

On a Strengthening Connective for Flexible Database Querying

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Abstract—In most of query languages, conjunctive and disjunctive combinations of conditions remain the usual way for aggregation. Fuzzy query languages also offer trade-off operators, such as means in order to compensate between elementary conditions. In this paper, we investigate a new type of condition basically founded on the interaction between two predicates, thus enriching the panoply of tools the user is provided with and enhancing the power of query languages.

Index Terms—Databases, fuzzy queries, fuzzy connectives.

I. INTRODUCTION

Query languages based on fuzzy sets include a wide range of connectives, among which conjunction/disjunction and their weighted version, mean operators and quantified statements such as “most of $P_1, \dots, P_n$ ” where P_i is a fuzzy condition. So doing, preferences are introduced at a double level: i) atomic fuzzy predicates which allow for stating that some values are preferred to others, and ii) connectors which permit to specify that some terms are more important than others or can be compensated by others.

Here, we would like to focus on an operator that has a counterpart in natural language, for which an interpretation is proposed in the context of information selection/retrieval in a database querying process. More precisely, it connects two predicates P_1 (or X is A) and P_2 (or Y is B) and it writes as follows: “ P_1 all the more as P_2 ”. A typical example of such a condition is: “find second hand cars that are all the cheaper as their mileage is high”. The idea is to express a reinforcement of P_1 as P_2 is more and more satisfied. This obviously defines a new type of condition based on a sophisticated interaction between the two predicates involved, which are not interchangeable. One originality lies in the fact that the connector “all the more as” is by nature non commutative as are the connectors “and possibly” [1] or “then” [2] proposed earlier by other authors. Let us notice that “ P_1 all the more as P_2 ” does not make sense in the Boolean context where there is no graduality in the satisfaction of a Boolean predicate, whereas this notion is at the heart of the interpretation of this type of new condition.

We consider an interpretation according to which “ P_1 all the more as P_2 ” basically depends on P_1 and is not a matter of “joint” satisfaction of the predicates P_1 and P_2 as it is the case when an implication-based approach is taken. Indeed, if the gradual rule [3] “ $P_2 \Rightarrow_{R-i} P_1$ ” is used ($\Rightarrow_{R-i}$ denoting

an R-implication [4]), the truth value obtained may be high although P_1 is weakly satisfied (it suffices that the level of satisfaction of P_1 attains that of P_2 to get full satisfaction). We rather start with the idea that if the grade of satisfaction for P_1 is low, the grade of satisfaction of “ P_1 all the more as P_2 ” must also be low. Our approach to interpreting the condition “ P_1 all the more as P_2 ” consists in strengthening P_1 more and more as P_2 is more and more satisfied. We consider that “ P_1 all the more as P_2 ” reduces to P_1 when P_2 is false and becomes “very P_1 ” when P_2 is true.

We situate our proposal as an enrichment of the SQLf query language [5]. As a consequence, the new condition is associated with a single truth value taken in the unit interval, which makes it possible to compose it with usual predicates (atomic or not).

The rest of the paper is structured as follows. Some basic notions about fuzzy predicates and queries are first recalled in Section 2. Then, different ways for strengthening a predicate are presented in Section 3. Section 4 is devoted to a detailed study of the new type of predicate “ P_1 all the more as P_2 ”. In Section 5, an inclusion operator derived from *all the more as* is suggested as well as its use in some advanced SQLf queries. The conclusion summarizes the contributions of the paper and draws some lines for future works.

II. REMINDERS ON FUZZY PREDICATES AND QUERIES

Regular sets allow for the definition of Boolean predicates. In an analogous way, gradual predicates (or conditions) can be associated with fuzzy sets [6] aimed at describing classes of objects with vague boundaries. Often, elementary fuzzy predicates correspond to adjectives of the natural language, such as *young*, *tall*, *cheap* or *well-paid*. A fuzzy predicate P can be modeled as a function μ_P (usually of triangular or trapezoidal shape) from one (or several) domain(s) X to the unit interval $[0, 1]$. The degree $\mu_P(x)$ represents the extent to which element x satisfies the vague predicate P (or equivalently the extent to which x belongs to the fuzzy set of objects which match the fuzzy concept P). An elementary fuzzy predicate can also compare two attributes using a gradual comparison operator such as *more or less equal*.

It is possible to alter the meaning of a given predicate using a modifier which is generally associated with an adverb (e.g., *very*, *more or less*, *relatively*). For instance, *very cheap* is more

restrictive than *cheap* and *fairly high* is less demanding than *high*. The meaning of the predicate *mod P* (where *mod* is a modifier) may be defined in a compositional way and different approaches have been advocated, among which: $\mu_{mod P}(x) = (\mu_P(x))^n$ (see [7]).

Atomic and modified predicates can be involved in compound predicates which go far beyond those used in regular queries. Conjunction (resp. disjunction) is interpreted by means of a triangular norm $\top$ (resp. co-norm $\perp$), for instance the minimum (resp. the maximum). As to negation, it is interpreted as: $\forall x, \mu_{\neg P}(x) = 1 - \mu_P(x)$. Weighted conjunction and disjunction as well as weighted means or OWA can be used to assign a different importance to each of the predicates (see [8], [4] for more details).

The operations from the relational algebra can be straightforwardly extended to fuzzy relations by considering fuzzy relations as fuzzy sets on the one hand and by introducing fuzzy predicates in the appropriate operations on the other hand. Indeed, a fuzzy relation is designed as a fuzzy subset of a Cartesian product of domains. Thus, any such fuzzy relation r can be seen as made of weighted tuples, denoted by μ/t , where μ expresses the extent to which tuple t belongs to the relation, i.e., is compatible with the concept conveyed by r . As an illustration, we give the definition of the fuzzy selection hereafter, where “cond” denotes a fuzzy predicate built according to what was said above:

$$\mu_{sel(r, cond)}(t) = \top(\mu_r(t), \mu_{cond}(t)).$$

The language called SQLf described in [5] extends SQL so as to support fuzzy queries and it serves as a background in the rest of the paper.

III. STRENGTHENING OPERATORS

A. Introduction

As mentioned previously, the basis for interpreting the condition “ P_1 all the more as P_2 ” is to strengthen predicate P_1 depending on the satisfaction of predicate P_2 . In particular, when P_2 is true (degree equal to 1), the final satisfaction will be that of *very* P_1 . So, we will look at various ways to move from P to *very* P along with their compliance with the following three common sense properties:

- S1: $\forall(x, y), \mu_P(x) \geq \mu_P(y) \Rightarrow \mu_{very P}(x) \geq \mu_{very P}(y)$,
- S2: $core(very P) \subseteq core(P)$,
- S3: $\forall x, ((\mu_P(x) \in]0, 1[\Rightarrow \mu_{very P}(x) < \mu_P(x)) \wedge (\mu_P(x) \in \{0, 1\} \Rightarrow \mu_{very P}(x) \leq \mu_P(x)))$.

In other words, property S1 states that the strengthening operator *very* does not change the ordering of elements, property S2 tells that *very* P is at least as demanding as P for the elements that are completely in agreement with predicate P , and property S3 imposes that for any element which somewhat (except fully) satisfies predicate P , predicate *very* P is strictly less satisfied (there is a true strengthening).

B. Different strengthening approaches

1) *Use of a triangular norm*: A well known way for reinforcing a predicate P is to define:

$$\mu_{very P}(x) = \mu_{\top^n(P)}(x)$$

where $\top$ stands for a non idempotent triangular norm. For instance, with the product and Łukasiewicz’ norm, one gets respectively:

$$\mu_{very P}(x) = \mu_{P^n}(x) = (\mu_P(x))^n$$

$$\mu_{very P}(x) = \mu_{P^n}(x) = \max(n \cdot \mu_P(x) - n + 1, 0).$$

In such a framework, the interpretation of the statement “ X is A all the more as Y is B ” applying to an element t , denoted by $atma(A, B)(t)$, will be such that:

$$atma(A, B)(t) = f_1(A, B, n, t) = \mu_{A^n}(t) \text{ if } \mu_B(t) = 1$$

with n a given powering coefficient.

The satisfaction of properties S1-S3 by this strengthening method is now examined:

- S1: if $\mu_P(x) \geq \mu_P(y)$ then $\mu_{P^n}(x) \geq \mu_{P^n}(y)$ due to the monotonicity of norms with respect to both arguments, then $\mu_{very P}(x) \geq \mu_{very P}(y)$
- S2: if $\mu_P(x) = 1$ then $\mu_{P^n}(x) = \top(1, \dots, 1) = 1$ due to the fact that 1 is a neutral element of any norm, and we have $core(P) = core(very P)$,
- S3: when $\mu_P(x) = 0, \mu_{\top^n(P)}(x) = \mu_P(x) = 0$; when $\mu_P(x) = 1, \mu_{\top^n(P)}(x) = \mu_P(x) = 1$; when $\mu_P(x) \in]0, 1[$, the strict inferiority of $\mu_{\top^n(P)}(x)$ with respect to $\mu_P(x)$ is not necessarily satisfied by a triangular norm $\top$, except if $\top$ is Archimedean; in such a case, by definition it is ensured that $\top(y, y) < y$ for any $y \in]0, 1[$ (see [4]). Then, we have: $\top(y, \dots, y) = \top^n(y) < y$ and $\mu_{very P}(x) = \mu_{\top^n(P)}(x) < \mu_P(x)$ when $\mu_P \in]0, 1[$.

So, we can conclude that if an Archimedean norm ($\top$) is used (for instance the product), the strengthening mechanism defined as $\mu_{very P}(x) = \mu_{\top^n(P)}(x)$ complies with all three desired properties.

2) *Translation*: A second approach to reinforcing a predicate is to translate its membership function. If P is an increasing (resp. a decreasing) predicate, the translation is to the right (resp. left). In the more general case of a predicate P with a trapezoidal membership function, two translations take place, to the right for its increasing part and to the left for its decreasing part. For an increasing predicate P (situation considered later on), one has the following formula:

$$\mu_{very P}(x) = \mu_P(x - \tau)$$

where τ is a positive value (shift). Then, the interpretation of the statement “ X is A all the more as Y is B ” applying to an element t will be such that:

$$atma(A, B)(t) = f_2(A, B, \tau, t) = \mu_A(t - \tau) \text{ if } \mu_B(t) = 1$$

with τ a given magnitude of the translation.

Such a mechanism satisfies properties S1-S3 due to the monotonicity of P and the very nature of a translation.

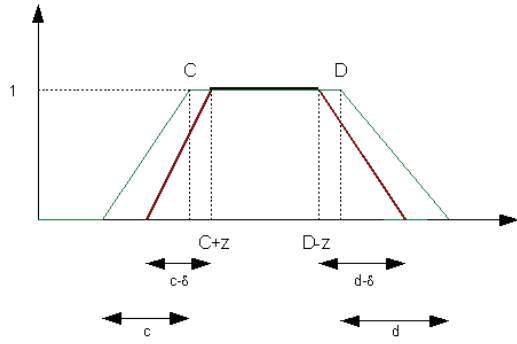


Fig. 1. Principle of the erosion operation

3) *Erosion*: The last strengthening technique considered here to build *very P* from *P* is called erosion [9] and it relies on the idea of a semantic proximity between the initial and the modified (reinforced) predicates. It makes use of a proximity relation E between two (numeric) values x and y which is reflexive and symmetric, and the quantity $\mu_E(x, y)$ can be viewed as a degree of approximate equality of x and y . This quantity can be defined as:

$$\mu_E(x, y) = \mu_Z(x - y)$$

which depends only on the difference between x and y . The parameter Z , called a tolerance indicator, is a fuzzy subset of the real line centered in 0 which has a trapezoidal membership function whose core is $[-z, z]$ and whose support is $[-z - \delta, z + \delta]$, denoted by $(-z, z, -\delta, \delta)$. So doing, the degree produced by $\mu_Z(x - y)$ evaluates the extent to which $(x - y)$ is close to 0 (according to Z). In other words, z is the maximal difference for which x and y are considered completely “approximately equal” and $z + \delta$ is the difference over which x and y are “not at all approximately equal”. Using this proximity relation E parameterized by Z , a fuzzy set (or a predicate) P may be eroded into *very P* as follows:

$$\mu_{veryP}(x) = \mu_{P \ominus Z}(x)$$

where $\ominus$ denotes Minkowski’s difference defined as [10]:

$$\mu_{F \ominus Z}(r) = \inf_s (\mu_Z(r - s) \rightarrow_{Gd} \mu_F(s)),$$

where $\rightarrow_{Gd}$ is the so-called Gödel implication ($p \rightarrow_{Gd} q = 1$ if $p \leq q$, q otherwise). Then, the interpretation of the statement “ X is A all the more as Y is B ” applying to an element t will be such that:

$$atma(A, B)(t) = f_3(A, B, Z, t) = \mu_{A \ominus Z}(t) \text{ if } \mu_B(t) = 1$$

with Z a given tolerance indicator for the erosion.

Fig. (1) illustrates the behavior of the erosion operator acting on a fuzzy predicate P whose trapezoidal membership function is (C, D, c, d) . Of course, one must have $z \leq \frac{D-C}{2}$, $\delta \leq c$ and $\delta \leq d$. If z (resp. δ) equals 0, the core (resp. support) of the initial predicate remains unchanged. Such a strengthening operator satisfies Properties S1-S3 due to the nature of the

erosion operator which corresponds to a difference in terms of membership functions (cf. Fig. (1)). Let us notice that erosion may modify the core and/or the support of the initial predicate, while translation does reduce both and the composition using any triangular norm leads to a reinforced predicate (*very P*) with the same core and support as the initial one (P).

IV. THE CONNECTOR “ALL THE MORE AS”

A. Axioms for this connector

We now review a bunch of axioms that can serve as a reasonable basis for defining the predicate “ X is A all the more as Y is B ”. The following properties will act as a mandatory requirement for any definition of the connector “all the more as”:

A1: decreasing monotonicity in the second argument, i.e.:

$$\begin{aligned} \mu_B(t.Y) > \mu_B(t'.Y) \wedge \mu_A(t.X) &= \mu_A(t'.X) \\ \Rightarrow atma(A, B)(t) &\leq atma(A, B)(t'), \end{aligned}$$

A2: increasing monotonicity in the first argument, i.e.:

$$\begin{aligned} \mu_A(t.X) > \mu_A(t'.X) \wedge \mu_B(t.Y) &= \mu_B(t'.Y) \\ \Rightarrow atma(A, B)(t) &\geq atma(A, B)(t'), \end{aligned}$$

A3: when X is not at all A , “ Y is B ” has no effect, i.e.:

$$\mu_A(t.X) = 0 \Rightarrow \forall \mu_B(t.Y), atma(A, B)(t) = 0,$$

A4: when Y is not at all B , the value of “ X is A all the more as Y is B ” is that of “ X is A ”, i.e.:

$$\begin{aligned} \mu_B(t.Y) = 0 \Rightarrow \\ \forall \mu_A(t.X), atma(A, B)(t) &= \mu_A(t.X), \end{aligned}$$

A5: when “ Y is B ” is somewhat satisfied, the value of the statement “ X is A all the more as Y is B ” must result in an effective strengthening of “ X is A ” (except for the extreme truth values 0 and 1), i.e.:

$$\begin{aligned} (\mu_A(t.X) \in]0, 1[\wedge \mu_B(t.Y) > 0) \\ \Rightarrow atma(A, B)(t) < \mu_A(t.X). \end{aligned}$$

B. A plausible definition

As mentioned before, the interpretation of the statement “ X is A all the more as Y is B ” will have two extreme points: $\mu_A(t.X)$ when $\mu_B(t.X)$ is false and $\mu_{veryA}(t.X)$ when $\mu_B(t.X)$ is completely satisfied. We suggest a linear transition between these two states, which has the advantage of being general for the three families of strengthening operators evoked before. This choice leads to the following definition of the connector “all the more as”:

$$\begin{aligned} atma(A, B)(t) = \\ (\mu_{veryA}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X). \end{aligned} \quad (1)$$

Let us remark that, in the particular case of the use of the norm product (strengthening of type 1), it could also be possible to have the following definition:

$$\begin{aligned} atma(A, B)(t) &= (\mu_A(t.X))^{((n-1) \cdot \mu_B(t.Y) + 1)} \\ &= (\mu_A(t.X))^{\mu_B(t.Y) + 1} \text{ for } n = 2. \end{aligned}$$

Example 1. Let us consider the relation *secondhandcars*(*#c*, *brand*, *name*, *year*, *price*, *mileage*, *horsepower*) with the tuples:

$t_1 = \langle 13, \text{Ford}, \text{Focus}, 2005, 6000, 40000, 75 \rangle$,
 $t_2 = \langle 264, \text{Renault}, \text{Clio}, 2002, 4000, 130000, 75 \rangle$,
 $t_3 = \langle 59, \text{Toyota}, \text{Prius}, 2006, 15000, 90000, 115 \rangle$,
 $t_4 = \langle 508, \text{Peugeot}, 307, 2004, 5500, 100000, 110 \rangle$,
 $t_5 = \langle 4, \text{Ford}, \text{Mondeo}, 2005, 6000, 125000, 110 \rangle$,
 $t_6 = \langle 78, \text{Renault}, \text{Megane}, 2006, 7000, 110000, 110 \rangle$,
 $t_7 = \langle 112, \text{Nissan}, \text{Primera}, 2008, 16000, 30000, 130 \rangle$

and the query looking for cars whose horsepower is less than 120 and whose price is all the cheaper as their mileage is high. The predicates “cheap” and “high” are assumed to be defined as follows:

$$\mu_{cheap}(p) = \begin{cases} 1 & \text{if } p \leq 5,000, \\ 0 & \text{if } p \geq 7,500, \\ \text{linear in-between,} \end{cases}$$

$$\mu_{high}(m) = \begin{cases} 0 & \text{if } m \leq 80,000, \\ 1 & \text{if } m \geq 120,000, \\ \text{linear in-between.} \end{cases}$$

Clearly, the seventh car does not qualify for the query since its horsepower exceeds the limit imposed by the user. With the first type of strengthening, the grades obtained by the first six cars are:

$0.6/t_1, 1/t_2, 0/t_3, 0.72/t_4, 0.36/t_5, 0.08/t_6$ with the product and $n = 2$,
 $0.6/t_1, 1/t_2, 0/t_3, 0.7/t_4, 0.2/t_5, 0.05/t_6$ with Łukasiewicz’ norm and $n = 2$,
 $0.6/t_1, 1/t_2, 0/t_3, 0.656/t_4, 0.216/t_5, 0.056/t_6$ with the product and $n = 3$,
 $0.6/t_1, 1/t_2, 0/t_3, 0.6/t_4, 0/t_5, 0.05/t_6$ with Łukasiewicz’ norm and $n = 3$.

If a translation (to the left since the predicate is decreasing) is used to define *very cheap*, the grades are:

$0.6/t_1, 1/t_2, 0/t_3, 0.6/t_4, 0.2/t_5, 0.05/t_6$ with $\tau = 100$,
 $0.6/t_1, 0.6/t_2, 0/t_3, 0.4/t_4, 0/t_5, 0.05/t_6$ with $\tau = 200$.

Last, with an erosion to reinforce *cheap*, the grades obtained are:

$0.6/t_1, 1/t_2, 0/t_3, 0.73/t_4, 0.33/t_5, 0.05/t_6$ with $Z = (0, 0, 0, 100)$,
 $3 \ 0.6/t_1, 1/t_2, 0/t_3, 0.68/t_4, 0.42/t_5, 0.05/t_6$ with $Z = (0, 100, 0, 0)$.

As can be seen, the results may significantly depend on the type of strengthening used as well as of the parameters chosen for a given type. $\diamond$

C. Compliance with the axioms

In this subsection, we show that, if the strengthening mechanism used to reinforce “X is A” complies with Properties

S1-S3 (see Section III) in expression (1), Axioms A1-A5 are satisfied.

A1: Let us take two tuples t and t' such that: i) $\mu_A(t.X) = \mu_A(t'.X)$ and ii) $\mu_B(t.Y) > \mu_B(t'.Y)$. We have:

$$\begin{aligned} &atma(A, B)(t) - atma(A, B)(t') \\ &= (\mu_{very A}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X) - \\ &\quad (\mu_{very A}(t'.X) - \mu_A(t'.X)) \cdot \mu_B(t'.Y) - \mu_A(t'.X) \\ &= (\mu_{very A}(t.X) - \mu_A(t.X)) \cdot (\mu_B(t.Y) - \mu_B(t'.Y)). \end{aligned}$$

The first term is negative or equal to 0 due to the validity of Property S3 and the second one is positive according to the hypotheses stated above. So, we have:

$$\begin{aligned} &atma(A, B)(t) - atma(A, B)(t') \leq 0, \text{ i.e.:} \\ &atma(A, B)(t) \leq atma(A, B)(t') \end{aligned}$$

and Property A1 holds.

A2: Let us take two tuples t and t' such that: i) $\mu_A(t.X) > \mu_A(t'.X)$ and ii) $\mu_B(t.Y) = \mu_B(t'.Y)$. We have:

$$\begin{aligned} &atma(A, B)(t) - atma(A, B)(t') \\ &= (\mu_{very A}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X) - \\ &\quad (\mu_{very A}(t'.X) - \mu_A(t'.X)) \cdot \mu_B(t'.Y) - \mu_A(t'.X) \\ &= (\mu_{very A}(t.X) - \mu_{very A}(t'.X)) \cdot \mu_B(t.Y) + \\ &\quad (\mu_A(t.X) - \mu_A(t'.X)) \cdot (1 - \mu_B(t.Y)). \end{aligned}$$

From Axiom S1, we have:

$$\begin{aligned} &\mu_A(t.X) > \mu_A(t'.X) \Rightarrow \\ &\mu_{very A}(t.X) \geq \mu_{very A}(t'.X) \end{aligned}$$

and the first term of the sum is positive or equal to 0. The first term of the product is positive according to the hypotheses made above and the second term is positive or equal to 0. It follows that the previous expression yields a result that is positive or equal to 0, so:

$$\begin{aligned} &atma(A, B)(t) - atma(A, B)(t') \geq 0, \text{ i.e.:} \\ &atma(A, B)(t) \geq atma(A, B)(t') \end{aligned}$$

and Property A2 holds.

A3: Let us consider a tuple t such that $\mu_A(t.X) = 0$. Due to the validity of S3, one has $\mu_{very A}(t.X) = 0$. Then:

$$\begin{aligned} &atma(A, B)(t) = \\ &(\mu_{very A}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t) = 0. \end{aligned}$$

A4: Let us now take a tuple t such that $\mu_B(t.Y) = 0$. We have:

$$\begin{aligned} &atma(A, B)(t) \\ &= (\mu_{very A}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t) \\ &= \mu_A(t). \end{aligned}$$

A5: Let us consider a tuple t such that: i) $\mu_A(t.X) \in]0, 1[$ and ii) $\mu_B(t.Y) > 0$. We have:

$$\begin{aligned} &atma(A, B)(t) = \\ &(\mu_{very A}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t) \end{aligned}$$

where $(\mu_{very A}(t.X) - \mu_A(t.X))$ is strictly negative, then $(\mu_{very A}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y)$ as well. Consequently $atma(A, B)(t) < \mu_A(t.X)$ and Axiom A5 holds.

Finally, it turns out that the strengthening approaches previously proposed (composition with an Archimedean norm, translation, erosion) are convenient for the interpretation of the statement “ X is A all the more as Y is B ”. When applied to a tuple t of a relation r , the preceding predicate is defined as follows:

- with an Archimedean norm $\top$:

$$atma(A, B)(t) = f_1(A, B, n, t)$$

$$= (\mu_{\top^n(A)}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X),$$
- with a translation of magnitude $\tau > 0$ to the right applying to an increasing initial predicate A :

$$atma(A, B)(t) = f_2(A, B, \tau, t)$$

$$= (\mu_A(t.X - \tau) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X),$$
- with the erosion operator associated with indicator $Z = (-z, z, -\delta, \delta)$:

$$atma(A, B)(t) = f_3(A, B, Z, t)$$

$$= (\mu_{A \ominus Z}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X).$$

D. The connector “all the less as”

By analogy with the operator “all the more as”, one may think of a predicate “ X is A all the less as Y is B ”. In such a case, one will not reinforce but weaken condition A (from “ A ” to “more or less A ”). The interpretation is based on the fact that one is all the less demanding on the fact that X is A as the condition “ Y is B ” is more and more satisfied, similarly to the interpretation of “ X is A all the more as Y is B ” where one is all the more demanding on the fact that X is A as the condition “ Y is B ” is more and more satisfied. The generic formulation of the degree of satisfaction of the condition “ X is A all the less as Y is B ” is then given by:

$$atla(A, B)(t) =$$

$$(\mu_{more\ or\ less\ A}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X).$$

By analogy with the strengthening case, this expression will have three instantiations, namely:

- with an Archimedean co-norm $\perp$:

$$atla(A, B)(t) = f'_1(A, B, n, t)$$

$$= (\mu_{\perp^n(A)}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X),$$
 - with a translation of magnitude $\tau > 0$ to the left applying to an increasing initial predicate A :

$$atla(A, B)(t) = f'_2(A, B, \tau, t)$$

$$= (\mu_A(t.X + \tau) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X),$$
 - with the dilation operator associated with indicator $Z = (-z, z, -\delta, \delta)$:

$$atla(A, B)(t) = f'_3(A, B, Z, t)$$

$$= (\mu_{A \oplus Z}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X),$$
- where $\oplus$ denotes the addition of fuzzy numbers.

Example 2. Let us consider the relation *secondhandcars* introduced in Example 1 with the following extension:

$t_8 = \langle 19, \text{Ford, Focus, 2005, 6000, 80000, 75} \rangle$,
 $t_9 = \langle 21, \text{Renault, Clio, 2002, 6000, 60000, 75} \rangle$,
 $t_{10} = \langle 132, \text{Citroen, C1, 2006, 4000, 50000, 60} \rangle$,
 $t_{11} = \langle 75, \text{Peugeot, 307, 2004, 7500, 50000, 90} \rangle$,
 $t_{12} = \langle 57, \text{Opel, Astra, 2008, 10000, 55000, 115} \rangle$

and the query looking for cars whose horsepower is more than 55 and whose price is all the less cheap as their mileage is low. The predicate *cheap* is defined as in Example 1, and *low* as follows:

$$\mu_{low}(m) = \begin{cases} 1 & \text{if } m \leq 50,000, \\ 1 & \text{if } m \geq 70,000, \\ \text{linear in-between.} \end{cases}$$

If we assume that the weakening is done using a translation (to the right in this case) whose magnitude is $\tau = 100$, we have to compute for each tuple t :

$$(\mu_{more\ or\ less\ cheap}(t.price - \tau) - \mu_{cheap}(t.price)) \cdot \mu_{low}(t.mileage) + \mu_{cheap}(t.price)$$

which leads to:

$$0.6/t_8 \text{ since } \mu_{low}(80000) = 0, \mu_{cheap}(6000) = 0.6,$$

$$\mu_{more\ or\ less\ cheap}(6000) = 1,$$

$$0.8/t_9 \text{ since } \mu_{low}(60000) = 0.5, \mu_{cheap}(6000) = 0.6,$$

$$\mu_{more\ or\ less\ cheap}(6000) = 1,$$

$$1/t_{10} \text{ since } \mu_{low}(50000) = 1, \mu_{cheap}(4000) = 1,$$

$$\mu_{more\ or\ less\ cheap}(4000) = 1,$$

$$0.4/t_{11} \text{ since } \mu_{low}(50000) = 1, \mu_{cheap}(7500) = 0,$$

$$\mu_{more\ or\ less\ cheap}(7500) = 0.4,$$

$$0/t_{12} \text{ since } \mu_{low}(55000) = 0.75, \mu_{cheap}(10000) = 0,$$

$$\mu_{more\ or\ less\ cheap}(10000) = 0.$$

By the way, it is worth noticing that the approach chosen for the interpretation of “price is cheap all the less as mileage is low” is not equivalent to “price is all the cheaper as mileage is not low”, which is not quite surprising. For tuple t_9 , this second interpretation would lead to the grade 0.4 $((0.2 - 0.6) * 0.5 + 0.6)$ instead of 0.8. $\diamond$

V. A DERIVED IMPLICATION AND ITS APPLICATION TO DATABASE QUERYING

A. Deriving an implication from “all the more as”

Let us come back to the expression of the predicate “ X is A all the more as Y is B ” and the reinforcement of predicate A using an Archimedean norm $\top$:

$$atma(A, B)(t) = f_1(A, B, n, t)$$

$$= (\mu_{\top^n(A)}(t.X) - \mu_A(t.X)) \cdot \mu_B(t.Y) + \mu_A(t.X).$$

One may observe that this formula rewrites:

$$(\top^n(q) - q) \cdot p + q$$

letting $p = \mu_B(t.Y)$, $q = \mu_A(t.X)$. This latter formula may be calculated for any two quantities p and q of the unit interval.

Let us now define:

$$I_{(\top, n)}(p, q) = \begin{cases} 1 & \text{if } p = 0, \\ (\top^n(q) - q) \cdot p + q & \text{otherwise,} \end{cases}$$

where $\top$ denotes a non idempotent norm. Operator $I_{(\top, n)}$ complies with the following four properties:

- $\forall q, p_1 < p_2 \Rightarrow I_{(\top, n)}(p_1, q) \geq I_{(\top, n)}(p_2, q)$
since the term $(\top^n(q) - q)$ is negative,
- $\forall p, q_1 < q_2 \Rightarrow I_{(\top, n)}(p, q_1) \leq I_{(\top, n)}(p, q_2)$
since $I_{(\top, n)}(p, q_2) - I_{(\top, n)}(p, q_1) = (q_2 - q_1) \cdot (1 - p) + (\top^n(q_2) - \top^n(q_1)) \cdot p$, where $(q_2 - q_1) > 0$ and $(\top^n(q_2) - \top^n(q_1)) \geq 0$ from Axiom S1,
- $\forall q, I_{(\top, n)}(0, q) = 1$ — by definition
- $\forall p, I_{(\top, n)}(p, 1) = 1$
since $I_{(\top, n)}(p, 1) = (\top^n(1) - 1) \cdot p + 1 = 1$.

So, this operator satisfies four of the most cited (see e.g., [4]) properties of fuzzy implications. The fact that $(1 \Rightarrow_f q = q)$ does not hold, but instead $(1 \Rightarrow_f q = \top^n(q))$ (for instance q^n with the product), may be considered acceptable in view of a “reinforcement-based implication”, which could be seen as a new type of fuzzy implication, in addition to R-implications and S-implications in particular (see [3]).

B. Application to inclusion queries

We now tackle queries calling on an inclusion. For instance, consider the fuzzy relation *Profile* whose schema is (*emp*, *skill*) describing the skills possessed by the employees of a company along with a given level (between 0 and 1). The query looking for the employees whose set of skills is included in that associated with John writes:

**select emp from Profile where emp $\neq$ “John”
group by emp having set(skill) included-in
(select skill from Profile where emp = “John”)**

in an SQL-like language (e.g., SQLf) equipped with a gradual operator (*included-in*) aimed at assessing the extent to which a fuzzy set is included in another. Usually, the interpretation of the inclusion operator is founded on an implication:

$$E \subseteq F \Leftrightarrow \forall x \in X, x \in E \Rightarrow x \in F$$

which, for the query above, should be extended with a fuzzy implication (denoted by $\rightarrow_f$) as:

$$\deg(E \subseteq F) = \min_{x \in X} \mu_E(x) \rightarrow_f \mu_F(x).$$

Different semantics of the inclusion are obtained depending on the family of fuzzy implication taken. An R-implication naturally extends the usual implication in the sense that if E is included in F according to Zadeh ($\forall x \in X, \mu_E(x) \leq \mu_F(x)$) the maximal degree (1) is returned, while an S-implication leads to a more demanding type of inclusion where full satisfaction is reached when the support of E is included in the core of F . Of course, the use of $I_{(\top, n)}$ gives birth to a third kind of semantics, more demanding than S-implications, for which 1 is obtained only when $q = 1$. Here, an element x of X must be in F all the more as it is in E according to

the behavior of the connective *all the more as* at the heart of implication $I_{(\top, n)}$.

Example 3. With the following extension of relation *Profile*:

1/⟨John, A⟩, 0.7/⟨John, B⟩, 0.3/⟨John, C⟩,
0.6/⟨Peter, A⟩, 1/⟨Peter, B⟩, 0.1/⟨Peter, D⟩,
0.1/⟨Mary, A⟩, 0.2/⟨Mary, B⟩, 0.4/⟨Mary, C⟩,
0.9/⟨Jebediah, A⟩, 1/⟨Jebediah, C⟩

the results of the previous query are:

- with Gödel implication ($p \rightarrow_{Gd} q = 1$ if $p \leq q$, q otherwise): {0.3/Mary, 0.3/Jebediah}
- with Kleene-Dienes implication ($p \rightarrow_{KD} q = \max(1 - p, q)$): {0.7/Peter, 0.6/Mary, 0.3/Jebediah} ,
- with $I_{(\top, n)}$ using $\top(x, y) = x \cdot y$ and $n = 2$:
{0.216/Mary, 0.09/Jebediah}. $\diamond$

VI. CONCLUSION

In this paper, we have proposed a new connective that gives birth to a new semantics for the aggregation of two predicates. The new predicate obtained writes “ X is A all the more as Y is B ” and the idea is to use a predicate B to reinforce predicate A into *very A*. The strengthening operates in a linear way, i.e., A applies when B is false and when B is true the result is $\mu_{very A}$. Strengthening A using a composition via an Archimedean norm, a translation or an erosion operation turns out to be appropriate to guarantee that reasonable properties hold for the new predicate “ X is A all the more as Y is B ”. Such a predicate can be used in selection operations, but it may also serve as a basis for defining an implication operator. So, we are provided with a means to define a new type of inclusion conveying the notion of reinforcement. This work opens several perspectives, especially that of implementing the new selection and inclusion operator in the SQLf language.

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