions on any configuration $B_r(x, X)$. It can either move the agent x to the left, which we notate as $[f(x, X)] = L$; or it can move x to the right, which we notate as $[f(x, X)] = R$; or f can fix x , in which case $[f(x, X)] = S$ is just the same as previous defined notation.

It turns out that on $\vec{\mathbb{Z}}$, local checkability is sufficient for solutions to P to exist. To prove this, given a pattern P over $\vec{\mathbb{Z}}$, we construct a local rule that generates it. Let q be a generator in standard form of P . We first construct a rule that generates q when exactly $|\text{Ag}(q)|$ are present close enough together. In particular, let $\mathcal{Y}_q$ be the set of configurations Y with $|\text{Ag}(q)|$ agents and such that $\text{dist}(\text{le}(Y), \text{re}(Y)) \leq |q|$. Let $Y \in \mathcal{Y}_q$. Then define $T \in \mathcal{D}_{|q|}$ by

$$[T(a, Y)] = \begin{cases} L & \text{if } a = a_Y^* \\ S & \text{otherwise} \end{cases}$$

where

$$a_Y^* = \text{re}(\{a \in \text{Ag}(Y) \mid (\text{pos}(a) \leq \text{pos}(a_Y')) \text{ and } (\text{dist}(a, a^-) > 1)\}).$$

in which

$$a_Y' = \text{re}(\{a \in \text{Ag}(Y) \mid (\text{dist}(a, a^+, Y) < \text{dist}(a, a^+, q)) \text{ or } (\text{dist}(a, a^-, Y) > \text{dist}(a, a^-, q))\}).$$

T has the three properties:

- $\Omega(T, s, Y) = \vec{\mathbb{Z}} \oplus q$, for all $Y \in \mathcal{Y}_q$ and $s \in S_Y$
- $[T(a, X)] \neq R$ for all $X \in \mathcal{C}_r$ and
- $\text{dist}(\text{le}(T_s^n(Y)), \text{re}(T_s^n(Y))) \leq |q|$ for all n , for all $Y \in \mathcal{Y}_q$ and $s \in S_Y$.

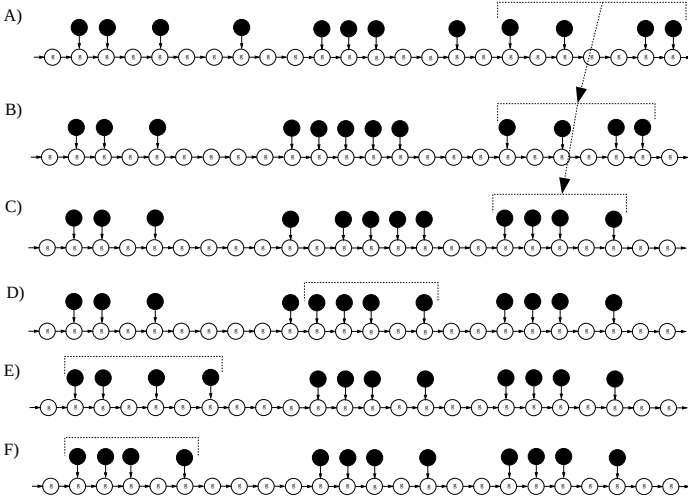


Figure 2: Illustration of the operation of the local rule constructed to solve pattern formation. A) Initial condition. B) Right-most agents comes together. C) Right-most group implements local rule T . In D) through F) this procedure repeats, propagating to the left.

In other words, T is like a solution to one unit of the pattern (Fig. 2). Our task is define a local rule which forces agents to come together closely enough to implement T , and properly draws boundaries between the units.

For $X \in \mathcal{C}$, and $a \in Ag(X)$, define $\tilde{B}(a)$ to be the set of agents b in X with $pos(b) \leq pos(a)$ such that $X[pos(b), pos(b) + |q|]$ contains $|Ag(q)|$ or more agents but $X[pos(b^{+j}), pos(b^{+j}) + |q|]$ does not for $j = 1, \dots, |Ag(q)| - 1$. If $\tilde{B}(a) \neq \emptyset$, then let $a' = re(\tilde{B}(a))$, the right-most agent in $\tilde{B}(a)$. Then define $\gamma_X : Ag(X) \rightarrow \{0, 1, \dots, |Ag(X)|\}$ by

$$\gamma_X(a) = \begin{cases} k+1 & \text{where } a = (a')^{+k}, \text{ if } \tilde{B}(a) \neq \emptyset \text{ and } k < |Ag(q)| \\ 0 & \text{otherwise} \end{cases}$$

Note that $\gamma_X(le(X)) \leq 1$ for all X . γ is a check which draws boundaries between units, each new unit starting again when γ_X is 1; and when not enough agents are close enough together for a unit to be declared, γ_X is zero.

We put γ and T together now. Define $f_T[P] \in \mathcal{D}_{2|q|+1}$ by: If $|Ag(X[pos(a) - 2|q| - 1, pos(a) - 1])| = 0$, then:

$$[f_T[P](a, X)] = \begin{cases} L, & \text{if } |X[pos(a), pos(a) + |q|]| \geq |Ag(q)| \\ & \text{and } \gamma_X(a) = 0 \\ [T(a, \mathbb{Z} \oplus X[pos(a), pos(a) + |q|])], & \text{if } \gamma_X(a) = 1 \\ S, & \text{otherwise} \end{cases}$$

and if $|Ag(X[pos(a) - 2|q| - 1, pos(a) - 1])| > 0$, then

$$[f_T[P](a, X)] = \begin{cases} L, & \text{if } \gamma_X(a) = 0 \text{ and } dist(a, a^+) \neq 2|q| + 1 \text{ or} \\ & X[pos(a), pos(a) + |q|] \cong q \text{ and} \\ & dist(a, a^-) = 2|q| + 1 \\ [T(a, \mathbb{Z} \oplus X[pos(a^+), pos(a^+) + |q| - 1])], & \\ \varphi_X(a) \neq 0, dist(a, a^+) \neq 2|q| + 1 & \\ S & \text{otherwise} \end{cases}$$

where $l = 1 - \gamma_X(a)$. The operation of $f_T[P]$ is described pictorially in figure 2. For each P , $f_T[P]$ is a local rule with information radius $2|q| + 1$. It can be shown that $f_T[P]$ is a solution to the connected P -pattern formation problem, yielding:

Theorem 1 *All connected patterns can be solved over $\mathbb{Z}$.*

Symmetry

Any configuration X possess a *symmetry group*, $Sym(X)$; these are the permutations σ on the set $\mathbb{N}$ of node identity numbers that induce a graph automorphism on X . The underlying spatial graphs also of course have symmetry groups (think of them as trivial configurations with no agents). For example, $Sym(\mathbb{Z}) = \mathbb{Z}$; for each $n \in \mathbb{Z}$, the symmetry σ_n corresponds to shifting all nodes in $\mathbb{Z}$ by n steps. $Sym(\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$, because in addition to having all of the shift symmetries which generate the $\mathbb{Z}$ summand, the undirected line also has a flip symmetry σ^* which generates the $\mathbb{Z}_2$ summand. For any configuration X over $\mathbb{Z}$ with $|Ag(X)| > 0$ (and finite), the symmetry group $Sym(X)$ is trivial. That is because none of the shift symmetries can take the “end-most” agent to anything other than itself. However, for all configurations Y over $\mathbb{Z}$ that are symmetric about their middle agent(s), $Sym(Y) = \mathbb{Z}_2$.

We can put a measure on the amount of symmetry that a given geometry $\mathcal{G}$ has by measuring the size of the symmetry groups for non-trivial configurations over $\mathcal{G}$. In this sense, $\mathbb{Z}$ has more symmetry than $\mathbb{Z}$ (which has none). This non-zero amount of symmetry places prevent emergeability on this geometry, as reflected in the following:

Theorem 2 *No patterns can be solved over $\mathbb{Z}$ in the synchronous model of concurrency.*

The proof of Theorem 2 relies heavily on the choice of the synchronous concurrency model. In fact, the small amount of symmetry-breaking which asynchrony allows is enough solve all patterns, via a construction analogous to that used in the case of $\mathbb{Z}$ in theorem 1:

Theorem 3 *All patterns can be solved over $\mathbb{Z}$ in the asynchronous concurrency model.*

Having considered $\mathbb{Z}$ and $\mathbb{Z}$, let us consider the set of ring graphs $\mathcal{R} = \{\mathcal{R}_m | m \in \mathbb{N}\}$, where $\mathcal{R}_m$ is the ring graph with m nodes.

Theorem 4 *Only trivial patterns over the ring lattices $\mathcal{R}$ can be solved.*

Theorem 4 says that the failure of existence of solutions to pattern formation is worse on $\mathcal{R}$ than it was on $\mathbb{Z}$, since failure holds in both the synchronous and asynchronous concurrency models. The essence of this failure is, once again, symmetry. Note that $Sym(\mathcal{R}) = \bigcup_{m \in \mathbb{N}} \mathbb{Z}_m$. Unlike $\mathbb{Z}$ which has no symmetries on non-trivial configurations, and $\mathbb{Z}$, which has at most the finite group $\mathbb{Z}_2$, $\mathcal{R}$ has infinitely many symmetries over non-trivial configurations, and it is precisely these symmetries that are exploited in the proof of theorem 4.

In summary, proposition 1 and theorems 1, 2, 3, and 4, suggest that two rough criteria are important for whether a given global property is emergeable: A) whether it is **locally checkable** and B) whether the property **respects the symmetry** of the underlying space.

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