

Merging and Splitting for Power Indices in Weighted Voting Games and Network Flow Games on Hypergraphs

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Abstract. The Banzhaf power index is a prominent measure of a player’s influence for coalition formation in weighted voting games, an important class of simple coalitional games that are fully expressive but compactly representable. For the normalized Banzhaf index, Aziz and Paterson [1] show that it is NP-hard to decide whether merging any coalition of players is beneficial, and that in unanimity games, merging is always disadvantageous, whereas splitting is always advantageous. We show that for the *probabilistic* Banzhaf index (which is considered more natural than the normalized Banzhaf index), the merging problem is in P for coalitions of size two, and is NP-hard for coalitions of size at least three. We also prove a corresponding result for the splitting problem. In unanimity games and for the probabilistic Banzhaf index (in strong contrast with the results for the normalized Banzhaf index), we show that splitting is always disadvantageous or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions of size at least three. In addition, we study the merging and splitting problems for threshold network flow games [3, 4] on hypergraphs.

1 Introduction

Weighted voting games are an important class of simple coalitional games; their importance is mainly due to the fact that they are fully expressive but at the same time compactly representable.

Power indices measure how influential a player is for forming winning coalitions in weighted voting games. The Banzhaf and the Shapley-Shubik power indices are most prominent among such measures. Roughly speaking, they indicate, respectively, in how many ways and in how many “orders of support” a player can swing the outcome of a coalition by joining or leaving it. For the Shapley-Shubik index, Bachrach and Elkind [2] study the problem of whether a player can increase its power by splitting into several new players and distributing its weight among them; such a player may have incentive to manipulate the game via introducing false names. Bachrach and Elkind also ask whether merging their weights can help any two players to increase their power. For the normalized Banzhaf index, Aziz and Paterson [1] show that it is NP-hard to decide whether merging any coalition of players is beneficial, and that in unanimity games, merging is always disadvantageous and splitting is always advantageous.

We show that for the *probabilistic* Banzhaf index (which Dubey and Shapley [6, p. 102] call “in many respects more natural” than the normalized Banzhaf index), the merging problem is in P for coalitions of size two, and is NP-hard for coalitions with at least three players. We also prove a corresponding result for the splitting problem. In unanimity games and with respect to the probabilistic

Banzhaf index, we show that splitting is always disadvantageous or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions with at least three players. This strongly contrasts with Aziz and Paterson’s result [1] for unanimity games with respect to the normalized Banzhaf index.

We also consider the merging and splitting problems for threshold network flow games, which have been introduced by Bachrach and Rosenschein [3, 4]. However, it seems that merging and splitting cannot be reasonably defined on graphs in such games. We propose to study threshold network flow games on hypergraphs where merging and splitting can be sensibly defined. Though our results here are rather preliminary, we hope to propose an interesting model and to obtain more results in future work.

2 Banzhaf Indices in Weighted Voting Games

A *simple coalitional game* consists of a set $N = \{1, \dots, n\}$ of players and a function $v : \mathfrak{P}(N) \rightarrow \{0, 1\}$ such that $v(\emptyset) = 0$, $v(N) = 1$, and $v(R) \leq v(S)$ whenever $R \subseteq S$ for coalitions $R, S \subseteq N$. A weighted voting game $G = (w_1, \dots, w_n; q)$ is a special form of simple coalitional game with a threshold q and nonnegative integer weights w_i , $1 \leq i \leq n$, where w_i is the i^{th} player’s weight. A coalition $S \subseteq N$ is *successful* (i.e., $v(S) = 1$) if $\sum_{i \in S} w_i \geq q$, and not successful (i.e., $v(S) = 0$) otherwise. An essential term is that of a pivotal (or “crucial” or “critical”) player. A player i is *pivotal for a coalition* $S \subseteq N - \{i\}$ if $S \cup \{i\}$ is successful, but S is not. Letting

$$d_G(S, i) = v(S \cup \{i\}) - v(S),$$

we have $d_G(S, i) = 1$ if player i is pivotal for S , and $d_G(S, i) = 0$ otherwise.

A power index measures a player’s influence in a weighted voting game. The *raw Banzhaf power index* was introduced in [5] and indicates the number of coalitions in which a player is pivotal:

$$\text{Banzhaf}^*(G, i) = \sum_{S \subseteq N - \{i\}} d_G(S, i)$$

for a weighted voting game G and a player i in G . However, since the ratios of these indices are more important than their magnitudes, it has become common to normalize them so as to add up to one. In fact, two different ways of normalization have been proposed for the Banzhaf index. The original definition of the *normalized* Banzhaf power index [5] was analyzed by Dubey and Shapley [6], who introduced another normalization, which they dubbed the *probabilistic* Banzhaf power index and which they viewed as being “in many respects more natural” than the normalized Banzhaf index.²

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² As pointed out by an ECAI-2010 reviewer, the normalized Banzhaf power index has its advantages as well, depending on which setting one considers.

In a nutshell, Dubey and Shapley [6] consider various natural and useful properties (or axioms) of power indices in simple games, and they show as Theorem 1 in [6, p. 104] that there exists a unique function ϕ on the set of simple games satisfying Axioms A1 through A4. In particular, Axiom A4 can be stated as follows: For any two simple games v and w on the set $N = \{1, 2, \dots, n\}$ of players, it holds that

$$\phi(v \vee w) + \phi(v \wedge w) = \phi(v) + \phi(w),$$

where the games $v \vee w$ and $v \wedge w$ are defined by

$$\begin{aligned} (v \vee w)(S) &= \max(v(S), w(S)), \\ (v \wedge w)(S) &= \min(v(S), w(S)). \end{aligned}$$

Moreover, $\phi(v) = \eta(v)$ for each simple game v , where $\eta(v) = (\eta_1(v), \eta_2(v), \dots, \eta_n(v))$ and $\eta_i(v)$ gives the number of coalitions for which player i is pivotal in game v (i.e., $\eta_i(v)$ is the raw Banzhaf index of player i in v). It is beyond the purpose of this paper to give a full explanation of these axioms (and even of A4), but we refer the reader to [6] for a careful, detailed discussion (including proofs of various equivalent conditions for Axiom A4, which show that this axiom can be presented in a variety of mathematical guises). However, we note here that Dubey and Shapley show that the probabilistic Banzhaf index satisfies Axiom A4, whereas the normalized one does not. As the normalized Banzhaf index lacks Axiom A4, they conclude: “This may be taken as an initial sign of trouble with the normalization [of the normalized Banzhaf index]” (see [6, Footnote 21]). They also note that the probabilistic Banzhaf index “is better behaved when analyzing convergence” [6, p. 116]. To cite another source, Bachrach and Rosenschein [4, p. 126] note that the normalized Banzhaf index “has certain undesirable qualities.”

Formally, the *normalized Banzhaf power index* is defined as

$$\text{Banzhaf}'(G, i) = \frac{\text{Banzhaf}^*(G, i)}{\sum_{j=1}^n \text{Banzhaf}^*(G, j)},$$

and the *probabilistic Banzhaf power index* is defined as

$$\text{Banzhaf}(G, i) = \frac{\text{Banzhaf}^*(G, i)}{2^{n-1}}.$$

The *raw Shapley-Shubik power index* [12]:

$$\text{Shapley-Shubik}^*(G, i) = \sum_{S \subseteq N - \{i\}} \|S\|! \cdot (n - \|S\| - 1)! \cdot d_G(S, i)$$

describes in how many coalitions a player is pivotal with respect to the order in which players enter coalitions. Normalizing the raw Shapley-Shubik power index by

$$\text{Shapley-Shubik}(G, i) = \frac{\text{Shapley-Shubik}^*(G, i)}{n!}$$

we obtain the *Shapley-Shubik power index*

We assume that the reader is familiar with the basic notions of complexity theory such as the complexity classes P, NP, #P (the class of functions that map instances of NP problems to the number of their solutions, see [14]), and PP (“probabilistic polynomial time,” see [9]) and with the notions of hardness and completeness (with respect to the polynomial-time many-one reducibility) for complexity classes. For more background and details we refer to, e.g., the textbooks [8, 10, 11].

Our reductions will make use of the following well-known NP-complete problem (see, e.g., [8]):

PARTITION	
<i>Given:</i>	A nonempty sequence $A = (a_1, a_2, \dots, a_k)$ of positive integers summing up to an even number.
<i>Question:</i>	Is there a subset $I' \subseteq I = \{1, 2, \dots, k\}$ such that $\sum_{i \in I'} a_i = \sum_{i \in I - I'} a_i$?

Let #PARTITION denote the function that maps any given instance A of PARTITION to the number of solutions of A , i.e.,

$$\#PARTITION(A) = \left\| \left\{ I' \mid \begin{array}{l} A = (a_1, a_2, \dots, a_k) \text{ and} \\ I' \subseteq I = \{1, 2, \dots, k\} \text{ and} \\ \sum_{i \in I'} a_i = \sum_{i \in I - I'} a_i \end{array} \right\} \right\|.$$

#PARTITION is the counting variant of PARTITION and is one member of the function class #P.

3 Beneficial Merging and Beneficial Splitting

Bachrach and Elkind [2] asked about the complexity of the problem of whether two players in a given weighted voting game can benefit from merging their weights by forming a single new player, i.e., of whether their joint power will increase in this new weighted voting game. For both the Shapley-Shubik power index and the normalized Banzhaf power index, Aziz and Paterson [1] provided an NP-hardness lower bound for the more general problem of whether any given *coalition* of players can benefit from merging (i.e., can increase their joint power by forming a new player whose weight is the sum of the coalition members’ weights). To define this problem formally for the probabilistic Banzhaf power index, we need the following notation. For a weighted voting game $G = (w_1, \dots, w_n; q)$ and a coalition $S \subseteq N = \{1, \dots, n\}$, define the new weighted voting game

$$G_{\&S} = \left(\sum_{i \in S} w_i, w_{j_1}, \dots, w_{j_{n-\|S\|}}; q \right),$$

where $\{j_1, \dots, j_{n-\|S\|}\} = N - S$.³ BENEFICIALMERGE_{Banzhaf} is the problem of whether a coalition can increase its power by merging their weights to form one new player. Formally, letting f be one of the power indices defined in Section 2, define the following problem:

BENEFICIALMERGE _f	
<i>Given:</i>	A weighted voting game $G = (w_1, \dots, w_n; q)$ and a coalition $S \subseteq \{1, \dots, n\}$.
<i>Question:</i>	Is it true that $f(G_{\&S}, 1) > \sum_{i \in S} f(G, i)$?

Aziz and Paterson [1] have proved NP-hardness of BENEFICIALMERGE_{Banzhaf}. Considering the probabilistic Banzhaf index, we now show that the complexity of BENEFICIALMERGE_{Banzhaf} depends on whether the input coalition S contains either two players or more than two players.

Theorem 1 BENEFICIALMERGE_{Banzhaf} is in P for instances (G, S) with $\|S\| = 2$, and is NP-hard for instances (G, S) with $\|S\| \geq 3$.

Proof. Given an instance (G, S) of BENEFICIALMERGE_{Banzhaf}, suppose $\|S\| = 2$. Without loss of generality (see Footnote 3), we

³ Note that the players’ order doesn’t matter when considering the normalized or probabilistic Banzhaf index.

may assume that $S = \{1, 2\}$. Thus $G_{\&S} = (w_1 + w_2, w_3, \dots, w_n; q)$. It is easy to see that

$$\text{Banzhaf}(G_{\&S}, 1) = \text{Banzhaf}(G, 1) + \text{Banzhaf}(G, 2),$$

so a coalition of two players can never benefit from merging.

To prove NP-hardness of $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ for $\|S\| \geq 3$, we give a reduction from the NP-complete problem PARTITION . From a given instance $A = (a_1, \dots, a_k)$ of PARTITION we construct in polynomial time the following $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ instance (G, S) :

$$\begin{aligned} G &= (w_1, \dots, w_n, 1, 1, 1, 1; q) \\ &= (8a_1, \dots, 8a_k, 1, 1, 1, 1; 1 + 4 \sum_{\ell=1}^k a_\ell); \\ S &= \{k+2, k+3, k+4\}. \end{aligned}$$

The last three players' raw Banzhaf indices each are equal to

$$\text{Banzhaf}^*(G, k+4) = \|\{S' \subseteq \{1, \dots, k+3\} \mid k+4 \text{ is pivotal for } S'\}\|. \quad (1)$$

The set of coalitions S' in (1) can be divided into four disjoint subsets: For any $\tilde{S} \subseteq \{1, \dots, k\}$ and any two distinct $a, b \in \{k+1, k+2, k+3\}$, distinguish between the set of coalitions S' with

- (i) $S' = \tilde{S}$,
- (ii) $S' = \tilde{S} \cup \{a\}$,
- (iii) $S' = \tilde{S} \cup \{a, b\}$, and
- (iv) $S' = \tilde{S} \cup \{k+1, k+2, k+3\}$.

By construction of G , player $k+4$ cannot be pivotal for any coalition S' in cases (ii), (iii), or (iv) but only for those in case (i): Player $k+4$ is pivotal for those $S' = \tilde{S} \subseteq \{1, \dots, k\}$ such that

$$1 + \sum_{j \in \tilde{S}} 8a_j \geq 1 + 4 \sum_{\ell=1}^k a_\ell \quad \text{and} \quad \sum_{j \in \tilde{S}} 8a_j < 1 + 4 \sum_{\ell=1}^k a_\ell,$$

which is true if and only if

$$\sum_{j \in \tilde{S}} 8a_j = 4 \sum_{\ell=1}^k a_\ell,$$

which in turn is true exactly if A is a yes instance of PARTITION . Thus, the number of coalitions for which player $k+4$ is pivotal equals the number of solutions of the PARTITION instance A . Denote this number by $X_A = \#\text{PARTITION}(A)$. Thus, the probabilistic Banzhaf index of player $k+4$ (and, analogously, of players $k+3$ and $k+2$) is

$$\text{Banzhaf}(G, k+4) = \text{Banzhaf}(G, k+3) = \text{Banzhaf}(G, k+2) = \frac{X_A}{2^{k+3}}.$$

Similarly, the probabilistic Banzhaf index of the first player in the new game

$$G_{\&S} = (3, 8a_1, \dots, 8a_k, 1; 1 + 4 \sum_{\ell=1}^k a_\ell)$$

can be calculated to be

$$\text{Banzhaf}(G_{\&S}, 1) = \frac{X_A}{2^{k+1}}.$$

Thus, we have

$$\text{Banzhaf}(G_{\&S}, 1) = \frac{X_A}{2^{k+1}} = \frac{4X_A}{2^{k+3}} > \frac{3X_A}{2^{k+3}} = \sum_{i=2}^4 \text{Banzhaf}(G, k+i)$$

if and only if $X_A > 0$, which in turn is true if and only if A is a yes instance of PARTITION . $\square$

Now, to formally define the splitting problem, let f be one of the power indices from Section 2, and define the following problem:

BENEFICIALSPLIT_f	
Given:	A weighted voting game $G = (w_1, \dots, w_n; q)$, a player $i \in \{1, \dots, n\}$, and an integer $m \geq 2$.
Question:	Can player i split into m players $i_1, \dots, i_m$, such that $\sum_{j=1}^m f(G', i_j) > f(G, i)$, where $G' = (w_1, \dots, w_{i-1}, w_{i_1}, \dots, w_{i_m}, w_{i+1}, \dots, w_n; q)$ with $\sum_{j=1}^m w_{i_j} = w_i$?

By similar methods as above we can prove the following theorem.

Theorem 2 $\text{BENEFICIALSPLIT}_{\text{Banzhaf}}$ is in P for instances $(G, i, 2)$, and is NP-hard for instances (G, i, m) with $m \geq 3$.

Regarding the lower bounds for $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ and $\text{BENEFICIALSPLIT}_{\text{Banzhaf}}$ stated in Theorems 1 and 2, a natural question arises: What about the upper bounds? Faliszewski and Hemaspaandra [7] established an upper bound for the beneficial merging problem originally proposed by Bachrach and Elkind [2] ("Can any two players increase their joint Shapley-Shubik index via merging?"): This problem is in the complexity class PP. Their proof technique can be used to show that $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ also belongs to PP.⁴ The proof is omitted. We conjecture that both $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ and $\text{BENEFICIALSPLIT}_{\text{Banzhaf}}$ are PP-complete.

Theorem 3 $\text{BENEFICIALMERGE}_{\text{Banzhaf}}$ is in PP.

4 Merging and Splitting in Unanimity Games

A weighted voting game $G = (w_1, \dots, w_n; q)$ is called a *unanimity game* if $q = \sum_{i=1}^n w_i$, i.e., if only the grand coalition can reach the threshold and win. Aziz and Paterson [1] show that for the normalized Banzhaf index in unanimity games, merging is always disadvantageous, whereas splitting is always advantageous. (Thus one can decide in polynomial time whether or not merging/splitting is beneficial.) In strong contrast, we show that in unanimity games with respect to the probabilistic Banzhaf index, splitting is always disadvantageous or neutral, whereas merging is neutral for size-two coalitions, yet advantageous for coalitions with at least three players.

Theorem 4 Let G be a unanimity game with player set N .

1. $(G, S) \notin \text{BENEFICIALMERGE}_{\text{Banzhaf}}$ for each $S \subseteq N$ with $\|S\| = 2$,
2. $(G, S) \in \text{BENEFICIALMERGE}_{\text{Banzhaf}}$ for each $S \subseteq N$ with $\|S\| \geq 3$.
3. $(G, i, m) \notin \text{BENEFICIALSPLIT}_{\text{Banzhaf}}$ for each $i \in N$ and $m \geq 2$.

Proof. The first statement follows immediately from Theorem 1.

To prove the second statement, note that in a unanimity game, any player i can be pivotal only for the coalition $S = N - \{i\}$, and i always is pivotal for this coalition. Thus the raw Banzhaf index of each i is

⁴ We note that the same proof cannot be transferred immediately to the beneficial merging problem with respect to the normalized Banzhaf index, since the normalization in the original game G differs from that in the game $G_{\&S}$.

always equal to one. It follows that $\text{Banzhaf}(G, i) = 1/2^{n-1}$ for each player i . If an arbitrary coalition S merges, the new game $G_{\&S} = (\sum_{i \in S} w_i, w_{j_1}, \dots, w_{j_{n-\|S\|}}; \sum_{i=1}^n w_i)$, where $\{j_1, \dots, j_{n-\|S\|}\} = N - S$, is also a unanimity game. Thus,

$$\text{Banzhaf}(G_{\&S}, i) = \frac{1}{2^{n-\|S\|}}.$$

Since $\|S\| \geq 3$, we have

$$\text{Banzhaf}(G_{\&S}, 1) - \sum_{i \in S} \text{Banzhaf}(G, i) = \frac{(2^{\|S\|-1} - \|S\|)}{2^{n-1}} > 0.$$

The third statement can be shown by similar arguments. In particular, for any possible split into players with integer weights, it holds that

$$\sum_{j=1}^m \text{Banzhaf}(G', i_j) - \text{Banzhaf}(G, i) = \frac{m - 2^{m-1}}{2^{n+m-2}} \leq 0.$$

This completes the proof. $\square$

5 Threshold Hypergraph Weighted Voting Games

In this section, we consider another representation of weighted voting games, namely, *threshold network flow games on hypergraphs*.

Bachrach and Rosenschein introduced and analyzed network flow games (on graphs) [3, 4]. A *threshold network flow game* $\mathcal{G}$ is defined as a weighted graph $G = (V, E, w : E \rightarrow \mathbb{N})$ with n agents⁵ that each control one edge in E , a source node $s \in V$ and a target node $t \in V$, and a threshold $k \in \mathbb{R}$ representing the data flow. The coalitional function is the success function $\text{succ}_{\mathcal{G}} : \mathfrak{P}(N) \rightarrow \{0, 1\}$, where a coalition of agents is *successful* if and only if a dataflow of size k can be sent from s to t over the edges represented by the agents in the coalition.

Power indices can be defined analogously as in the case of weighted voting games (see Section 2). Determining the raw Banzhaf index is #P-complete [3], while the Shapley-Shubik index is only known to be at least as hard as problems in NP [4].

If one wants to analyze the problems of merging and splitting in network flow games, the question arises of how these problems can be defined in that setting. Since agents control single edges, merging two or more agents would yield one new agent who has control over more than one edge and so would be different than the remaining agents. Similarly, splitting an agent into several subagents would mean to “split” the original agent’s edge, and it is unclear how that could be defined.

Our approach for solving this issue is to consider threshold network flow games on hypergraphs rather than on graphs.⁶ A hyperedge in a hypergraph is any subset of the vertex set (so a graph is the special case of a hypergraph with hyperedges of size two only). This makes it possible that, in a game defined on a hypergraph, a coalition of agents can be merged into a single new agent who controls the edge that corresponds to the union of vertices belonging to

the edges of the coalition’s original agents. Similarly, if we consider such a game on a hypergraph, it is possible for an agent to split into several subagents by partitioning this agent’s edge into subsets that each are controlled by one of the new subagents.

Definition 5 (Threshold hypergraph network flow games) A threshold hypergraph network flow game (a THNFG, for short) $\mathcal{G}$ consists of a weighted hypergraph $H = (V, E)$ with vertex set V and a set $E = \{e_1, \dots, e_n\}$ of n hyperedges, which represent the n agents, and a weight function $w : E \rightarrow \mathbb{N}$, where we write $w_i = w(e_i)$, a source node $s \in V$ and a target node $t \in V$, and a threshold $k \in \mathbb{R}$ representing the data flow.

The success function $\text{succ}_{\mathcal{G}} : \mathfrak{P}(N) \rightarrow \{0, 1\}$ is defined by

$$\text{succ}_{\mathcal{G}}(S) = \begin{cases} 1 & \text{if a dataflow of size } k \text{ from } s \text{ to } t \text{ is possible} \\ & \text{in } H|_{\{e_i \mid i \in S\}}, \\ 0 & \text{otherwise.} \end{cases}$$

where $H|_{\{e_i \mid i \in S\}}$ denotes the subhypergraph of H induced by the edge set $\{e_i \mid i \in S\}$.

Power indices in THNFGs can be defined analogously to those in weighted voting games.

Example 6 The THNFG with four agents in Figure 1 may help to visualize the definitions and theorems given below. It shows the hypergraph (in the standard bipartite graph representation for hypergraphs) related to the following game:

$$\begin{aligned} \mathcal{G} &= (H, s, t, w, k) \\ &= ((\{v_1, \dots, v_5\}, \{\{v_4, v_5\}, \{v_1, v_3, v_4\}, \{v_3, v_5\}, \{v_2\}\}), \\ &\quad v_1, v_5, (1, 2, 3, 4), 5). \end{aligned}$$

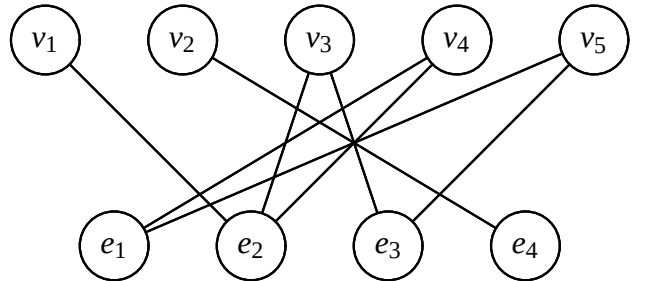


Figure 1. A THNFG for Example 6

Possible applications of THNFGs can be found in grid computing where many—often thousands of—computers collaborate to solve a common task by distributing various subtasks to certain clusters of computers (which represent the agents, respectively the hyperedges). Connecting a number of such computer clusters corresponds to forming a coalition of agents. Modeled as a game, a coalition (i.e., a set of clusters) is successful if it is connected and its total weight (i.e., the sum of the computing power of the computers in the clusters) exceeds a certain threshold so as to allow a sufficient network flow from the source to the target, and in this case these clusters can be assigned the desired subtask.

As another example of a possible application of THNFGs, we propose to use this model for smart power grids that deliver electricity from suppliers to consumers. Success of a coalition would mean

⁵ We use “agent” and “player” synonymously.

⁶ It might be tempting—and, indeed, an anonymous STAIRS reviewer suggested this approach—to allow an agent to control more than a single edge, and if one does so, merging and splitting for threshold network flow games could be formulated on graphs rather than on hypergraphs. However, Bachrach and Rosenschein’s model [3, 4] allows each agent to control only a single edge. We feel it is more natural to extend their model to hypergraphs—keeping their restriction—to be able to define merging and splitting in terms of threshold network flow games.

here that a certain threshold need be exceeded according to the consumers' current demands, thus allowing a sufficiently large power flow.

For functions, there are several notions of reducibility and consequently there are several notions of hardness and completeness for complexity classes of functions such as #P. We define below parsimonious reductions [13] and the more flexible many-one reductions [15] (see [7] for a more detailed discussion). Intuitively, parsimonious reductions preserve the number of solutions, whereas many-one reductions are the functional analogue of (polynomial-time) many-one reductions between sets.

Definition 7 Let f and g be two functions mapping from Σ^* to $\mathbb{N}$.

1. We say f parsimoniously reduces to g if there exists a polynomial-time computable function ϕ such that for each $x \in \Sigma^*$, $f(x) = g(\phi(x))$.
2. We say f many-one-reduces to g if there exist two polynomial-time computable functions, ϕ and ψ , such that for each $x \in \Sigma^*$, $f(x) = \psi(g(\phi(x)))$.
3. We say g is #P-parsimonious-hard if every function $f \in \#P$ parsimoniously reduces to g . If g is #P-parsimonious-hard and $g \in \#P$, then g is #P-parsimonious-complete. The notions of #P-many-one-hardness and #P-many-one-completeness are defined analogously.

Theorem 8 Computing the raw Banzhaf power index in THNFGs is #P-parsimonious-complete.

Proof. Given a THNFG

$$\mathcal{G} = ((\{v_1, \dots, v_m\}, \{e_1, \dots, e_n\}), s, t, (w_1, \dots, w_n), k)$$

and an agent i in $\mathcal{G}$, the raw Banzhaf power index can be computed in #P, since there exists a nondeterministic Turing machine M that, on input $(\mathcal{G}, i)$, nondeterministically guesses a coalition $S \subseteq N - \{i\}$ and verifies deterministically whether

$$\text{succ}_{\mathcal{G}}(S \cup i) - \text{succ}_{\mathcal{G}}(S) = 1.$$

For each coalition S guessed, this verification is successful exactly if i is pivotal for S . Thus, the number of accepting paths in $M(\mathcal{G}, i)$ is exactly the raw Banzhaf index $\text{Banzhaf}^*(\mathcal{G}, i)$. M is an NP machine, since the success of a coalition S can be determined by testing the following three properties in polynomial time each:

1. The hypergraph $H|_{\{e_i | i \in S\}}$ contains s and t .
2. There exists a path from s to t in $H|_{\{e_i | i \in S\}}$, i.e., there exists a subset $T \subseteq S$, such that $H|_{\{e_i | i \in S\}}$ contains s and t and is connected.
3. The sum of weights is not below the threshold, i.e.,

$$\sum_{i \in T} w_i \geq k$$

with T as in (2).

#P-parsimonious-hardness can be shown by a reduction from the raw Banzhaf power index in weighted voting games, which is #P-parsimonious-complete [7]. Given a weighted voting game $G = (w_1, \dots, w_n, q)$ and a player i in G , create the THNFG

$$\mathcal{G} = ((\{v_0, v_1, \dots, v_{n+1}\}, \{e_1, \dots, e_n\}), v_0, v_{n+1}, (w_1, \dots, w_n), q),$$

where $e_i = \{v_0, v_i, v_{n+1}\}$ for each i , $1 \leq i \leq n$, with the same weights and threshold, and consider the same player i in $\mathcal{G}$. Since for every

nonempty coalition S the corresponding subhypergraph $H|_{\{e_i | i \in S\}}$ always contains $s = v_0$ and $t = v_{n+1}$ and is connected, it obviously holds that

$$\text{succ}_G(S) = \text{succ}_{\mathcal{G}}(S).$$

Therefore, $\text{Banzhaf}^*(G, i) = \text{Banzhaf}^*(\mathcal{G}, i)$.

Consequently, computing the raw Banzhaf is #P-parsimonious-complete. $\square$

Similarly, the following results that are known for weighted voting games also hold for THNFGs.

Theorem 9 Computing the Shapley-Shubik power index in THNFGs is #P-many-one-complete.

Proof. That the problem is in #P follows from similar argumentats as those in Theorem 8. By means of the same reduction as in Theorem 8, a weighted voting game G and a player i are mapped to a THNFG $\mathcal{G}$ and the same player i . It holds that

$$\text{Shapley-Shubik}^*(G, i) = \text{Shapley-Shubik}^*(\mathcal{G}, i).$$

This reduction is parsimonious. Since it is #P-many-one-complete to compute the Shapley-Shubik index in weighted voting games [7], computing the Shapley-Shubik index in THNFGs is #P-many-one-hard, and thus #P-many-one-complete. $\square$

Faliszewski and Hemaspaandra [7] studied the power compare problem POWERCOMPARE_f in weighted voting games, where f is some power index. This problem can be defined analogously for THNFGs:

THNFG-POWERCOMPARE _f	
Given:	Two THNFGs $\mathcal{G} = (H, s, t, w, k)$, and $\mathcal{G}' = (H', s', t', w', k')$, and a player i occurring in both games.
Question:	Is it true that $f(\mathcal{G}, i) > f(\mathcal{G}', i)$?

Theorem 10 THNFG-POWERCOMPARE_f is PP-hard for $f \in \{\text{Shapley-Shubik}, \text{Banzhaf}\}$.

Proof. For both the probabilistic Banzhaf and the Shapley-Shubik index, PP-completeness of POWERCOMPARE_f in weighted voting games has been shown by Faliszewski and Hemaspaandra [7]. Using the reduction from the proof of Theorem 8, we obtain PP-hardness of THNFG-POWERCOMPARE_f. An instance (G, G', i) of POWERCOMPARE_f in weighted voting games is mapped to an instance $(\mathcal{G}, \mathcal{G}', i)$ of THNFG-POWERCOMPARE_f, where $\mathcal{G}$ and $\mathcal{G}'$ are constructed as in Theorem 8. Since $\text{succ}_G = \text{succ}_{\mathcal{G}}$ and $\text{succ}_{G'} = \text{succ}_{\mathcal{G}'}$, it follows that

$$f(G, i) > f(G', i) \iff f(\mathcal{G}, i) > f(\mathcal{G}', i).$$

This completes the proof. $\square$

Now, for a given power index f , the problems THNFG-BENEFICIALMERGE_f and THNFG-BENEFICIALSPLIT_f can be defined for THNFGs as follows:

THNFG-BENEFICIALMERGE _f	
Given:	A THNFG $\mathcal{G} = (H, s, t, w, k)$ and a coalition $S \subseteq \{1, \dots, n\}$.
Question:	Defining a new agent Merge(S) controlling the edge $e_{\&S} = \bigcup_{i \in S} e_i$ with $w_{\&S} = \sum_{i \in S} w_i$ and the corresponding new game $\mathcal{G}_{\&S} = (H_{\&S}, s, t, w_{\&S}, k)$ with $H_{\&S} = (V, (E - \{e_i \mid i \in S\}) \cup e_{\&S})$, does it hold that $f(\mathcal{G}_{\&S}, \text{Merge}(S)) > \sum_{i \in S} f(\mathcal{G}, i)$?

THNFG-BENEFICIALSPLIT _f	
Given:	A THNFG $\mathcal{G} = (H, s, t, w, k)$, an agent $i \in \{1, \dots, n\}$, and an integer $m \geq 2$.
Question:	Can agent i split into m agents $\{i_1, \dots, i_m\}$ such that $\sum_{j=1}^m f(\mathcal{G}', i_j) > f(\mathcal{G}, i)$, where $\mathcal{G}' = (H', s, t, w', k)$ with $H' = (V, (E - \{e_i\}) \cup \{e_{i_1}, \dots, e_{i_m}\})$, $\bigcup_{j=1}^m e_{i_j} = e_i$, $e_{i_j} \cap e_{i_k} = \emptyset$ for $j \neq k$, and w_{i_j} , $1 \leq j \leq m$, such that $\sum_{j=1}^m w_{i_j} = w_i$?

Analyzing the complexity of these two problems, in contrast to weighted voting games, merging and splitting might be advantageous for the probabilistic Banzhaf index, also for size-two coalitions, or, respectively, a split into two players.

As an example, consider the THNFG in Example 6. When merging the players of coalition $S = \{1, 2\}$, neither of the power indices increases in the new game

$$\mathcal{G}_{\&\{1,2\}} = ((\{v_1, \dots, v_5\}, \{\{v_1, v_3, v_4, v_5\}, \{v_3, v_5\}, \{v_2\}\}), v_1, v_5, (3, 3, 4), 5)$$

in comparison to the sum of power indices of the first two players in $\mathcal{G}$. However, merging the agents of coalition $S = \{1, 4\}$ is beneficial for all three power indices. The new game is

$$\mathcal{G}_{\&\{1,4\}} = ((\{v_1, v_2, \dots, v_5\}, \{\{v_4, v_5\}, \{v_1, v_3, v_4\}, \{v_3, v_5\}\}), v_1, v_5, (5, 2, 3), 5).$$

For comparison, letting

$$G = (1, 2, 3, 4; 5)$$

be the corresponding weighted voting game, and, respectively,

$$G_{\&\{1,4\}} = (5, 1, 3; 5),$$

merging the agents of $S = \{1, 4\}$ is neutral.

Analogously to weighted voting games, unanimity games can be defined on hypergraphs. In order to satisfy the property that only the grand coalition $S = N$ is successful, an adequate definition is the following.

Definition 11 A THNFG $\mathcal{G} = (H, s, t, w, k)$ is called a unanimity game if $k = \sum_{i=1}^n w_i$ and H is connected.

It is easy to see that Theorem 4 also holds for unanimity THNFGs.

6 Conclusions and Future Work

In this paper, we complemented the results of Aziz and Paterson [1] for the normalized Banzhaf power index to the probabilistic Banzhaf power index in weighted voting games. The latter was introduced by Dubey and Shapley [6], who argued that, in certain ways, it is more natural than the normalized Banzhaf power index originally defined by Banzhaf [5]. In particular, we showed that for the probabilistic Banzhaf power index the merging problem is in P for coalitions of size two, and is NP-hard for coalitions of size at least three, and we provided corresponding results for the splitting problem. The most interesting open question in this regard is to close the gap between NP-hardness (Theorem 1) and membership in PP (Theorem 3) for BENEFICIALMERGE_{Banzhaf}.

In addition, we studied these two problems for threshold network flow games on hypergraphs.

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