

Complexity of Merging and Splitting for the Probabilistic Banzhaf Power Index in Weighted Voting Games¹

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Abstract. The Banzhaf power index is a prominent measure of a player’s influence for coalition formation in weighted voting games, an important class of simple coalitional games that are fully expressive but compactly representable. For the normalized Banzhaf index, Aziz and Paterson [1] show that it is NP-hard to decide whether merging any coalition of players is beneficial, and that in unanimity games, merging is always disadvantageous, whereas splitting is always advantageous. We show that for the *probabilistic* Banzhaf index (which is considered more natural than the normalized Banzhaf index), the merging problem is in P for coalitions of size two, and is NP-hard for coalitions of size at least three. We also prove a corresponding result for the splitting problem. In unanimity games and for the probabilistic Banzhaf index (in strong contrast with the results for the normalized Banzhaf index), we show that splitting is always disadvantageous or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions of size at least three.

1 Introduction

Weighted voting games are an important class of simple coalitional games that are fully expressive but compactly representable. Power indices measure how influential a player is for forming winning coalitions in weighted voting games. The Banzhaf and the Shapley-Shubik power indices are most prominent among such measures. Roughly speaking, they indicate, respectively, in how many ways and in how many “orders of support” a player can swing the outcome of a coalition by joining or leaving it. For the Shapley-Shubik index, Bachrach and Elkind [2] study the problem of whether a player can increase its power by splitting into several new players and distributing its weight among them; such a player may have incentive to manipulate the game via introducing false names. Bachrach and Elkind also ask whether merging their weights can help any two players to increase their power. For the normalized Banzhaf index, Aziz and Paterson [1] show that it is NP-hard to decide whether merging any coalition of players is beneficial, and that in unanimity games, merging is always disadvantageous and splitting is always advantageous.

In this note, we show that for the *probabilistic* Banzhaf index (which Dubey and Shapley [5, p. 102] call “in many respects more natural” than the normalized Banzhaf index), the merging problem is in P for coalitions of size two, and is NP-hard for coalitions with at least three players. We also prove a corresponding result for the splitting problem. In unanimity games and with respect to the probabilistic Banzhaf index, we show that splitting is always disadvantageous

or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions with at least three players. This strongly contrasts with Aziz and Paterson’s result [1] for unanimity games with respect to the normalized Banzhaf index.

2 Banzhaf Indices in Weighted Voting Games

A *simple coalitional game* consists of a set $N = \{1, \dots, n\}$ of players and a function $v : \mathcal{P}(N) \rightarrow \{0, 1\}$ such that $v(\emptyset) = 0$, $v(N) = 1$, and $v(R) \leq v(S)$ whenever $R \subseteq S$ for coalitions $R, S \subseteq N$. A weighted voting game $G = (w_1, \dots, w_n; q)$ is a special form of simple coalitional game with a threshold q and nonnegative integer weights w_i , $1 \leq i \leq n$, where w_i is the i^{th} player’s weight. A coalition $S \subseteq N$ is *successful* (i.e., $v(S) = 1$) if $\sum_{i \in S} w_i \geq q$, and not successful (i.e., $v(S) = 0$) otherwise. An essential term is that of a pivotal (or “crucial” or “critical”) player. A player i is *pivotal for a coalition* $S \subseteq N - \{i\}$ if $S \cup \{i\}$ is successful, but S is not. Letting $d_G(S, i) = v(S \cup \{i\}) - v(S)$, we have $d_G(S, i) = 1$ if player i is pivotal for S , and $d_G(S, i) = 0$ otherwise.

A power index measures a player’s influence in a weighted voting game. The *raw Banzhaf power index* was introduced in [4] and indicates the number of coalitions in which a player is pivotal: $\text{Banzhaf}^*(G, i) = \sum_{S \subseteq N - \{i\}} d_G(S, i)$ for a weighted voting game G and a player i in G . However, since the ratios of these indices are more important than their magnitudes, it has become common to normalize them so as to add up to one. In fact, two different ways of normalization have been proposed. The original definition of the *normalized* Banzhaf power index was analyzed by Dubey and Shapley [5], who introduced another normalization, which they dubbed the *probabilistic* Banzhaf power index and which they viewed as being “in many respects more natural” than the normalized Banzhaf index.³ In a nutshell, they consider various useful properties of power indices in simple games (e.g., what they call Axiom A4, see [5, p. 104]) and show that the probabilistic Banzhaf index satisfies this property, whereas the normalized one does not. As the normalized Banzhaf index lacks Axiom A4, they conclude: “This may be taken as an initial sign of trouble with the normalization [of the normalized Banzhaf index]” (see [5, Footnote 21]). They also note that the probabilistic Banzhaf index “is better behaved when analyzing convergence” [5, p. 116]. Bachrach and Rosenschein [3, p. 126] note that the normalized Banzhaf index “has certain undesirable qualities.”

Formally, the *normalized Banzhaf power index* is defined as $\text{Banzhaf}^d(G, i) = \text{Banzhaf}^*(G, i) / \sum_{j=1}^n \text{Banzhaf}^*(G, j)$, and the *probabilistic Banzhaf power index* is defined as $\text{Banzhaf}(G, i) = \text{Banzhaf}^*(G, i) / 2^{n-1}$.

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³ As pointed out by an ECAI-2010 reviewer, the normalized Banzhaf power index has its advantages as well, depending on which setting one considers.

3 Beneficial Merging and Beneficial Splitting

Bachrach and Elkind [2] asked about the complexity of the problem of whether two players in a given weighted voting game can benefit from merging their weights by forming a single new player, i.e., of whether their joint power will increase in this new weighted voting game. For both the Shapley-Shubik power index and the normalized Banzhaf power index, Aziz and Paterson [1] provided an NP-hardness lower bound for the more general problem of whether any given *coalition* of players can benefit from merging (i.e., can increase their joint power by forming a new player whose weight is the sum of the coalition members' weights). To define this problem formally for the probabilistic Banzhaf power index, we need the following notation. For a weighted voting game $G = (w_1, \dots, w_n; q)$ and a coalition $S \subseteq N = \{1, \dots, n\}$, define the new weighted voting game $G_{\&S} = (\sum_{i \in S} w_i, w_{j_1}, \dots, w_{j_{n-\|S\|}}; q)$, where $\{j_1, \dots, j_{n-\|S\|}\} = N - S$.⁴ $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ is the problem of whether, given a weighted voting game $G = (w_1, \dots, w_n; q)$ and a coalition $S \subseteq \{1, \dots, n\}$, it holds that $\text{Banzhaf}(G_{\&S}, 1) > \sum_{i \in S} \text{Banzhaf}(G, i)$.

Theorem 1 $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ is in P for instances (G, S) with $\|S\| = 2$, and is NP-hard for instances (G, S) with $\|S\| \geq 3$.

Proof. Given an instance (G, S) of $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$, suppose $\|S\| = 2$. Without loss of generality (see Footnote 4), we may assume that $S = \{1, 2\}$. Thus $G_{\&S} = (w_1 + w_2, w_3, \dots, w_n; q)$. It is easy to see that $\text{Banzhaf}(G_{\&S}, 1) = \text{Banzhaf}(G, 1) + \text{Banzhaf}(G, 2)$, so a coalition of two players can never benefit from merging.

To prove NP-hardness of $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ for $\|S\| \geq 3$, we give a reduction from the NP-complete problem PARTITION: Given a nonempty sequence $(a_1, \dots, a_k)$ of positive integers such that $\sum_{i=1}^k a_i$ is an even number, does there exist a subset $I \subseteq \{1, \dots, k\}$ such that $\sum_{i \in I} a_i = \sum_{i \in \{1, \dots, k\} - I} a_i$? From a given instance $A = (a_1, \dots, a_k)$ of PARTITION we construct in polynomial time the following $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ instance (G, S) : $G = (8a_1, \dots, 8a_k, 1, 1, 1, 1; 1 + 4 \sum_{\ell=1}^k a_\ell)$ and $S = \{k+2, k+3, k+4\}$.

The last three players' raw Banzhaf indices each are equal to $\text{Banzhaf}^*(G, k+4) = \|\{S' \subseteq \{1, \dots, k+3\} \mid k+4 \text{ is pivotal for } S'\}\|$. The set of coalitions S' in this equation can be divided into four disjoint subsets: For any $\tilde{S} \subseteq \{1, \dots, k\}$ and any two distinct $a, b \in \{k+1, k+2, k+3\}$, distinguish between the set of coalitions S' with (i) $S' = \tilde{S}$, (ii) $S' = \tilde{S} \cup \{a\}$, (iii) $S' = \tilde{S} \cup \{a, b\}$, and (iv) $S' = \tilde{S} \cup \{k+1, k+2, k+3\}$. By construction of G , player $k+4$ cannot be pivotal for any coalition S' in cases (ii), (iii), or (iv) but only for those in case (i): player $k+4$ is pivotal for those $S' = \tilde{S} \subseteq \{1, \dots, k\}$ such that $1 + \sum_{j \in \tilde{S}} 8a_j \geq 1 + 4 \sum_{\ell=1}^k a_\ell$ and $\sum_{j \in \tilde{S}} 8a_j < 1 + 4 \sum_{\ell=1}^k a_\ell$ if and only if $\sum_{j \in \tilde{S}} 8a_j = 4 \sum_{\ell=1}^k a_\ell$, which in turn is true exactly if A is a yes instance of PARTITION. Thus, the number of coalitions for which player $k+4$ is pivotal equals the number of solutions of the PARTITION instance A . Denote this number by $X_A = \#\text{PARTITION}(A)$. Thus, the probabilistic Banzhaf index of player $k+4$ (and, analogously, of players $k+3$ and $k+2$) is $\text{Banzhaf}(G, k+4) = \text{Banzhaf}(G, k+3) = \text{Banzhaf}(G, k+2) = X_A / 2^{k+3}$. Similarly, the probabilistic Banzhaf index of the first player in the new game $G_{\&S} = (3, 8a_1, \dots, 8a_k, 1; 1 + 4 \sum_{\ell=1}^k a_\ell)$ can be calculated to be $\text{Banzhaf}(G_{\&S}, 1) = X_A / 2^{k+1}$. Thus, we have $\text{Banzhaf}(G_{\&S}, 1) = \frac{X_A}{2^{k+1}} = \frac{4X_A}{2^{k+3}} > \frac{3X_A}{2^{k+3}} = \sum_{i=2}^4 \text{Banzhaf}(G, k+i)$ if and only if $X_A > 0$, which in turn is true if and only if A is a yes instance of PARTITION. $\square$

⁴ Note that the players' order doesn't matter for the (probabilistic) Banzhaf index considered in this paper.

$\text{BENEFICIALSPLIT}_{\text{Banzhaf}}$, the beneficial splitting problem for the probabilistic Banzhaf power index, is formally defined as follows: Given a weighted voting game $G = (w_1, \dots, w_n; q)$, a player $i \in \{1, \dots, n\}$, and an integer $m \geq 2$, can i split into m players $i_1, \dots, i_m$ such that $\sum_{j=1}^m \text{Banzhaf}(G_{i,m}, i_j) > \text{Banzhaf}(G, i)$, where $G_{i,m} = (w_1, \dots, w_{i-1}, w_{i_1}, \dots, w_{i_m}, w_{i+1}, \dots, w_n; q)$ with $\sum_{j=1}^m w_{i_j} = w_i$?

By similar methods as above we can prove the following theorem.

Theorem 2 $\text{BENEFICIALSPLIT}_{\text{Banzhaf}}$ is in P for instances $(G, i, 2)$, and is NP-hard for instances (G, i, m) with $m \geq 3$.

Regarding the lower bounds for $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ and $\text{BENEFICIALSPLIT}_{\text{Banzhaf}}$ stated in Theorems 1 and 2, a natural question arises: What about the upper bounds? Faliszewski and Hemaspaandra [6] established an upper bound for the beneficial merging problem originally proposed by Bachrach and Elkind [2] ("Can any two players increase their joint Shapley-Shubik index via merging?"): This problem is in the complexity class PP ("probabilistic polynomial time"). Their proof technique can be used to show that $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ also belongs to PP.⁵ The proof is omitted due to space. We conjecture that both $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ and $\text{BENEFICIALSPLIT}_{\text{Banzhaf}}$ are PP-complete.

Theorem 3 $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ is in PP.

4 Merging and Splitting in Unanimity Games

A weighted voting game $G = (w_1, \dots, w_n; q)$ is called a *unanimity game* if $q = \sum_{i=1}^n w_i$, i.e., if only the grand coalition can reach the threshold and win. Aziz and Paterson [1] show that for the normalized Banzhaf index in unanimity games, merging is always disadvantageous, whereas splitting is always advantageous. (Thus one can decide in polynomial time whether or not merging/splitting is beneficial.) In strong contrast, we show that in unanimity games with respect to the probabilistic Banzhaf index, splitting is always disadvantageous or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions with at least three players.

Theorem 4 Let G be a unanimity game with player set N .

1. $(G, S) \notin \text{BENEFICIALMERGE}_{\text{Banzhaf}}$ for each $S \subseteq N$ with $\|S\| = 2$,
2. $(G, S) \in \text{BENEFICIALMERGE}_{\text{Banzhaf}}$ for each $S \subseteq N$ with $\|S\| \geq 3$.
3. $(G, i, m) \notin \text{BENEFICIALSPLIT}_{\text{Banzhaf}}$ for each $i \in N$ and $m \geq 2$.

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⁵ We note that the same proof cannot be transferred immediately to the beneficial merging problem with respect to the normalized Banzhaf index, since the normalization in the original game G differs from that in the game $G_{\&S}$.