

A Comparative Study of Sequential and Simultaneous Auctions

Shaheen S. Fatima
Department of Computer Science
University of Liverpool
Liverpool L69 7ZF, U.K
shaheen@csc.liv.ac.uk

ABSTRACT

This paper compares the sequential and simultaneous auctions for the following scenario. There are multiple similar objects for sale, each object is sold in a separate auction, and each bidder needs only one object. We use English auction rules and first determine equilibrium bidding strategies for the simultaneous and sequential cases. We do this for both common and private value objects by treating a bidder's information about these values as *uncertain*. We then consider the case where the private and common values have a uniform distribution and compare the two mechanisms in terms of three key properties: a bidder's ex-ante expected profit, the auctioneer's expected cumulative revenue, and the total expected surplus. For both common and private value objects, our study shows the following result. The expected cumulative revenue and the expected total surplus is higher for the sequential mechanism. However, a bidder's ex-ante expected profit depends on the number of objects being auctioned and the number of participating bidders, and it is sometimes higher for the sequential mechanism and sometimes for the simultaneous one.

Categories and Subject Descriptors

I.2.11 [Distributed Artificial Intelligence]: Multiagent Systems;
K.4.4 [Computers and Society]: Electronic Commerce

General Terms

Algorithms, Design, Economics

Keywords

Game-theory, Auctions, Multi-object auctions.

1. INTRODUCTION

Multiple objects can be auctioned using a number of mechanisms. Now the bidding behaviour is different for different mechanisms. So the outcome of such auctions (i.e., who wins and at what price) is also different for different mechanisms. Hence, the auctioneer and the bidding agents want to know which mechanism will be

the most beneficial to them. To this end, we study and compare two different mechanisms: *sequential* and *simultaneous*. To date, considerable research effort has been devoted to the study of these mechanisms. But the main limitation of existing work is that it has studied sequential and simultaneous auctions in isolation and so far there has been very little work on the comparison of these two mechanisms in terms of their outcomes. Our objective is therefore to provide a comparative analysis of sequential (SEQ) and simultaneous (SIM) auctions in an incomplete information setting. We do this for both common value (CV) and private value (PV) objects. For both CV and PV objects, our study shows the following result. The expected cumulative surplus and the auctioneer's cumulative revenue is higher for the SEQ mechanism. However, the mechanism that generates a higher ex-ante expected profit for the bidders depends on the number of objects being auctioned and the number of participating bidders, and it is sometimes higher for SEQ mechanism and sometimes for the SIM one.

Section 2 determines equilibrium for the SEQ and SIM mechanisms. Section 3 compares the outcomes for the two mechanisms. Section 4 concludes.

2. EQUILIBRIUM BIDDING STRATEGIES

Equilibrium bidding strategies for a single object with both common and private values have been obtained in [4]. We therefore briefly summarize its key results and then determine the equilibrium for our m objects case.

2.1 Single object

We first describe the CV case. For a single object with CV V_1 , the equilibrium obtained in [4] is as follows. Let f^n denote the first order statistic of the surplus for the n bidders and s^n the second order statistic. Then the bidder with the highest surplus wins and the winner's expected profit is $EP_w(n) = E(f^n) - E(s^n)$. The expected surplus is $ES(n) = E(V_1)$. Finally, the expected revenue is $ER(n) = ES(n) - EP_w(n)$.

On the other hand, for PV objects [6], the winner's expected profit is $\overline{EP}_w(n) = E(\overline{f}^n) - E(\overline{s}^n)$. The expected surplus is $\overline{ES}(n) = E(\overline{f}^n)$ and the expected revenue is $\overline{ER}(n) = E(\overline{s}^n)$ where $\overline{f}^n$ and $\overline{s}^n$ denote the first and second order statistic for the private value of the object.

2.2 Multiple sequential auctions

The m similar objects are sold in m separate English auctions conducted sequentially one after another. Each bidder needs only one object and draws its CV/PV for an auction just before that auction begins. Since a bidder needs only one object, it participates in the series of auctions until it wins an object. After that, it does not take

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee.

AAMAS'06 May 8-12, 2006, Hakodate, Japan

Copyright 2006 ACM 1-59593-303-4/06/0005 ...\$5.00.

part in any of the remaining auctions¹. In our model, there is a single probability distribution function for the CVs and PVs of all the objects and each bidder receives its signals for an object before the auction for that object begins (like in [5, 1]).

The English auction rules are as follows. The auctioneer continuously raises the price, and bidders publicly reveal when they withdraw from the auction. Bidders who drop out from an auction cannot re-enter that auction. A bidder's strategy for the j th (for $1 \leq j \leq m-1$) auction depends on its expected profit from the $(m-j)$ future auctions. Since the m th auction is the last one, a bidder's strategy for this auction is the same as the single object case. But the bidding behaviour for the first $(m-1)$ auctions is different from the single object case.

We first obtain the equilibrium bids for CV objects and then extend it to PVs. For the former case, a bidder's strategy for an individual auction depends on its probability of winning any one of the remaining auctions that are yet to be conducted. If the number of bidders for the first auction is n , then let $\beta(y, j, m, n)$ denote a bidder's ex-ante probability of winning the y th (for $j \leq y \leq m$) auction in the series from the j th to the m th auction before the j th one begins. This probability is:

$$\beta(y, j, m, n) = [\Pi_{k=j}^{y-1} (1 - 1/(n - y + k + 1))] \times (1/(n - y + 1))$$

For CV objects, let $EP_w(j, m, n)$ denote the winner's expected profit for the j th auction in a series of m auctions with n bidders for the first one. Also, let $\alpha(j, m, n)$ denote a bidder's ex-ante expected profit for winning any one auction in the series from the j th (for $1 \leq j \leq m$) to the m th auction where

$$\alpha(j, m, n) = \sum_{y=j}^m \beta(y, j, m, n) EP_w(y, m, n)$$

For CV objects, the n -tuple of strategies $(B(\cdot), \dots, B(\cdot))$ with $B(\cdot)$ defined below constitutes a symmetric equilibrium for the j th (for $1 \leq j < m$) auction at a stage where k bidders have dropped out (see [3] for details):

$$\begin{aligned} B_0^j(x_{ij}) &= E(v_{ij} | S_{ij} = x_{ij}) - \alpha(j+1, m, n) \\ B_k^j(x_{ij}; b_1, \dots, b_k) &= \frac{n-j+1-k}{n-j+1} E(v_{ij} | S_{ij} = x_{ij}) \\ &+ \frac{1}{n-j+1} \sum_{y=0}^{k-1} E(v_{ij} | B_y(x_{ij}; b_1, \dots, b_y) = b_{y+1}) \\ &- \alpha(j+1, m, n) \end{aligned}$$

where x_{ij} is bidder i 's surplus for the j th object.

Let $ER(j, m, n)$ denote the expected revenue from the j th auction in a series of m CV auctions with n bidders for the first one. Also, let $ES(j, m, n)$ denote the expected surplus for the j th auction. Then for the j th (for $1 \leq j \leq m$) auction, the winner's expected profit ($EP_w(j, m, n)$), the expected surplus ($ES(j, m, n)$), and the expected revenue ($ER(j, m, n)$) are:

$$\begin{aligned} EP_w(j, m, n) &= E(f^{n-j+1}) - E(s^{n-j+1}) + \alpha(j+1, m, n) \\ EP_w(m, m, n) &= E(f^{n-m+1}) - E(s^{n-m+1}) \\ ES(j, m, n) &= E(V_j) \\ ER(j, m, n) &= ES(j, m, n) - EP_w(j, m, n) \end{aligned}$$

¹This model is an extension of [1]: [1] analyses a sequence of two private value auctions while we analyse $n \geq 2$ auctions for common and private values.

We now find the cumulative revenue, the cumulative surplus, and a bidder's ex-ante expected profit from all m auctions. For $n > m$, let $X_{seq}(m, n)$ denote the cumulative revenue and $Y_{seq}(m, n)$ a bidder's ex-ante profit from all m auctions where $X_{seq}(m, n) = \sum_{j=1}^m ER(j, m, n)$ and $Y_{seq}(m, n) = \alpha(1, m, n)$. In general (i.e., $n > m$ or $n \leq m$), the cumulative revenue ($ECR_{seq}(m, n)$) is as follows:

$$ECR_{seq}(m, n) = \begin{cases} 0 & \text{if } n = 0, \text{ or } n = 1 \\ X_{seq}(m, n) & \text{if } n > m \\ X_{seq}(n-1, n) & \text{if } n \leq m \end{cases}$$

This is because if $n \leq m$, then the number of bidders for the first $n-1$ auctions is at least 2. For the n th auction there is only one bidder and so it wins the object for nothing resulting in a revenue of zero. For all the remaining $m-n$ auctions, there are no bidders and so the revenue is zero again.

Let $EEP_{seq}(m, n)$ denote a bidder's ex-ante expected profit for all the m auctions where

$$EEP_{seq}(m, n) = \begin{cases} E(V) & \text{if } n = 1 \\ Y_{seq}(m, n) & \text{if } n > m \\ Y_{seq}(n-1, n) + \beta(n, 1, n, n)E(V) & \text{if } n \leq m \end{cases}$$

This is because if $(n \leq m)$ for the first $n-1$ auctions there are at least two bidders. But for the n th auction, since there is only one bidder, it gets the object for nothing. For this auction, the winner's expected profit is the expectation of the CV. Since the CVs of all the objects are drawn from a single distribution function (F), we drop the subscript in $E(V)$.

Finally, the expected cumulative surplus is:

$$EC S_{seq}(m, n) = \begin{cases} mE(V) & \text{if } n \geq m \\ nE(V) & \text{if } n < m \end{cases}$$

We now turn to PV objects. For these objects, we denote the first and second order statistic of the PV for the j th auction as $\bar{f}^{n-j+1}$ and $\bar{s}^{n-j+1}$ respectively. For these objects, a bidder's bid for an auction is obtained by subtracting its ex-ante expected profit for the future auctions from its bid for the current one. Hence, for the j th auction, the winner's expected profit (denoted $\overline{EP}_w(j, m, n)$) is:

$$\begin{aligned} \overline{EP}_w(j, m, n) &= E(\bar{f}^{n-j+1}) - E(\bar{s}^{n-j+1}) + \bar{\alpha}(j+1, m, n) \\ \overline{EP}_w(m, m, n) &= E(\bar{f}^{n-m+1}) - E(\bar{s}^{n-m+1}) \end{aligned}$$

where $\bar{\alpha}(j, m, n)$ is a bidder's ex-ante expected profit for winning any one auction in the series from the j th (for $1 \leq j \leq m$) to the m th auction and is:

$$\bar{\alpha}(j, m, n) = \sum_{y=j}^m \beta(y, j, m, n) \overline{EP}_w(y, m, n)$$

The expected revenue and the expected surplus are:

$$\begin{aligned} \forall_{j=1}^{m-1} \overline{ER}(j, m, n) &= E(\bar{s}^{n-j+1}) - \bar{\alpha}(j+1, m, n) \\ \overline{ER}(m, m, n) &= E(\bar{s}^{n-m+1}) \\ \forall_{j=1}^m \overline{ES}(j, m, n) &= \overline{ER}(j, m, n) + \overline{EP}_w(j, m, n) \end{aligned}$$

For $n > m$, let $\overline{X}_{seq}(m, n)$ denote the expected cumulative revenue for all the m objects and $\overline{Y}_{seq}(m, n)$ a bidder's ex-ante expected profit from all the m auctions where $\overline{X}_{seq}(m, n)$ is equal to $\sum_{j=1}^m \overline{ER}(j, m, n)$ and $\overline{Y}_{seq}(m, n) = \bar{\alpha}(1, m, n)$. In general, (i.e., for $n > m$ or $n \leq m$), the cumulative revenue is:

$$\overline{ECR}_{seq}(m, n) = \begin{cases} 0 & \text{if } n = 0, \text{ or } n = 1 \\ \overline{X}_{seq}(m, n) & \text{if } n > m \\ \overline{X}_{seq}(n-1, n) & \text{if } n \leq m \end{cases}$$

and a bidder's ex-ante expected profit ($\overline{EEP}_{seq}(m, n)$) is:

$$\overline{EEP}_{seq}(m, n) = \begin{cases} E(\bar{f}^1) & \text{if } n = 1 \\ \bar{Y}_{seq}(m, n) & \text{if } n > m \\ \bar{Y}_{seq}(n-1, n) + \beta(n, 1, n, n)E(\bar{f}^1) & \text{if } n \leq m \end{cases}$$

Finally, the expected cumulative surplus is:

$$\overline{ECS}_{seq}(m, n) = \begin{cases} \sum_{j=1}^m E(\bar{f}^{n-j+1}) & \text{if } n \geq m \\ \sum_{j=1}^n E(\bar{f}^{n-j+1}) & \text{if } n < m \end{cases}$$

2.3 Multiple simultaneous auctions

As before, we have $m > 1$ similar objects, n risk-neutral bidders, and each bidder needs only one object. Given this, the SIM auctions game is played as follows. The m objects are sold in m different auctions conducted simultaneously and independently of each other. Each bidder randomly selects one of the m auctions and bids in it. It is obvious that after a bidder selects an auction, its equilibrium bids are the same as that for a single object.

This game has been analysed in [2] for private value objects for the limiting case where both m and n tend to infinity. The analysis is done by assuming complete information. Here, we extend this analysis to a more realistic case where both m and n are finite, the objects are either CV or PV, and there is uncertainty about these values. For this SIM auctions game, the equilibrium outcome depends on how the n bidders arrive at the m auctions. We let T_m^n denote the number of different ways in which n bidders can be distributed between m auctions where

$$T_m^n = \begin{cases} 1 & \text{if } n = 0 \text{ or } m = 1 \\ \sum_{k=0}^n C(n, k) & \text{if } m = 2 \\ \sum_{k=0}^n C(n, k) T_{m-1}^{n-k} & \text{if } m > 2 \end{cases}$$

For m SIM auctions, the probability that k bidders arrive at an auction ($P(m, k)$) is:

$$P(m, k) = (C(n, k) T_{m-1}^{n-k}) / T_m^n$$

Consider any one auction. The number of bidders for this auction can vary between 0 and n . If no bidders arrive at the auction, the object remains unsold. If only one bidder arrives, it gets the object for nothing. If $k \geq 2$ bidders arrive, then the expected revenue is greater than zero.

Consider CV objects. For these objects, the expected cumulative revenue from all the m auctions ($ECR_{sim}(m, n)$) is:

$$ECR_{sim}(m, n) = \sum_{k=0}^n P(m, k) [ER(k) + ECR_{sim}(m-1, n-k)]$$

Also, for an individual auction, the winner's expected profit is:

$$EP_w(m, n) = P(m, 1)[E(V)] + \sum_{k=2}^n P(m, k)[E(f^k) - E(s^k)]$$

and hence a bidder's ex-ante expected profit ($EEP_{sim}(m, n)$) is:

$$EEP_{sim}(m, n) = P(m, 1)[E(V)] + \sum_{k=2}^n P(m, k)(1/k)[E(f^k) - E(s^k)]$$

Finally, the total expected surplus (denoted $ECS_{sim}(m, n)$) is:

$$ECS_{sim}(m, n) = \sum_{k=1}^n P(m, k)[ES(k) + ECS_{sim}(m-1, n-k)]$$

Now consider PV objects. For these objects, the expected cumulative revenue is:

$$\overline{ECR}_{sim}(m, n) = \sum_{k=0}^n P(m, k) [\overline{ER}(k) + \overline{ECR}_{sim}(m-1, n-k)]$$

Also, for an auction, the winner's expected profit is:

$$\overline{EP}_w(m, n) = P(m, 1)[E(\bar{f}^1)] + \sum_{k=2}^n P(m, k)[E(\bar{f}^k) - E(\bar{s}^k)]$$

and hence a bidder's ex-ante expected profit is:

$$\overline{EEP}_{sim}(m, n) = P(m, 1)[E(\bar{f}^1)] + \sum_{k=2}^n P(m, k)(1/k)[E(\bar{f}^k) - E(\bar{s}^k)]$$

Finally, the total expected surplus is:

$$\overline{ECS}_{sim}(m, n) = \sum_{k=1}^n P(m, k) [\overline{ES}(k) + \overline{ECS}_{sim}(m-1, n-k)]$$

3. A COMPARISON OF THE TWO AUCTION MECHANISMS

We now compare the two mechanisms in terms of three key properties: the expected cumulative revenue, the expected cumulative surplus, and a bidder's ex-ante expected profit. To do this, we consider the uniform distribution function for both PVs and CVs. For this distribution, our study shows that from an auctioneer's perspective, the SEQ mechanism is better because it generates a higher revenue. From a global perspective too, the SEQ mechanism is better (because it generates a higher expected cumulative surplus). However, from a bidder's perspective, the ex-ante expected profit depends on the number of participating bidders and the number of objects being auctioned, and it is sometimes higher for the SEQ mechanism and sometimes for the SIM one.

4. CONCLUSIONS

This paper analyses two key auction mechanisms for multiple objects: *sequential* and the *simultaneous*. This analysis is done for both common and private value objects by treating a bidder's information about these values as uncertain. We used the English auction rules and determined equilibrium for the two mechanisms. We then considered the case where the private and common values have a uniform distribution and compared the mechanisms in terms of their expected cumulative revenues, their expected cumulative surplus, and a bidder's ex-ante expected payoff. For both common and private values, our study shows that, the sequential mechanism is better both from the auctioneer's and also from a global perspective. However, from a bidder's perspective, the choice of a mechanism depends on the number of objects and the number of bidders.

5. REFERENCES

- [1] D. Bernhardt and D. Scoones, A note on sequential auctions, *American Economic Review*, 84(3):653-657, 1994.
- [2] R. Engelbrecht-Wiggans and R. J. Weber, An example of a multi-object auction game, *Management Science*, 25(12):1272-1277, 1979.
- [3] S. S. Fatima and M. Wooldridge and N. R. Jennings, Sequential auctions for objects with common and private values, In *Proceedings of the Fourth International Conference on Autonomous Agents and Multi-Agent Systems*, pages 635-642, Utrecht, Netherlands, 2005.
- [4] J. K. Goeree and T. Offerman, Competitive bidding in auctions with private and common values, *The Economic Journal* 113(489):598-613, 2003.
- [5] A. Ortega-Reichert, Models of competitive bidding under uncertainty, *Stanford University Technical Report*, 8, 1968.
- [6] W. Vickrey, Counterspeculation, auctions and competitive sealed tenders, *Journal of Finance* 16:8-37, 1961.