

Systematicity, Accessibility, and Universal Properties

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Abstract. Human cognition is a mixture of the systematic and the non-systematic. One thing we can do systematically can be described as follows. If we know about multiplication, and the facts of basic multiplication, and we know conceptually what division is, then we can utilise the facts of multiplication that we know in order to solve division problems that correspond to those facts. For example, once children know that $4 \times 7 = 28$, and once they understand about division, they can work out that $28/4 = 7$. Aizawa has defined standards for what counts as an explanation of systematicity. In this paper, in accordance with Aizawa's framework, we apply concepts from category theory to this problem, and resolve it by identifying the unique natural transformation that underpins this example of systematicity, and others in the same class.

1 Introduction

Systematicity is a property of cognitive systems whereby the capacity for certain cognitive abilities implies that capacity for certain other cognitive abilities. In their seminal paper [1], Fodor and Pylyshyn introduced systematicity to cognitive science/artificial intelligence as something that needed to be explained to have a theory of cognition. As the title of their article suggests, the authors emphasised the problem that systematicity poses for connectionism, in particular. So what is systematicity? The usual way of describing it is by example: if a person has the capacity to infer John as the lover from the statement *John loves Mary* then they will also have the capacity to infer Mary as the lover from the statement *Mary loves John*, and conversely. There are many other examples. This type of phenomenon is termed systematicity.

Naturally, connectionists responded to the explanatory challenge raised by [1], an early and prominent instance being [2]. Systematicity grew into an industry, initially focusing on whether connectionist architectures could be devised that met the systematicity requirements posed by Fodor and Pylyshyn. Yet, despite

numerous attempts at addressing the systematicity problem in a way that did not simply implement a classical theory, no consensus was reached on an explanation (see [5]). Perhaps most interestingly, although connectionism has generally failed to provide an explanation for systematicity, so too has the classical approach (again, see [5]).

1.1 Characterising Systematicity

Before exploring the reasons why connectionism (and classicism) have not provided a complete explanation for systematicity, we give more detail how systematicity can be characterised. McLaughlin [3] noted that although there had been no clear, non-circular definition (as classes of examples) of systematicity, one can nonetheless provide useful schemas characterising instances of systematicity. McLaughlin provided five such schemas, all of which are of the form: *Ceteris paribus*, a cognizer is able to mentally represent that P if and only if the cognizer is able to mentally represent that $\tau(P)$, where τ is some transformation of propositions P . In McLaughlin's first collection of systematic schemata (which he calls SG1), P is aRb , where R is a relational predicate, and $\tau(P)$ is bRa . McLaughlin's SG1 takes in the *John loves Mary* example.

One could go a little further with McLaughlin [3], as in [4], and formulate a characterisation of systematicity: *in general, an instance of systematicity is when a cognizer has cognitive capacity c_1 if and only if the cognizer has cognitive capacity c_2* . In such a case, c_1 and c_2 are systematically related. This characterisation admits another type of systematicity, which we address in this paper.

1.2 Explaining Systematicity: Are Classical Systems Systematic?

The essential aspect of the systematicity problem is to provide an account whereby systematicity entails as a necessary, not just possible consequence of the assumptions and principles of the theory. So, although it is possible to devise a connectionist network that can represent *John loves Mary* if and only if it can represent *Mary loves John*, it is also possible to devise a connectionist network that can represent *John loves Mary* without being able to represent *Mary loves John*. So, without some principled reason for preferring the systematic over the non-systematic network, connectionism fails to *explain* systematicity [1].

Importantly, the classical explanation for systematicity also falls short in the same way, as analysed by Aizawa in great detail [5]. The ins and outs of his analysis are lengthy, but perhaps it can be summarised as follows. Aizawa argued that classicism failed to fully explain systematicity, because it is possible to design classical systems that will, for example, generate (or make inferences about) *John loves Mary* without being able to generate/infer about *Mary loves John*. Thus, context-free grammars, viewed as a way of producing sentences with compositional structure are normally chosen with grammar rules that support compositionality: $S \rightarrow \text{Person loves Person}$ and $\text{Person} \rightarrow \text{John}|\text{Mary}$ generate both *John loves Mary* and *Mary loves John*, and so on [6]. However, this does involve choices. If we had chosen $S \rightarrow \text{Person1 loves Person2}$, $\text{Person1} \rightarrow \text{John}|\text{Mary}$

and *Person2* $\rightarrow$ *Mary* then *Mary loves John* is no longer generated. Arguing along these lines, Aizawa inferred that systematicity is not a necessary property of classical systems, either. He characterised the problem by noting that the choice of grammar rules is *ad hoc*—the classical AI approach does not provide a *principled* way to choose between the alternative grammar rule sets. One rule set “works” and one doesn’t, but that is not principle, it’s model-fitting.

So now we are facing a problem: neither connectionism nor classicism provides a complete explanation for systematicity. Note that dynamic systems and Bayesian approaches suffer the same shortcoming. Like connectionism and classicism, dynamic systems and Bayesian models could be devised with the systematicity property, but equally dynamic systems and Bayesian models can also be devised without the systematicity property. What is required is an explanation that produces only the systematic models, and none has been forthcoming classicists, connectionists, Bayesian or dynamic systems modellers.

1.3 A Category Theory Explanation: Synopsis

In the face of this problem, we turned to an alternative approach that uses a branch of mathematics called Category Theory to explain systematicity without the need to introduce additional assumptions [6,16,4]. In Section 2, we provide some formal background to our category theory explanation; here, we summarise the general form of our explanation, as we will use it later in explaining yet another type of systematicity.

Category theory provides a formal notion of a *universal construction*: a construction that is a component of all constructions within the context of interest. For example, suppose we need a construction for representing *John loves Mary*, and inferring John as the lover. Connectionist and classical frameworks provide many constructions for this purpose. The problem with these methods is that there is no guarantee that the method for representing *John loves Mary* is the same as that for *Mary loves John*. A universal construction (in this case a *categorical product*), by contrast, means that there is exactly one way each instance is constructed, i.e., via that universal construction. Thus, all capacities within a systematically related group are connected via that arrow in a unique way.

We used the notion of universal construction to account for systematicity with regard to the capacity to represent and infer from relationally and recursively definable capacities [6,16,4]. In this paper, we sketch how our explanation also extends to another form of systematicity, called *accessibility* (introduced next), in terms of universal constructions. For reasons of space, we omit some background category theory relating to systematicity, (see [6,16,4]).

1.4 Accessibility

First, we characterise *accessibility*. When a child learns multiplication of small natural numbers, they (eventually) learn a largish set of facts like $9 \times 7 = 63$. Just as with the pizza example (above), not only can they make the required

inference given $9 \times 7 = \square$ (i.e., fill in the empty square), but they can also make related inferences given $\square \times 7 = 63$, $9 \times \square = 63$, and $9 \square 7 = 63$.

Once they are familiar with the concept of division of natural numbers, children can access these facts without further memorisation in order to do division of natural numbers, at least where the quotient is again a whole number: $63/7 = 9$ and $63/9 = 7$. Again, we are accessing the data via multiple routes.

We originally termed this phenomenon *omni-directional access*, but later adopted the less cumbersome term *accessibility* to refer to the same phenomenon. A fuller description of this cognitive phenomenon can be found in [7].

It is easy to see that accessibility is an instance of systematicity, in the broad sense defined by McLaughlin [3] and outlined above. We are saying, for example, that, given an suitable set of facts, one can answer questions like *Who ate a pizza?* if and only if they can answer questions like *What did Fred eat?* This fits the template definition of systematicity: i.e. when a cognizer has cognitive capacity c_1 if and only if the cognizer has cognitive capacity c_2 .

Our aim in this paper is to outline a theory of accessibility that meets the requirements of systematicity. In Section 2, we give category theory background for our explanation of accessibility, which follows in Section 3. Our explanation has general implications for supposedly alternative notions of (connectionist) compositionality, presented in Section 4. Discussion is provided in Section 5.

2 Category Theory and Universality

In this section we will introduce enough category theory to provide a mathematical framework for our theory of accessibility and to allow us to demonstrate systematicity for this theory.

Category theory is a branch, or maybe an arch, of mathematics that deals with structure. It has previously been applied in a number of areas of computer science, though more scantily in cognitive science. In computer science, it is conspicuous in the design of aspects of modern functional programming languages, e.g. [8], but it has long been known in connection with automata theory e.g. [9], and it is relevant in standard and non-standard logics [10]. In cognitive science, while some of the fundamental ideas had been mentioned in the cognitive psychology literature as early as 1963 [11], the first detailed application of categorical ideas (commutative diagrams) appears to have been [12].

2.1 Categories: States and Transitions

Category theory in mathematics deals with common structures. It has been applied in theoretical physics (string theory) and in functional programming. From a computational perspective, a cognitive system is often regarded as a collection of cognitive states together with cognitive processes realising transitions between those states. In category theory, formal constructions reside in a *category* of some description, which consists of a collection of *objects* and a collection of *maps* between objects obeying certain rules. To help ground category theory concepts

for the purposes of explaining systematicity, we will often interpret objects as sets of states and arrows as functions mapping states to states.

Definitions of basic category theory are available in a range of introductory texts on the subject, e.g. [13,14,15], but are recalled here for convenience.

Definition. A category $\mathbf{C}$ consists of a set of *objects* and a set of *arrows*, with the following properties: (i) every arrow f is associated with two objects, its *domain*, say a , and *codomain*, say b . This is usually written as $f : a \rightarrow b$; (ii) Given two arrows $f : a \rightarrow b$ and $g : b \rightarrow c$, where the codomain of f is the domain of g , we can form the composition $g \circ f : a \rightarrow c$; (iii) This composition is associative: if $f : a \rightarrow b$, $g : b \rightarrow c$, and $h : c \rightarrow d$ are composable arrows, then $h \circ (g \circ f) = (h \circ g) \circ f$. (iv) For every object $a \in \mathbf{C}$ there is an *identity* arrow 1_a . For identity arrows 1_a and 1_b and any arrow $f : a \rightarrow b$, $1_b \circ f = f = f \circ 1_a$.

Examples. 1. **Set**, the category of sets and functions: composition is composition of functions and identity arrows are identity functions from a set to itself.

2. **Vect**, the category of vector spaces and linear mappings: composition is composition of linear mappings and identity arrows are identity linear mappings from a vector space to itself.

3. A partially ordered set $(P, \leq)$ is a category, with objects the elements of the set P . Each relation $p \leq q$ is a unique arrow between p, q . Since $p \leq p$, there are identity arrows. Since $\leq$ is transitive, composition is defined.

4. **Poset**, the collection of partially ordered sets, is a category, with posets as objects and order-preserving maps as arrows.

5. A monoid M (that is, a set closed under an associative binary operation and possessing an identity element) is a category. There is a single object, and the arrows are the members of the monoid. The identity element is the identity arrow. (A familiar monoid is the set of all strings over an alphabet A .)

6. **Mon**, the collection of monoids, is a category, with monoids as objects and monoid homomorphisms as arrows.

Arrows are also often referred to as *maps* or *morphisms* (from *homomorphism*), typically in categories like **Set**, **Vect**, **Poset** and **Mon** where they are realised as functions between (structured) sets.

An arrow $f : a \rightarrow b$ is called an *isomorphism* if there is an arrow $g : b \rightarrow a$ such that $f \circ g = 1_b$ and $g \circ f = 1_a$.

2.2 Functors and Natural Transformations: Natural Constructions

If we regard categories as cognitive (sub)systems, then we can regard functors as relations between cognitive systems, possibly constructing cognitive processes from the components of other systems, and natural transformations as relations between such constructions. Functors preserve structure from one category to another, suggesting that they are important for an explanation of systematicity over structurally-related cognitive capacities. Natural transformations are important because they provide a formal basis for distinguishing constructions

that are “natural” in a principled (rather than some arbitrary informal) way, suggesting that they are important for meeting Aizawa’s explanatory standard.

Definition. If $\mathbf{C}$ and $\mathbf{D}$ are categories, then a *functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ maps objects of a, b of $\mathbf{C}$ to objects Fa, Fb of $\mathbf{D}$ and also maps arrows $f : a \rightarrow b$ of $\mathbf{C}$ to arrows $Ff : Fa \rightarrow Fb$ of $\mathbf{D}$, in a way preserves identity arrows and respects composition of arrows: that is (i) for $c \in \mathbf{C}$, $F1_c = 1_{Fc}$ and (ii) given $f : a \rightarrow b$, $g : b \rightarrow c$, $F(g \circ f) = Fg \circ Ff$.

Examples. One functor from **Vect** to **Set** is the forgetful functor that maps the vector space to its underlying set, and linear maps to functions. A functor from **Set** to **Vect** is the free vector space functor which maps a set to a vector space with the members of that set as the basis, and maps a function between sets to a linear map that maps the basis vectors as indicated by the function. We’ll have more relevant examples of functors later.

The functors from $\mathbf{C} \rightarrow \mathbf{D}$ actually form the objects of a category. The arrows of this category are called *natural transformations*.

Definition. Given two functors $F, G : \mathbf{C} \rightarrow \mathbf{D}$, a *natural transformation* $\eta : F \rightarrow G$ is a collection of arrows (“components”) in $\mathbf{D}$: one for every object $c \in \mathbf{C}$: $\eta_c : Fc \rightarrow Gc$ such that for every arrow $f : c_1 \rightarrow c_2$ in $\mathbf{C}$, $Gf \circ \eta_{c_1} = \eta_{c_2} \circ Ff$. This is summarised in a *commutative diagram*, in which compositions of arrows indicated by paths with the same start point and same end point are equal:

$$\begin{array}{ccc} Fc_1 & \xrightarrow{\eta_{c_1}} & Gc_1 \\ Ff \downarrow & & \downarrow Gf \\ Fc_2 & \xrightarrow{\eta_{c_2}} & Gc_2 \end{array} \quad (1)$$

Since natural transformations are to be arrows, they must be composable: if $\eta : F \rightarrow G$ and $\mu : G \rightarrow H$ are natural transformations, then $\mu \circ \eta : F \rightarrow H$ has components $(\mu \circ \eta)_c : Fc \rightarrow Hc$ for each $c \in \mathbf{C}$. The identity natural transformation $1_F : F \rightarrow F$ has components $1_c : Fc \rightarrow Fc$.

Hopefully, we’ve made it fairly clear why natural transformations are reasonable as morphisms of functors. Examples of natural transformations can be found in [13,14,15], and also in [6] (list reversal example) and [16] (in attached Text S2: clock example). This is addressed further in Section 4.

2.3 Universal Constructions

The final concept we need, which is central to our explanation of systematicity, is *universal construction*. A universal construction is a construction that captures an essential property shared by all instances in the category (system) of interest. In terms of cognitive processes, a universal construction is a process common to

all processes, suggesting how systematically-related capacities are intrinsically connected. Moreover, all processes are composed of the universal process in a unique way, so no further assumptions are needed to guarantee systematicity, meeting Aizawa's explanatory standard.

The most general definition of universal properties is not very transparent, and goes beyond our current needs; it can be found in [13,16,4]. Here we shall describe the universality property of the more-familiar Cartesian products (and of categorical products in general). A Cartesian product is an instance of the more general categorical product, which we employed to explain systematicity with regard to particular relations (e.g., John loves Mary) in [6]. Here, we show how this construct is used to explain systematic connections between relations such as multiplication and division.

Definition. cf. [13,18]. A *product* of two objects a, b in a category $\mathbf{C}$ is a triple $\langle p, \pi_1, \pi_2 \rangle$ where $\pi_1 : p \rightarrow a$ and $\pi_2 : p \rightarrow b$, such that for any triple $\langle r, \rho_1, \rho_2 \rangle$ with $\rho_1 : r \rightarrow a$ and $\rho_2 : r \rightarrow b$, there is a unique arrow $u : r \rightarrow p$ such that $\pi_1 \circ u = \rho_1$ and $\pi_2 \circ u = \rho_2$. ρ_1, ρ_2 are called *projections* onto the factors a, b .

A categorical product is an instance of a universal construction, which we will use later to address accessibility. In the category **Set**, the Cartesian product $a \times b$ of two sets a, b has this property, with π_1 and π_2 being the projections $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. The product object, in any category, is often written as $a \times b$, rather than p . With this convention, the product condition is that there is a *unique* arrow u such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & r & & \\
 & \swarrow \rho_1 & \downarrow u & \searrow \rho_2 & \\
 a & & a \times b & & b \\
 & \nwarrow \pi_1 & & \nearrow \pi_2 &
 \end{array} \tag{2}$$

It is easy to check that if $\langle r, \rho_1, \rho_2 \rangle$ is also a product, then u is the unique isomorphism between r and $a \times b$ (that makes the diagram commute)—that is, products are unique up to a unique isomorphism. This uniqueness property is a feature of all universal constructions.

The Cartesian product of two categories can easily be checked to be a category, and we will use such constructions later in this paper.

3 A Uniqueness Property for Multiplication-Division

In order to meet Aizawa's requirement, we need to show why there is a unique correspondence multiplication and division. In fact, multiplication and division of natural numbers can be viewed as relations - for example, $(8, 3, 24)$ belongs to the multiplication relation because $8 \times 3 = 24$. If $F_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = F_2$ are the sets of factors for multiplication of digits, and $P = \{a \times b \mid a \in F_1, b \in F_2\}$ is the set of products of digits, then the multiplication relation is a subset of the product $F_1 \times F_2 \times P$, and similarly the division relation (for the sets of numbers) is

a subset of the product $P \times F_1 \times F_2$. The correspondence between these products is the function $\tau : F_1 \times F_1 \times P \rightarrow P \times F_1 \times F_2 : (f_1, f_2, p) \mapsto (p, f_1, f_2)$ (the notation $\mapsto$ (maps to) is a way that is sometimes convenient for expressing the action of a function: $h : x \mapsto y$ means $h(x) = y$). Our aim is to show that this function is natural and unique.

We'll do this in a more general setting, so that our argument can be applied to this transformation of other ternary relations. Given sets (or objects in any category with products) A , B , and C , we can form their product $A \times B \times C$. Technically, this product will, by a slight elaboration of the definition of product above, be a quadruple $\langle A \times B \times C, \pi_1, \pi_2, \pi_3 \rangle$, where $\pi_1 : A \times B \times C \rightarrow A : (a, b, c) \mapsto a$ is the projection onto A , and similarly π_2, π_3 are the projections onto B, C . This triple product has a uniqueness property like that of the product of two objects: given another such quadruple $\langle R, \rho_1, \rho_2, \rho_3 \rangle$, there is a unique arrow $u : R \rightarrow A \times B \times C$ such that $\pi_1 \circ u = \rho_1$, $\pi_2 \circ u = \rho_2$ and $\pi_3 \circ u = \rho_3$.

We shall use a construction analogous to one in [15]: we can also form the product $(C \times A \times B, \sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1, \sigma_2, \sigma_3$ are respectively the projections onto C, A , and B . Since $C \times A \times B$ is a product of A, B , and C , there is a unique map $v : A \times B \times C \rightarrow C \times A \times B$ such that $\sigma_2 \circ v = \pi_1$, $\sigma_3 \circ v = \pi_2$ and $\sigma_1 \circ v = \pi_3$. Note the cross-over of subscripts. In the same way, since $A \times B \times C$ is also a product of A, B , and C , there is a unique map $w : C \times A \times B \rightarrow A \times B \times C$ such that $\pi_3 \circ w = \sigma_1$, $\pi_1 \circ w = \sigma_2$ and $\pi_2 \circ w = \sigma_3$.

Since $\sigma_1 \circ v = \pi_3$ and $\pi_3 \circ w = \sigma_1$, $\sigma_1 \circ v \circ w = \sigma_1$ and similarly $\sigma_2 \circ v \circ w = \sigma_2$ and $\sigma_3 \circ v \circ w = \sigma_3$, so by the uniqueness property of $C \times A \times B$, $v \circ w = 1_{C \times A \times B}$. A similar argument shows that $w \circ v = 1_{A \times B \times C}$. Thus u is an isomorphism, and so it is the unique isomorphism $A \times B \times C \rightarrow C \times A \times B$ satisfying $\pi_3 \circ w = \sigma_1$, $\pi_1 \circ w = \sigma_2$ and $\pi_2 \circ w = \sigma_3$.

Thus there is a unique correspondence between the two products. In fact, this correspondence is natural in A, B , and C as well. To see this, we have to set up functors and a natural transformation between them. If $\mathbf{C}$ is the category, then the two functors are $\times : \mathbf{C} \times \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C} : (A, B, C) \mapsto A \times B \times C$ and $\hat{\times} : \mathbf{C} \times \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C} : (A, B, C) \mapsto C \times A \times B$. The transformation $\tau : \times \rightarrow \hat{\times}$ that we are going to show is natural, has components $\tau_{A,B,C} : \times(A, B, C) \rightarrow \hat{\times}(A, B, C)$, i.e. $\tau_{A,B,C} : A \times B \times C \rightarrow C \times A \times B$. These components are of course chosen to be the map that we have just shown is a unique isomorphism for each A, B , and C . In a category where objects have elements, these components are given by $\tau_{A,B,C}(a, b, c) = (c, a, b)$. To prove naturality, we need to show that for arrows $\alpha : A \rightarrow A_1$, $\beta : B \rightarrow B_1$, and $\gamma : C \rightarrow C_1$, the following diagram commutes:

$$\begin{array}{ccc}
 A \times B \times C & \xrightarrow{\tau_{A,B,C}} & C \times A \times B \\
 \alpha \times \beta \times \gamma \downarrow & & \downarrow \gamma \times \alpha \times \beta \\
 A_1 \times B_1 \times C_1 & \xrightarrow{\tau_{A_1,B_1,C_1}} & C_1 \times A_1 \times B_1
 \end{array} \tag{3}$$

In a category where objects have elements, we assume $a \in A, b \in B, c \in C$:

$$\begin{aligned}
 (\gamma \times \alpha \times \beta) \circ \tau_{A,B,C}(a, b, c) &= \gamma \times \alpha \times \beta(c, a, b) \\
 &= (\gamma(c), \alpha(a), \beta(b)) \\
 &= \tau_{A_1, B_1, C_1}(\alpha(a), \beta(b), \gamma(c)) \\
 &= \tau_{A_1, B_1, C_1} \circ (\alpha \times \beta \times \gamma)(a, b, c)
 \end{aligned}$$

That is, $(\gamma \times \alpha \times \beta) \circ \tau_{A,B,C} = \tau_{A_1, B_1, C_1} \circ (\alpha \times \beta \times \gamma)$, so the diagram commutes, and τ is natural in A, B , and C . Thus we have shown:

Proposition. The map $\tau_{A,B,C} : A \times B \times C \rightarrow C \times A \times B$ is a unique isomorphism natural in A, B , and C . It is unique with respect to the product structures of its domain and codomain.

A Cartesian product of two sets consists of *all* pairwise combinations of elements the two sets. Thus multiplication (as a relation) is not a whole Cartesian product: e.g. $1 \times 2 \neq 3$, but triple $(1, 2, 3) \in A \times B \times C$. Restriction to a subset of the Cartesian product is achieved by using a *pullback* (or *constrained product*), where additional morphisms restrict the construction to a subset of the Cartesian product. Pullbacks are a further instance of a universal construction, and their role in explaining systematicity parallels that of products. For reasons of space, we omit the details here. A detailed explanation of pullbacks, pullback functors and their roles in explaining (quasi-)systematicity, where some but not all possible combinations of constituents are systematically related, is in [16].

4 Natural Constructions

The argument against the classical explanation for systematicity is that it relies on stipulating specific modes of classical compositionality (i.e., grammars), since not all modes account for systematicity [5]. The general retort is to appeal to “canonical” constructions. But classicism has no principle that dictates such constructions, whatever they may be [5]. In this section, we show how the category theory notion of a natural transformation provides this principle, and how constructions that are not natural transformations do not support systematicity.

Some constructions seem more intuitively “natural” than others. To illustrate, suppose we have pairs of ordered blocks. A “natural” construction is to compose a representation of each pair of blocks from symbols representing each block, such that the first symbol in the representation of the pair corresponds to the first block and the second symbol corresponds to the second block. For example, the composition ab represents the pair of blocks (a, b) . An alternative construction is to compose a representation of each pair as a number constructed from numbers representing each block. *Gödel numbering* is one possibility, proposed as an alternative non-classical form of compositionality [17]. For example, $2^n 3^m$ represents the pair (n, m) . Given that the numbers representing pairs of blocks are composed from prime powers (where each unique exponent n, m ,

corresponds to a unique block, and the prime numbers 2, 3, correspond to the first and second positions in the pair), we know from the *fundamental theorem of arithmetic* that the numbers corresponding to the individual blocks are recoverable by a prime decomposition procedure. Although both constructions are isomorphisms, there is a sense in which the second construction is less “natural” than the first. A Godel numbering scheme provides a unique way of getting from the object represented (such as the pair (n, m)) to the representation $(2^n 3^m)$.

To illustrate, consider an instance of Diagram 1, involving Godel numbers:

$$\begin{array}{ccc}
 A \times B & \xrightarrow{g_{A,B}} & G(A \times B) \\
 \downarrow 1_A \times i & & \downarrow G(1_A \times i) \\
 A \times B' & \xrightarrow{g_{A,B'}} & G(A \times B')
 \end{array} \tag{4}$$

where F (in Diagram 1) is instantiated as the functor $\mathbf{Set} \times \mathbf{Set} \rightarrow \mathbf{Set}$ that maps pairs of sets (A, B) to $A \times B$ and pairs of arrows (functions in this case) $p : A \rightarrow A'$ and $q : B \rightarrow B'$ to $p \times q : A \times B \rightarrow A' \times B'$, where $(p \times q)(a, b) = (p(a), q(b))$. In our case, $B \subset B'$, and we have the inclusion map $i : B \rightarrow B' : b \mapsto b$, and in the vertical arrows in Diagram 4, $(1_A, i)$ is mapped to $1_A \times i : A \times B \rightarrow A \times B'$.

The functor $G : \mathbf{Set} \times \mathbf{Set} \rightarrow \mathbf{Set}$ is a “change of basis” functor whereby maps between sets of elements become maps between sets of corresponding Godel numbers, where each unique element has a unique Godel number. For example, suppose A and B are sets of people’s names (e.g., *John* and *Mary*), so that $A \times B$ is a Cartesian product that includes pairs of names, e.g., $(\textit{John}, \textit{Mary})$. Each name is assigned a unique number j , e.g., *John* is 1, *Mary* is 2, and so on. Functor G constructs a set of Godel numbers for each argument set by using the number assigned to each name (j) as the exponent to a prime number assigned to each tuple position: e.g., the first tuple position is 2, the second is 3, the third is 5, and so on. Thus, the pair $(a_j, b_k) \in A \times B$ is assigned the unique Godel number $2^j 3^k$, from which j and k are recoverable by prime number decomposition. As each element is assigned a unique Godel number, the action of functor G on arrows is essentially a change of basis. So identities and compositions are preserved.

Transformation g is a family of maps, one for each set, sending each element $(a, b) \in A \times B$ to a corresponding Godel number. The critical point here is that each map in g is only locally-consistent: i.e., as already noted, we are guaranteed to have a bijection $g_{A,B} : A \times B \rightarrow G(A \times B)$, but this bijection is generally not unique. Another map $g_{A,B'} : A \times B' \rightarrow G(A \times B')$, where $B' \supset B$, is also a bijection, yet together they do not necessarily make Diagram 4 commute (i.e., they do not necessarily constitute a natural transformation).

To see why, suppose we have sets $A = \{\textit{John}\}$, and $B = \{\textit{Mary}\}$, so that $g_{A \times B} : (\textit{John}, \textit{Mary}) \mapsto 2^1 3^2 = 18$. Now, we introduce a new name *Akiko*, which is an element of the set $B' = B \cup \{\textit{Akiko}\}$, which is assigned 3 for the purpose of constructing Godel numbers via G . Suppose, however, that g assigns elements to the available indices (j) based on local information only (e.g., within-set lexical order, not the globally assigned index used by G). In this

case, we have for map $g_{A,B'}$, $Akiko < John < Mary$, where $Akiko$ is assigned 1, and $John$ and $Mary$ are assigned 2 and 3 respectively. Although this locally defined ordering scheme affords the needed bijections, these bijections are not consistent across objects. We have the mapping $g_{A,B'} : (John, Akiko) \mapsto 2^2 3^1 = 24$, $(John, Mary) \mapsto 2^2 3^3 = 108$. However, Diagram 4 does not commute for i , since $G(i) \circ g_{A,B}(John, Mary) = G(i)(6) = 18 \neq 108 = g_{A,B'}(John, Mary) = g_{A \times B'} \circ i(John, Mary)$.

We can, of course, modify the Godel numbering scheme above to make Diagram 4 commute by reordering the new element $Akiko$: e.g., assign the order $John < Mary < Akiko$ to set B' . To do so, however, requires prior knowledge of all possible elements in all possible sets, which the cognitive system cannot have in general. Or, conversely, upon knowing the existence of a new element, the cognitive system must update all construction schemes involving the new element, clearly violating the systematicity requirement: i.e., prior to the update the cognitive system is in a state whereby it has the capacity for some but all related instances. Hence, such a scheme fails to explain systematicity.

Characteristically, natural constructions make no reference to an object's contents; constructions that are not natural do. Also, the specific assignment of elements to (in this case) numbers is arbitrary: while the Godel numbering scheme maps items uniquely, we could easily provide an alternative mapping, such as the function $(j, k) \mapsto 5^j 7^k$, thus the mapping is *ad hoc*.

5 Discussion and Conclusions

Our argument in section 3 was expressed in terms of (Cartesian) products, because they are somewhat familiar and have the needed universal mapping property. As already noted, though, where the relations in question involve a subset of the Cartesian product, a pullback can be employed and the explanation parallels the explanation using products. In some cases, however, the form of accessibility in question is not a function. For example, in the presence of other facts such as *John ate ice-cream*, the link from person to what they ate is no longer a function, in the usual sense, since John is now associated with both pizza and ice-cream. One possible way of addressing such cases is to consider another kind a universal construction, called an *adjunction* (see [6,16]), that connects the category **Set** of sets with functions as arrows, and the category **Rel** of sets with *relations* as arrows.

Accessibility is a property of human cognition that we believe is important in itself, but also in distinguishing between the capabilities of different theories and classes of cognitive models. It may also be important to bear in mind when designing human-computer interfaces as well as intelligent tutoring systems, etc. Systematicity is interesting as a frequent property of cognitive systems and models, and also as a desirable property for AI applications. We have shown in this paper that accessibility can be analysed in terms of systematicity by means of a simple construction that is natural and universal in the category-theoretic sense of these terms. This provides guidelines for determining, for a given model, in

principle, whether it meets the standard of systematicity of accessibility. Not all cognitive models will specifically draw on accessibility, but those that do should benefit from our contribution.

Systematicity (hence accessibility) is important for AI and cognitive science: the advantage of a systematicity property is that one gets a *collection* of intelligent capacities for free once one has a universal construction. Naturality provides a general, formal category-theory-based notion of content independence in cognitive science and artificial intelligence, where mental capacity extends to a variety of situations that have a common form, despite their specific differences.

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