

Games with Ambiguous Payoffs and Played by Ambiguity and Regret Minimising Players

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Abstract. In real life games, a player's belief about the consequence of a strategy is often ambiguous due to out-of-control factors in the environment where the games are played. However, existing work cannot handle this situation. To address the issue, we introduce a new kind of games, called ambiguous games, and incorporate human cognitive factors of ambiguity aversion and minimising regret to propose a concept of solution to such a game. Moreover, we also study how ambiguity degrees of belief about payoffs impact the outcomes of a game, and find the condition under which a player should release more or less ambiguous information to his opponents.

1 Introduction

Game theory is a powerful tool for analysing the interactions between decision-makers in many domains, such as voting, biology, economics, management science, and artificial intelligence [5,18]. In game theory, it is assumed that a player's belief about the consequences of a strategy taken is accurate. However, in real world, it is not always the case [2,10]. For example, two competitive cell phone retailers can choose two different brands, which are their pure strategies. In real life, the payoffs of each retailer are often impacted by some uncontrol and uncertain factors [14], such as buyers' tastes and economical environments, and so it is impossible to assign a precise payoff to a single consequence of a strategy profile. This kind of uncertainty about consequence of a strategy is so called ambiguity. That is, in a game the possible consequence of each strategy profile is in a set, and we do not have a precise probability for each of all possible consequences of a strategy but could have for their subsets [14,22,24].

Although the ambiguity cannot be avoidable in games played in real world, existing work cannot handle properly the situation. In fact, there are mainly two kinds of games (the more detailed discussion can be founded in related the work section). One is based on the assumption that the precise probabilities can be estimated for each single possibility of the consequences of a strategy, including Bayesian games [8] as well as the concepts of trembling hand perfect equilibrium [20] and proper equilibrium [16]. The another is fuzzy game [2,10,12,13], which assumes that the payoffs are fuzzy. However, it is also assumed each possibility of a fuzzy payoff can be estimated accurately.

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The aim of this paper is to address the problem. The main tool to handle ambiguity is D-S theory [21]. In [24], based on the theory we construct a decision model, in which we use cognitive factors of ambiguity-avoiding and regret minimising to set a proper preference ordering over strategies with interval-valued expected utilities. In [24], we show that our model can solve classic Ellsberg paradox [4] under ambiguity, and the preference ordering is a pre-order (i.e., complete and transitive). So, it is proper to employ the model for choosing the best response problem of games under ambiguity.

The main contributions of this paper are: (i) we can solve games with ambiguous payoffs; and (ii) we discover the rule of how ambiguity degrees of the players' beliefs about consequences impact the outcome of an ambiguous game.

The remainder of this paper is organized as follows. Section 2 briefs D-S theory and our ambiguity decision model. Section 3 defines static games under ambiguity and introduces the solution concept to the games. Section 4 studies how the ambiguity degrees of belief about possible consequences influence the outcome of an ambiguous game. Section 5 relates our work to others'. Finally, Section 6 concludes the paper and points out the future work.

2 Preliminaries

This section recaps D-S theory [21] and our decision model under ambiguity [24].

2.1 Basics of D-S Theory

Definition 1. *Let Θ be a frame of discernment. (i) Function $m : 2^\Theta \rightarrow [0, 1]$ is called a basic probability assignment or a mass function over Θ if $m(\emptyset) = 0$ and $\sum_{A \subseteq \Theta} m(A) = 1$. (ii) Let $A \subseteq \Theta$, mass function m is called a simple support function if $m(A) = s$, and $m(\Theta) = 1 - s$; we also say the mass function is focused on A , and s is the focused mass.*

D-S theory is different from probability theory. For example, a student lost his keys. Unfortunately, he has no ideas where he lost exactly, and locations a, b , and c are all possible. In D-S theory, it can be modeled by a mass function: $m(A) = 0$ if $A \neq \Theta$, and $m(\Theta) = 1$, where $\Theta = \{a, b, c\}$. While in probability theory, we have to find a probability function p such that $p(a) = s_1$, $p(b) = s_2$, and $p(c) = s_3$, where $s_1 + s_2 + s_3 = 1$. And this probability function p needs to be justified by lots of evidence. Obviously, this is much harder than the treatment of D-S theory. Another difference between D-S theory and probability theory is that mass function does not have to be additive, while probability one has to. Actually, probability function is a special case of mass function. In fact, if $m(A) > 0$ and $\sum m(A) = 1$, where A is a singleton (i.e., $|A| = 1$), then m is a probability function.

Generally speaking, a mass function represents a piece of ambiguous evidence. Intuitively, the bigger the cardinality of set A the more ambiguous in a simple support function focused on A . The concept of ambiguity degree of a mass function is formally defined as follows [3]:

Definition 2. Let m be a mass function over Θ and $|A|$ be the cardinality of set A . Then the ambiguity degree of m , denoted as δ_m , is given by

$$\delta_m = \frac{\sum_{A \subseteq \Theta} m(A) \log_2 |A|}{\log_2 |\Theta|}. \quad (1)$$

For the key missing example that we mentioned above, a mass function can model complete ignorance (absolute ambiguity), i.e., $m(A) = 0$ if $A \neq \Theta$, and $m(\Theta) = 1$. By Definition 2, we have $\delta_m = 1$. On the other side, if we have sufficient evidence to obtain a mass function, denoted as m , such that $m(\{a\}) = s_1$, $m(\{b\}) = s_2$, and $m(\{c\}) = s_3$, where $s_1 + s_2 + s_3 = 1$. In this case, m is a probability function and by formula (1) we have ambiguity degree $\delta_m = 0$.

2.2 The Ambiguity and Regret Aversion Model

Given a decision problem, assume that the decision maker knows the utility of each single consequence, but he is unsure which consequence that an action could cause. In other words, he only knows a mass function over the power set of consequences. Formally, such a problem can be defined as follows [24]:

Definition 3. A decision problem under uncertainty of ambiguity (or call an ambiguity decision problem) is a 4-tuple of (C, S, U, M) , where:

- (i) C is the set of all choices;
- (ii) S is the set of all consequences of choices;
- (iii) $U = \{u_c \mid c \in C, \forall s \in S, u_c(s) \in \mathbb{R}\}$, i.e., the utility of consequence $s \in S$ that be caused by strategy choice $c \in C$ is $u_c(s) \in \mathbb{R}$ (real number); and
- (iv) $M = \{m_c \mid c \in C\}$, i.e., the decision maker's uncertainty about the consequences that strategy choice c could cause is represented by mass function m_c over the discernment frame $\Theta = \{s_1, \dots, s_n\}$, where $s_1, \dots, s_n \in S$.

Based on the concept of mass function, the point-valued expected utility formula can be extended to the expected utility interval [23]:

Definition 4. Given ambiguity decision problem (C, S, U, M) , for choice c that is specified by mass function m_c over Θ , its expected utility interval is $EUI(c) = [\underline{U}(c), \overline{U}(c)]$, where:

$$\underline{U}(c) = \sum_{A \subseteq \Theta} m_c(A) \min\{u_c(s) \mid s \in A\}, \quad (2)$$

$$\overline{U}(c) = \sum_{A \subseteq \Theta} m_c(A) \max\{u_c(s) \mid s \in A\}. \quad (3)$$

Let $EUI(c_1) = [\underline{U}(c_1), \overline{U}(c_1)]$ and $EUI(c_2) = [\underline{U}(c_2), \overline{U}(c_2)]$ be the expected utility intervals, then according to [13],

$$EUI(c_1) + EUI(c_2) = [\underline{U}(c_1) + \underline{U}(c_2), \overline{U}(c_1) + \overline{U}(c_2)]. \quad (4)$$

Then, how to make a choice with interval-valued expected utility? We have the following [24]:

Definition 5. Let $EUI(x) = [\underline{U}(x), \overline{U}(x)]$ be the expected utility interval of choice $x \in \{c_1, c_2\}$, and $\delta_{m'}$ be the ambiguity degree of mass function m' corresponding to c_2 . Then the ambiguity-avoiding maximum regret of choice c_1 against choice c_2 is

$$\mathfrak{R}_{c_1}^{c_2} = \overline{\varepsilon}(c_2) - \underline{U}(c_1), \quad (5)$$

where

$$\overline{\varepsilon}(c_2) = \underline{U}(c_2) + (1 - \delta_{m'}) (\overline{U}(c_2) - \underline{U}(c_2)), \quad (6)$$

called the ambiguity-discounted upper bound of the expected utility of choice a' . The preference ordering $\succeq$ over choices is defined as follow:

$$c_1 \succeq c_2 \Leftrightarrow \mathfrak{R}_{c_1}^{c_2} \leq \mathfrak{R}_{c_2}^{c_1}. \quad (7)$$

By Definition 5, we have

$$c_1 \succeq c_2 \Leftrightarrow \overline{\varepsilon}(c_2) - \underline{U}(c_1) \leq \overline{\varepsilon}(c_1) - \underline{U}(c_2) \Leftrightarrow \underline{U}(c_1) + \overline{\varepsilon}(c_1) \geq \underline{U}(c_2) + \overline{\varepsilon}(c_2). \quad (8)$$

3 Game Definition

This section will define the ambiguous game and its solution concept.

Definition 6. An ambiguous game G is a 5-tuple of (N, S, Θ, M, U) , where:

- (i) N is a set of natural numbers, to denote the set of players of the game;
- (ii) $S = \{S_i\}_{i \in N}$, where S_i is the finite set of all pure strategies of player i ;
- (iii) Θ is the set of all possible consequences for all strategy profiles.
- (iv) $M = \{M_i\}_{i \in N}$, where $M_i = \{m_{i,j} \mid \forall \rho_j \in \times_{k \in N} S_k, m_{i,j}(B_{i,j}) = \xi, m_{i,j}(\Theta) = 1 - \xi, B_{i,j} \subset \Theta\}$;
- (v) $U = \{u_i\}_{i \in N}$, where u_i is the payoff function from the set of consequences to the set of real numbers.

In the above definition, item (iv) means that for a pure-strategy profile each player has a simple support function over the power set of the consequence set.

Given an ambiguous game, suppose each player calculates the expected payoff interval of a pure-strategy profile by formulas (2) and (3). Then we can transfer it to a normal static game in classic game theory [5,17]. In particular, we have the following theorem, which provides the way to calculate the sum, denoted as $\Xi_i(\rho_j)$, of the lower and ambiguity-discounted upper expected payoffs (see Definition 5) to any pure-strategy profile ρ_j for a specific ambiguous game.

Theorem 1. Given game (N, S, Θ, M, U) , suppose for any pure-strategy profile ρ_j , $m_{i,j}(B_{i,j}) = \xi_{i,j}$, and $m_{i,j}(\Theta) = 1 - \xi_{i,j}$, $B_{i,j} \subset \Theta$, $j = 1, 2, \dots, n$ ($n = |\times_{k \in N} S_k|$). Let $t_{i,j} = \min\{u_i(s_i), \dots, u_i(s_j)\}$, $T_{i,j} = \max\{u_i(s_i), \dots, u_i(s_j)\}$, $s_i, \dots, s_j \in B_{i,j}$, $y = \min\{u_i(s_1), \dots, u_i(s_n)\}$, and $Y = \max\{u_i(s_1), \dots, u_i(s_n)\}$, $s_1, \dots, s_n \in \Theta$. Then

$$\Xi_i(\rho_j) = \underline{U}_i(\rho_j) + \overline{\varepsilon}_i(\rho_j), \quad (9)$$

where

$$\underline{U}_i(\rho_j) = \xi_{i,j}t_{i,j} + (1 - \xi_{i,j})y, \quad (10)$$

$$\bar{\varepsilon}_i(\rho_j) = \xi_{i,j}t_{i,j} + (1 - \xi_{i,j})y + \xi_{i,j}(1 - \log_{|\Theta|}|B_{i,j}|)((T_{i,j} - t_{i,j})\xi_{i,j} + (1 - \xi_{i,j})(Y - y)). \quad (11)$$

Proof. Since player i has simple support function $m_{i,j}$, by formulas (2) and (3), for player i the lower and upper expected payoffs of profile ρ_j to mass function $m_{i,j}$ are as follows:

$$\underline{U}_i(\rho_j) = \xi_{i,j}t_{i,j} + (1 - \xi_{i,j})y, \quad \bar{U}_i(\rho_j) = \xi_{i,j}T_{i,j} + (1 - \xi_{i,j})Y.$$

By formula (1), discounting factor $\delta_{i,j}$ to the upper expected payoff of profile ρ_j is

$$\delta_{i,j} = \frac{\xi_{i,j}\log_2|B_{i,j}| + (1 - \xi_{i,j})\log_2|\Theta|}{\log_2|\Theta|}.$$

Thus, by formula (6), we can obtain the ambiguity-discounted upper expected payoff of profile ρ_j for player i as formula (11). Then, the expected payoff interval of profile ρ_j is $[\underline{U}_i(\rho_j), \bar{\varepsilon}_i(\rho_j)]$. So, we have $\Xi_i(\rho_j) = \underline{U}_i(\rho_j) + \bar{\varepsilon}_i(\rho_j)$. $\square$

Theorem 1 shows that a point-valued $\Xi_i(\rho_j)$ can be derived given any pure-strategy profile ρ_j by formula (9). So, we can obtain a normal game matrix with precise numbers. That is, we can obtain a traditional static game [5,17].

Definition 7. Given ambiguous game $G = (N, S, \Theta, M, U)$, a static game, denoted by $G' = (N, S, U')$, is called an induced game of G if the element of its payoff matrix is obtained by using formula (9) from game G .

A mixed strategy α_i for player i is a probability distribution over set S_i . Let $\Xi_i(\alpha_i, \alpha_{-i})$ be the sum of the lower expected payoff and ambiguity-discounted upper expected payoff to mixed-strategy profile $\alpha = (\alpha_i, \alpha_{-i})$.

Given an ambiguous game, the following lemma provides the way to calculate the sum of the lower and ambiguity-discounted upper expected payoffs to any mixed-strategy profile of α .

Lemma 1. Given game (N, S, Θ, M, U) and its mixed-strategy profile α , suppose for any pure-strategy profile ρ_j , $m_{i,j}(B_{i,j}) = \xi_{i,j}$, and $m_{i,j}(\Theta) = 1 - \xi_{i,j}$, $B_{i,j} \subset \Theta$, $j = 1, 2, \dots, n$ ($n = |\times_{k \in N} S_k|$). Let $t_{i,j} = \min\{u_i(s_i), \dots, u_i(s_j)\}$, $T_{i,j} = \max\{u_i(s_i), \dots, u_i(s_j)\}$, $s_i, \dots, s_j \in B_{i,j}$, $y = \min\{u_i(s_1), \dots, u_i(s_n)\}$, and $Y = \max\{u_i(s_1), \dots, u_i(s_n)\}$, $s_1, \dots, s_n \in \Theta$. Then

$$\Xi_i(\alpha) = \sum_j \left(\prod_i \alpha_i(\rho_j) \right) \varphi(\xi_{i,j}), \quad (12)$$

where

$$\varphi(\xi_{i,j}) = 2\xi_{i,j}t_{i,j} + 2(1 - \xi_{i,j})y + \xi_{i,j}(1 - \log_{|\Theta|}|B_{i,j}|)((T_{i,j} - t_{i,j})\xi_{i,j} + (1 - \xi_{i,j})(Y - y)). \quad (13)$$

Proof. By Theorem 1, the expected payoff interval of profile ρ_j is $[\underline{U}_i(\rho_j), \bar{\varepsilon}_i(\rho_j)]$. So, by formula (4), we have

$$\Xi_i(\alpha) = \sum_j \left(\prod_i \alpha_i(\rho_j) \right) (\underline{U}_i(\rho_j) + \bar{\varepsilon}_i(\rho_j)) = \sum_j \left(\prod_i \alpha_i(\rho_j) \right) \varphi(\xi_{i,j}),$$

where $\varphi(\xi_{i,j})$ is given by (13). □

Definition 8. *Mixed-strategy profile α^* in game $G = (N, S, \Theta, M, U)$ is an ambiguous equilibrium of mixed strategy if $\forall i \in N$, for every mixed-strategy α_i of player i , $\Xi_i(\alpha_i^*, \alpha_{-i}^*) \geq \Xi_i(\alpha_i, \alpha_{-i}^*)$, where $(\alpha_i^*, \alpha_{-i}^*) = \alpha^*$.*

Given game $G = (N, S, \Theta, M, U)$, suppose $G' = (N, S, U')$ is its induced game. By Definitions 7 and 8, we have $E_i(\alpha_i^*, \alpha_{-i}^*) \geq E_i(\alpha_i, \alpha_{-i}^*) \Leftrightarrow \Xi_i(\alpha_i^*, \alpha_{-i}^*) \geq \Xi_i(\alpha_i, \alpha_{-i}^*)$. So, mixed-strategy profile α^* is an ambiguous equilibrium of G if and only if it is a mixed-strategy Nash equilibrium of G' . However, it should be noted that our model does not assume that we could have that all the payoffs are precise under ambiguity, and the point-valued payoffs in the induced game in our model is calculated from an interval that carries the factor of ambiguity.

In the following, we will give an example to illustrate our model. Suppose two competitive retailers 1 and 2 sell cell phones and can choose two different brands, C and D, which are their pure strategies. Since buyers' tastes, economical environments, and other factors are often uncertain and out of sellers' control, sellers' belief about the consequences of their choices are ambiguous. For example, they can only know that the possible consequence of each pure-strategy profile is: bad sales and high costs (s_1), bad sales and low costs (s_2), good sales and high costs (s_3), or good sales and low costs (s_4). Then, what is the outcome of this game?

Then this game can be modeled as $G = (N, S, \Theta, M, U)$, where $N = \{1, 2\}$, $S = \{S_1, S_2\}$, $S_1 = S_2 = \{C, D\}$, $\Theta = \{s_1, s_2, s_3, s_4\}$ and $u_1(s_i) = u_2(s_i) = i$ ($i = 1, 2, 3, 4$). Since the players are ambiguous about the payoffs of pure-strategy profiles, for retailer 1, with respect to four pure-strategy profiles $\rho_1, \dots, \rho_4$, suppose there are four simple support functions over Θ ,¹ respectively, as follows:

$$\begin{aligned} \rho_1 &= (C, C) : m_{1,1}(B_{1,1}) = m_{1,1}(\{s_4\}) = 0.7, \quad m_{1,1}(\Theta) = 0.3; \\ \rho_2 &= (C, D) : m_{1,2}(B_{1,2}) = m_{1,2}(\{s_1\}) = 0.6, \quad m_{1,2}(\Theta) = 0.4; \\ \rho_3 &= (D, C) : m_{1,3}(B_{1,3}) = m_{1,3}(\{s_1\}) = 0.5, \quad m_{1,3}(\Theta) = 0.5; \\ \rho_4 &= (D, D) : m_{1,4}(B_{1,4}) = m_{1,4}(\{s_2\}) = 0.6, \quad m_{1,4}(\Theta) = 0.4. \end{aligned}$$

Similarly, for retailer 2, with respect to four pure-strategy profiles, suppose there are four simple support functions over Θ , respectively, as follows:

$$\rho_1 = (C, C) : m_{2,1}(B_{2,1}) = m_{2,1}(\{s_1\}) = 0.8, \quad m_{2,1}(\Theta) = 0.2;$$

¹ The mass function of $m_{1,1}(B_{1,1}) = 0.7$ and $m_{1,1}(\Theta) = 0.3$ implies $Bel(B_{1,1}) = 0.7$ and $Pl(B_{1,1}) = 1$ according to [21]. In other words, since $Bel(B_{1,1}) \leq P(B_{1,1}) \leq Pl(B_{1,1}) = 1$, actually the mass function means that we can only estimate the lower bound probability of $B_{1,1}$ (a kind of imprecise probability). Similarly, we can understand other similar mass functions in this paper.

$$\begin{aligned}\rho_2 &= (C, D) : m_{2,2}(B_{2,2}) = m_{2,2}(\{s_2\}) = 0.6, \quad m_{2,2}(\Theta) = 0.4; \\ \rho_3 &= (D, C) : m_{2,3}(B_{2,3}) = m_{2,3}(\{s_3\}) = 0.8, \quad m_{2,3}(\Theta) = 0.2; \\ \rho_4 &= (D, D) : m_{2,4}(B_{2,4}) = m_{2,4}(\{s_1\}) = 0.5, \quad m_{2,4}(\Theta) = 0.5.\end{aligned}$$

Let $((p, 1 - p), (q, 1 - q))$ be a mixed-strategy profile. By formula (12), we have

$$\begin{aligned}\Xi_1 &= 3.1pq + 3.73pq + p(1 - q) + 1.72p(1 - q) + q(1 - p) \\ &\quad + 1.75q(1 - p) + 1.6(1 - p)(1 - q) + 2.32(1 - p)(1 - q); \quad (14)\end{aligned}$$

$$\begin{aligned}\Xi_2 &= pq + 1.48pq + 1.6p(1 - q) + 2.32p(1 - q) + 2.60q(1 - p) \\ &\quad + 3.08q(1 - p) + (1 - p)(1 - q) + 1.75(1 - p)(1 - q). \quad (15)\end{aligned}$$

Thus, we have $\frac{\partial \Xi_1}{\partial p} = 5.28q - 1.2$ and $\frac{\partial \Xi_2}{\partial q} = 2.93 - 4.37p$. Let $\frac{\partial \Xi_1}{\partial p} = 0$ and $\frac{\partial \Xi_2}{\partial q} = 0$, then we obtain $p = 0.6705$ and $q = 0.2273$. Thus, $\alpha^* = ((0.6705, 0.3295), (0.2273, 0.7727))$ is the ambiguous equilibrium of mixed strategy of the game. And at this equilibrium point, by formulas (14) and (15) we can obtain $\Xi_1(\alpha^*) = 3.6541$ and $\Xi_2(\alpha^*) = 3.5345$.

4 Properties

In above example, each focused mass $\xi_{i,j}$ is a constant. This section will discuss what if the focused masses are variables. Then we study how the ambiguity degrees in beliefs about possible consequences affect the outcome of an ambiguous game.

The following theorem studies the monotonicity of function $\Xi_i(\xi_{i,1}, \dots, \xi_{i,m})$, and finds the maximum value of this function.

Theorem 2. *Given game (N, S, Θ, M, U) , suppose α is a mixed-strategy profile, and $m_{i,j}(B_{i,j}) = \xi_{i,j}$, and $m_{i,j}(\Theta) = 1 - \xi_{i,j}$, $B_{i,j} \subset \Theta$, $j = 1, 2, \dots, n$ ($n = |\times_{k \in N} S_k|$). Let $t_{i,j} = \min\{u_i(s_i), \dots, u_i(s_j)\}$, $T_{i,j} = \max\{u_i(s_i), \dots, u_i(s_j)\}$, $s_i, \dots, s_j \in B_{i,j}$, $y = \min\{u_i(s_1), \dots, u_i(s_n)\}$, $Y = \max\{u_i(s_1), \dots, u_i(s_n)\}$, $s_1, \dots, s_n \in \Theta$, and*

$$a_{i,j} = \frac{2t_{i,j} - 2y + (1 - \log_{|\Theta|}|B_{i,j}|)(Y - y)}{2(1 - \log_{|\Theta|}|B_{i,j}|)(Y - y - T_{i,j} + t_{i,j})}.$$

Then player i has the maximum value of $\Xi_i(\xi_{i,1}, \dots, \xi_{i,m})$ at $(\hat{\xi}_{i,1}, \dots, \hat{\xi}_{i,m})$, where

$$\hat{\xi}_{i,j} = \begin{cases} a_{i,j} & \text{if } a_{i,j} < 1, \\ 1 & \text{if } a_{i,j} \geq 1. \end{cases}$$

Proof. Let focused masses $\xi_{i,1}, \dots, \xi_{i,m}$ be variables in the assumption of Lemma 1. Then for any mixed-strategy profile α , by formula (12) we have

$$\frac{\partial \Xi_i}{\partial \xi_{i,j}} = \sum_j \left(\prod_i \alpha_i(\rho_j) \right) \frac{d\varphi}{d\xi_{i,j}}, \quad (16)$$

where

$$\frac{d\varphi}{d\xi_{i,j}} = 2t_{i,j} - 2y + 2\xi_{i,j}(1 - \log_{|\Theta|}|B_{i,j}|)((T_{i,j} - t_{i,j} - Y + y) + (1 - \log_{|\Theta|}|B_{i,j}|)(Y - y)).$$

Let $\frac{\partial \Xi_i}{\partial \xi_{i,j}} = 0$, by formula (16), $\frac{d\varphi}{d\xi_{i,j}} = 0$. Then we have $\xi_{i,j} = \hat{\xi}_{i,j} = a_{i,j}$. Since $B_{i,j} \subset \Theta$, we have $y \leq t_{i,j} \leq T_{i,j} < Y$ and $1 \leq |B_{i,j}| < |\Theta|$. Thus, $0 < 1 - \log_{|\Theta|}|B_{i,j}| \leq 1$. So, $\hat{\xi}_{i,j} > 0$, and by the definition of mass function, $\hat{\xi}_{i,j}$ should be in $(0, 1]$. For any $\hat{\xi}_{i,j}$, if $0 < \hat{\xi}_{i,j} < 1$, then $(\hat{\xi}_{i,1}, \dots, \hat{\xi}_{i,m})$ is a critical point.

Further, we have

$$\begin{aligned} \frac{\partial^2 \Xi_i}{\partial^2 \xi_{i,j}}(\hat{\xi}_{i,1}, \dots, \hat{\xi}_{i,m}) &= 2(1 - \log_{|\Theta|}|B_{i,j}|)(T_{i,j} - t_{i,j} - Y + y), \\ \frac{\partial^2 \Xi_i}{\partial \xi_{i,j} \partial \xi_{i,k}}(\hat{\xi}_{i,1}, \dots, \hat{\xi}_{i,m}) &= 0, j \neq k. \end{aligned}$$

Let $r_{i,j} = 2(1 - \log_{|\Theta|}|B_{i,j}|)(T_{i,j} - t_{i,j} - Y + y) < 0$. So, the Hessian of function Ξ_i at the critical point, the matrix formed from the second-order partial derivatives of the function [25], is a diagonal matrix as follows:

$$A_m = \begin{pmatrix} 2r_{i,1} & 0 & \cdots & 0 \\ 0 & 2r_{i,2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2r_{i,m} \end{pmatrix}.$$

Since for any $r_{i,j}$, $2r_{i,j} < 0$, the characteristic value of A_m is negative, which means A_m is negative definitely. Consequently, player i has an extremum of $\Xi_i(\xi_{i,1}, \dots, \xi_{i,m})$ at the critical point.

For any $\tilde{\xi}_{i,j}$, if $0 \leq \tilde{\xi}_{i,j} < \hat{\xi}_{i,j}$, since Ξ_i is differentiable at point $(\tilde{\xi}_{i,1}, \dots, \tilde{\xi}_{i,m})$, we have

$$\frac{\partial \Xi_i}{\partial v}(\tilde{\xi}_{i,1}, \dots, \tilde{\xi}_{i,m}) = \sum_j \frac{\partial \Xi_i}{\partial \tilde{\xi}_{i,j}} \cos \beta_{i,j},$$

where $\mathbf{v} = (\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,m})$ is a direction, $\mathbf{v}_{i,j} = \cos \beta_{i,j}$ is the angle between $\mathbf{v}$ and the positive direction of coordinate $\xi_{i,j}$ in a Cartesian coordinate system, and $\frac{\partial \Xi_i}{\partial v}$ is the directional derivative in direction $\mathbf{v}$. So, we have

$$\begin{aligned} \frac{\partial \Xi_i}{\partial v}(\tilde{\xi}_{i,1}, \dots, \tilde{\xi}_{i,m}) &= \sum_j \left(\prod_i \alpha_i(\rho_j) \right) \varphi'(\tilde{\xi}_{i,j}) \cos \beta_{i,j} \\ &> \sum_j \left(\prod_i \alpha_i(\rho_j) \right) \varphi'(\hat{\xi}_{i,j}) \cos \beta_{i,j} = 0. \end{aligned}$$

Similarly, for any $\tilde{\xi}_{i,j}$, if $\hat{\xi}_{i,j} \leq \tilde{\xi}_{i,j} \leq 1$, then we have $\frac{\partial \Xi_i}{\partial v}(\tilde{\xi}_{i,1}, \dots, \tilde{\xi}_{i,m}) < 0$. Thus, function $\Xi_i(\tilde{\xi}_{i,1}, \dots, \tilde{\xi}_{i,m})$ is strictly increasing on the any direction $\mathbf{v}$ if

$\tilde{\xi}_{i,j} \in [0, \hat{\xi}_{i,j})$, and is strictly decreasing on the any direction $\mathbf{v}$ if $\tilde{\xi}_{i,j} \in [\hat{\xi}_{i,j}, 1]$. Hence, for any $a_{i,j}$, $a_{i,j} < 1$, player i has the maximum value of $\Xi_i(\xi_{i,1}, \dots, \xi_{i,m})$ at point $(\hat{\xi}_{i,1}, \dots, \hat{\xi}_{i,m})$, i.e., $(a_{i,1}, \dots, a_{i,m})$.

If $\hat{\xi}_{i,j} \geq 1$, as shown as above, function Ξ_i is strictly increasing on the any direction if $\tilde{\xi}_{i,j} \in [0, \hat{\xi}_{i,j})$, and so the function is also strictly increasing on $\tilde{\xi}_{i,j} \in [0, 1]$. Hence, the function has the maximum value at point $(\hat{\xi}_{i,1}, \dots, \hat{\xi}_{i,m})$, where $\hat{\xi}_{i,j} = a_{i,j}$ if $a_{i,j} < 1$, or $\hat{\xi}_{i,j} = 1$ if $a_{i,j} \geq 1$. $\square$

If function Ξ is a univariate function, then we can derive the following corollary by Theorem 2.

Corollary 1. *Under the assumption of Theorem 2, let $\xi_{i,j}$ be a variable, and any $\xi_{i,k}$ ($j \neq k$) a constant. Then,*

(i) *Player i has the maximum value of Ξ_i under the following condition:*

$$\xi_{i,j} = \begin{cases} \hat{\xi}_{i,j} = \frac{2t_{i,j} - 2y + (1 - \log_{|\Theta|} |B_{i,j}|)(Y - y)}{2(1 - \log_{|\Theta|} |B_{i,j}|)(Y - y - T_{i,j} + t_{i,j})} & \text{if } \hat{\xi}_{i,j} < 1, \\ 1 & \text{if } \hat{\xi}_{i,j} \geq 1. \end{cases}$$

(ii) *If $B_{i,j} = \{b_{i,j}\}$ ($b_{i,j} \in \Theta$), then player i has the maximum value of Ξ_i under the following condition:*

$$\xi_{i,j} = \begin{cases} \hat{\xi}_{i,j} = \frac{2u_i(b_{i,j}) + Y - 3y}{2(Y - y)} \geq \frac{1}{2} & \text{if } u_i(b_{i,j}) < \frac{Y + y}{2}, \\ 1 & \text{if } u_i(b_{i,j}) \geq \frac{Y + y}{2}. \end{cases}$$

Proof. Item (i) of this corollary can be derived by Theorem 2 directly. Now we check item (ii) of the corollary. Since $|\{b_{i,j}\}| = 1$, we have $\log_{|\Theta|} |B_{i,j}| = 0$ and $t_{i,j} = T_{i,j} = u_i(b_{i,j})$. So, we have

$$\hat{\xi}_{i,j} = \frac{2t_{i,j} - 2y + (1 - \log_{|\Theta|} |B_{i,j}|)(Y - y)}{2(1 - \log_{|\Theta|} |B_{i,j}|)(Y - y - T_{i,j} + t_{i,j})} = \frac{2u_i(b_{i,j}) + Y - 3y}{2(Y - y)}.$$

Since $u_i(b_{i,j}) \geq y$, we have $2u_i(b_{i,j}) + Y - 3y \geq Y - y$. Thus, $\hat{\xi}_{i,j} \geq \frac{1}{2}$. If $\hat{\xi}_{i,j} < 1$, that is, $\frac{Y + y}{2} > u_i(b_{i,j})$. In this case, by the first part of this corollary, Ξ_i has the maximum at point $\hat{\xi}_{i,j} \in [\frac{1}{2}, 1)$. If $\hat{\xi}_{i,j} \geq 1$, that is, $\frac{Y + y}{2} \leq u_i(b_{i,j})$. So, Ξ_i has the maximum at point $\hat{\xi}_{i,j} = 1$. $\square$

Corollary 1 explores the maximum of function Ξ_i in the case of if focused mass $\xi_{i,j}$ is the only one variable. In particular, item (ii) of Corollary 1 shows that for any player i if $u_i(b_{i,j}) \geq \frac{Y + y}{2}$, the bigger the focused mass $\xi_{i,j}$ the bigger the sum of the lower and ambiguity-discounted upper payoffs. So, the less ambiguity degree the better for player i . In this case, function Ξ_i has a maximum if $\xi_{i,j} = 1$ (the degree of ambiguity is zero). If $u_i(b_{i,j}) < \frac{Y + y}{2}$, Ξ_i is strictly increasing when $\xi_{i,j}$ is in $[0, \hat{\xi}_{i,j}]$, and is strictly decreasing when $\xi_{i,j}$ is in $(\hat{\xi}_{i,j}, 1]$. That is, if ξ is in $[0, \hat{\xi}_{i,j}]$, the bigger ambiguity degree the better for player i ; if $\xi_{i,j}$ is in $(\hat{\xi}_{i,j}, 1]$, the smaller ambiguity degree the better for player i .

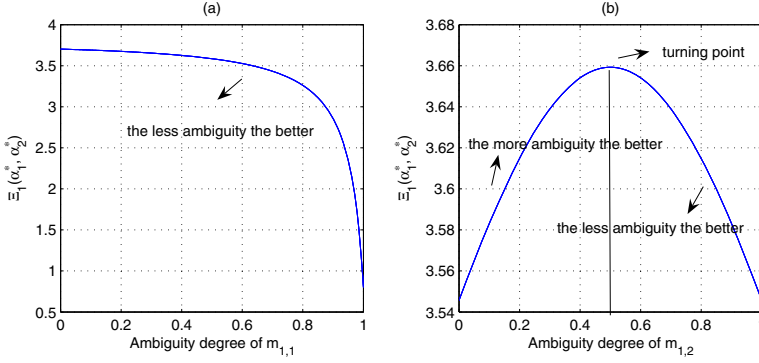


Fig. 1. Ambiguity degree and Sum of Interval

The concept of equilibrium is one of the most important concepts in game theory. Since Theorem 2 and its corollary hold for any mixed-strategy profile, they hold also for the ambiguous equilibrium profile of mixed strategy particularly.

Now we return to that selling cell-phone example. For mass function $m_{1,1}$, let $m_{1,1}(\{b_{1,1}\}) = m_{1,1}(\{s_4\}) = \xi$, where ξ be a variable (by formula (1), the ambiguity degree of function $m_{1,1}$, $\delta_{m_{1,1}} = 1 - \xi$), and other focused masses remain as the forenamed. Then similarly we can obtain the ambiguous equilibrium of mixed strategy α^* and $\Xi_1(\alpha^*)$. The latter is a function of variable ξ . Noting $u_1(s_i) = i$, $y = \min\{u_1(s_1), \dots, u_1(s_4)\} = 1$, and $Y = \max\{u_1(s_1), \dots, u_1(s_4)\} = 4$, and thus we have $\frac{Y+y}{2} = 2.5$. So, $u_1(b_{1,1}) = u_1(s_4) = 4 > \frac{Y+y}{2}$, by Corollary 1, retailer 1 has the maximum value of Ξ_1 at point $\xi = 1$ ($\delta_{m_{1,1}} = 0$). Fig. 1(a) shows that relation between the ambiguity degree of $m_{1,1}$ and $\Xi_1(\alpha^*)$. Again, let $m_{1,2}(\{b_{1,2}\}) = m_{1,1}(\{s_1\}) = \xi'$, where ξ' is a variable, then $u_1(b_{1,2}) = u_1(s_1) < \frac{Y+y}{2}$. By Corollary 1, retailer 1 has the maximum value of Ξ_1 at point $\xi' = \frac{2u_1(b_{1,2}) + Y - 3y}{2(Y - y)} = \frac{2 \times 1 + 4 - 3 \times 1}{2(4 - 1)} = 0.5$ (by formula (1), $\delta_{m_{1,2}} = 0.5$). Fig. 1(b) shows that relation between the ambiguity degree of $m_{1,2}$ and $\Xi_1(\alpha^*)$. The calculation and Fig. 1 illustrate under which condition the smaller degree of ambiguity the better for a player, and under which condition it is *not*.

5 Related Work

Harsanyi [8] is the first person who studied games played under uncertainty. He defined Bayesian games, which are played by different types of players with a probability distribution over the type set. Using expected utility method, a Bayesian game can be solved by the idea of Nash equilibrium. Clearly, it assumes that precise probabilities are available and a strategy taking is accurate. However, in real world, it is not always the case [11,21,22]. This problem does not exist in our model.

To handle uncertainty of strategy taking in games, the concepts of trembling hand perfect equilibrium [20] and proper equilibrium [16] are also introduced. Unfortunately, these two concepts also assume the probability of taking each

single strategy can be accurately estimated. And the main concern of the two concepts is the probability of strategy mistaking. This is not our concern. Ours is the uncertainty about the consequences of strategies. A stochastic game [15] is a dynamic game with probabilistic transitions, which also assumes the probabilities of the moves to a new random state are precise. Similarly, in [6], the uncertainty on the consequences of strategy is dealt with. However, these are just precise probabilities of each single consequence, and no ambiguity is involved like we did in this paper.

Currently, fuzzy games are analysed [2,10,12,13]. In these games, the fuzziness is mainly concerned with fuzzy payoffs. However, its meaning is different from that of ambiguity. In fact, for each possibility of a fuzzy payoff, it is assigned an accurate value in $[0, 1]$, indicating how possible it is in the fuzzy payoff set. While the meaning of ambiguity is that we are uncertain about the probability of each possibility, but we know probabilities of some subsets of possibilities [22]. So, fuzzy games cannot deal with the ambiguity that we did in this paper.

Choquet expected utility theory is another kind of a non-expected utility theory. In [1,9,19], the expected payoff of a strategy is calculated by the Choquet expected utility theory, and some solution concepts of games are discussed. More recently, Halpern [7] studied strategic games with a new decision rule of iterated regret minimization to reflect how people actually play games with regret attitude. However, these two kinds of work still assume that the consequences in a game are sure, but we relax the assumption.

6 Conclusion

This paper provides a model of static games played by players with ambiguous beliefs about consequences of their strategies. To solve this kind of games, we introduce a new solution concept. Moreover, we reveal how ambiguity degrees of players' belief about consequences impact the outcomes of a game. We find a turning point that the smaller or bigger degree of ambiguity the better for a player. In the future, it is interesting to extend our theory to extensive games.

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