

On the Geometry of Join and Meet Calculations for General Type-2 Fuzzy Sets

Jerry M. Mendel, *Life Fellow, IEEE*
Signal and Image Processing Institute
University of Southern California
Los Angeles, CA, USA
mendel@sipi.usc.edu

Abstract—The union and intersection of general type-2 fuzzy sets (T2 FSs) are fundamental computations for such FSs. In the past, algorithms were developed for the union and intersection computations using vertical-slice and horizontal-slice representations of T2 FSs. The vertical-slice representation of a T2 FS traces its origins back to Zadeh [32] and requires computing the join or meet, whereas the horizontal-slice representation of a T2 FS is very recent, traces its origins to Liu [14], and requires computing the join and meet only for interval T2 FSs that are raised to level alpha, for which closed-form formulas are available [15]. In this paper, by studying the join and meet geometrically for general T2 FSs, we show that for many situations closed-form formulas exist for them. We also show that the formulas for computing the union and intersection of general T2 FSs that were derived from the horizontal-slice representation of a T2 FS [19] can also be obtained directly from geometrical interpretations of formulas that were derived by Karnik and Mendel [12] for the vertical-slice representation of a T2 FS.

Keywords—Type-2 fuzzy sets, union, intersection, join, meet

I. INTRODUCTION

The use of general type-2 fuzzy sets (T2 FSs) in T2 fuzzy systems has the potential to let such fuzzy systems outperform comparable T2 fuzzy systems that only use interval type-2 fuzzy sets (IT2 FSs). This is because a general T2 FS makes full use of its third dimension, whereas an interval T2 FS does not. Consequently, there is now renewed attention at all aspects of general T2 FSs and fuzzy systems, including the computations of the three basic set-theoretic operations of intersection, union and complement [1]-[32]. This paper focuses only on the intersection and union operations, because they can be difficult to compute for general T2 FSs, whereas the complement is straightforward to compute.

Consider two T2 FSs sets [1] $\tilde{A} (\mu_{\tilde{A}} : X \times [0,1] \rightarrow [0,1])$ and $\tilde{B} (\mu_{\tilde{B}} : X \times [0,1] \rightarrow [0,1])$. $\tilde{A}$ and $\tilde{B}$ are described by 3D membership functions $\mu_{\tilde{A}}(x, u)$ ($x \in X, u \in U = [0,1]$) and $\mu_{\tilde{B}}(x, w)$ ($x \in X, w \in W = [0,1]$), respectively. They can also be expressed in terms of their respective *secondary membership functions*, $\mu_{\tilde{A}}(u|x)$ and $\mu_{\tilde{B}}(w|x)$, and *secondary grades*, $f_x(u)$ and $g_x(w)$, as:

$$\tilde{A} = \int_x \mu_{\tilde{A}}(u|x)/x = \int_x [\int_{u=[0,1]} f_x(u)/u] / x \quad (1)$$

and¹

$$\tilde{B} = \int_x \mu_{\tilde{B}}(w|x)/x = \int_x [\int_{w=[0,1]} g_x(w)/w] / x \quad (2)$$

Note that $\mu_{\tilde{A}}(u|x)$ and $\mu_{\tilde{B}}(w|x)$ are T1 FSs, and are also called *vertical slices* of $\tilde{A}$ and $\tilde{B}$, respectively; hence (1) and (2) are called *vertical-slice representations* of $\tilde{A}$ and $\tilde{B}$.

It is well-known² [20] that the union of $\tilde{A}$ and $\tilde{B}$ is another T2 FS whose MF can be computed from:

$$\tilde{A} \cup \tilde{B} = \int_x \mu_{\tilde{A} \cup \tilde{B}}(v|x)/x \quad (3)$$

where

$$\begin{aligned} \mu_{\tilde{A} \cup \tilde{B}}(v|x) &= \int_{u=[0,1]} \int_{w=[0,1]} f_x(u) \star g_x(w) / (v = u \vee w) \\ &\equiv \mu_{\tilde{A}}(u|x) \sqcup \mu_{\tilde{B}}(w|x) \quad x \in X \end{aligned} \quad (4)$$

in which $\sqcup$ denotes the *join* operation. The use of the notation $\mu_{\tilde{A}}(u|x) \sqcup \mu_{\tilde{B}}(w|x)$ to indicate the join between the secondary MFs $\mu_{\tilde{A}}(u|x)$ and $\mu_{\tilde{B}}(w|x)$ is, of course, a *shorthand notation* for the operations in the middle of (4). What (4) says is that to perform the join between two secondary MFs, $\mu_{\tilde{A}}(u|x)$ and $\mu_{\tilde{B}}(w|x)$, $v = u \vee w$ must be computed between every possible pair of primary memberships u and w , such that $u \in U$ and $w \in W$, and that the secondary grade of $\mu_{\tilde{A} \cup \tilde{B}}(v|x)$ must be computed as the *t*-norm operation between the corresponding secondary grades of $\mu_{\tilde{A}}(u|x)$ and $\mu_{\tilde{B}}(w|x)$, $f_x(u)$ and $g_x(w)$, respectively. Note that at each value of x the join involves T1 FSs (i.e.,

¹ When a secondary MF, e.g. $\mu_{\tilde{A}}(u|x)$, has support that does not have gaps (the support of a T1 FS is the set of values of its primary variable for which that MF is > 0), then it has been customary in the past to denote that support as J_x^u , and to call it the *primary membership*. This notation, which has been called into question in [1], is not used herein because it is not needed. Note, also, that in the past the notation $\mu_{\tilde{A}}(x)$ has been used (e.g., [15]) as a shorthand notation for $\mu_{\tilde{A}}(u|x)$. We now prefer to use $\mu_{\tilde{A}}(u|x)$ because it shows the explicit dependence of $\mu_{\tilde{A}}$ on its functional argument u .

² Some of the material in this section is taken from [16] and [19].

secondary MFs), and that the join must be computed for $\forall x \in X$. If more than one combination of u and w gives the same point $u \vee w$, then in the join we keep the one with the largest membership grade. Usually, the maximum t -conorm is used, as suggested in [32] and [20].

In general, evaluating the join is difficult to do for arbitrary T2 FSs. Karnik and Mendel [12] have shown that for n convex and normal (secondary) T1 FSs, $F_1, \dots, F_n$, characterized by MFs $f_1(\theta|x), \dots, f_n(\theta|x)$, respectively, where $f_1(v_1|x) = \dots = f_n(v_n|x) = 1$, and the $f_i(\theta|x)$ are re-ordered so that $v_1 \leq v_2 \leq \dots \leq v_n$, the MF of $\sqcup_{i=1}^n F_i$, using maximum t -conorm ($\vee$) and either minimum or product t -norm (T), can be expressed as ($\forall x \in X$):

$$\mu_{\sqcup_{i=1}^n F_i}(\theta|x) = \begin{cases} T_{i=1}^n f_i(\theta|x) & \theta < v_1 \\ T_{i=k+1}^n f_i(\theta|x) & v_k \leq \theta \leq v_{k+1} \quad 1 \leq k \leq n-1 \\ \vee_{i=1}^n f_i(\theta|x) & \theta > v_n \end{cases} \quad (5)$$

Coupland and John [4]-[8] have taken a very new and novel approach to computing the join. Because their approach is also used for computing the meet, we describe it later in this section.

It is also well known [20] that the intersection of $\tilde{A}$ and $\tilde{B}$ is another T2 FS whose MF can be computed from:

$$\tilde{A} \cap \tilde{B} = \int_x \mu_{\tilde{A} \cap \tilde{B}}(v|x)/x \quad (6)$$

where

$$\begin{aligned} \mu_{\tilde{A} \cap \tilde{B}}(v|x) &= \int_{U=[0,1]} \int_{W=[0,1]} f_x(u) \star g_x(w) / (v = u \wedge w) \\ &\equiv \mu_{\tilde{A}}(u|x) \sqcap \mu_{\tilde{B}}(w|x) \quad x \in X \end{aligned} \quad (7)$$

in which $\sqcap$ denotes the *meet* operation. The use of the notation $\mu_{\tilde{A}}(u|x) \sqcap \mu_{\tilde{B}}(w|x)$ to indicate the meet between the secondary MFs $\mu_{\tilde{A}}(u|x)$ and $\mu_{\tilde{B}}(w|x)$ is another *shorthand notation*, but this time for the operations in the middle of (7). What (7) says is that to perform the meet between two secondary MFs, $\mu_{\tilde{A}}(u|x)$ and $\mu_{\tilde{B}}(w|x)$, $v = u \wedge w$ must be performed between every possible pair of primary memberships u and w , such that $u \in U$ and $w \in W$, and the secondary grade of $\mu_{\tilde{A} \cap \tilde{B}}(v|x)$ must be computed as the t -norm operation between the corresponding secondary grades of $\mu_{\tilde{A}}(u|x)$ and $\mu_{\tilde{B}}(w|x)$, $f_x(u)$ and $g_x(w)$, respectively. This must be done for $\forall x \in X$. If more than one combination of u and w gives the same point $u \wedge w$, then in the meet (just as in the join) we keep the one with the largest membership grade.

Note that in (7) there are two t -norms, $\star$ and $\wedge$. Although they are usually chosen to be the same, they do not have to be. See Starczewski [23] for very interesting discussions about this.

In general, evaluating the meet is also difficult to do for arbitrary T2 FSs, especially when the product t -norm is used in both places in (7). Karnik and Mendel [20] have also shown

that for n convex and normal T1 FSs, $F_1, \dots, F_n$, characterized by MFs $f_1(\theta|x), \dots, f_n(\theta|x)$, respectively, where $f_1(v_1|x) = \dots = f_n(v_n|x) = 1$, and the $f_i(\theta|x)$ are re-ordered so that $v_1 \leq v_2 \leq \dots \leq v_n$, the MF of $\sqcap_{i=1}^n F_i$ using maximum t -conorm and minimum t -norm can be expressed as ($\forall x \in X$):

$$\mu_{\sqcap_{i=1}^n F_i}(\theta|x) = \begin{cases} \vee_{i=1}^n f_i(\theta|x) & \theta < v_1 \\ \wedge_{i=k+1}^n f_i(\theta|x) & v_k \leq \theta \leq v_{k+1} \quad 1 \leq k \leq n-1 \\ \wedge_{i=1}^n f_i(\theta|x) & \theta > v_n \end{cases} \quad (8)$$

The usual derivations of (4) and (7) utilize Zadeh's extension principle. Mendel and John [18] show how (4) and (7) can easily be derived without having to use the Extension Principle.

Because (4) and (7) compute the join and meet of T2 FSs one vertical slice at a time, we shall refer to them as *vertical-slice computations*.

Coupland and John [7, 8] have shown how to compute the join and meet of $\tilde{A}$ and $\tilde{B}$ using methods from computational geometry (e.g., a modified Weiler-Atherton clipping algorithm, and a Bentley-Ottman plane sweep algorithm). Their approach is based on modeling a secondary MF geometrically as a [9] "set of connected straight line segments that need not be equally-spaced across the domain," and is limited so far to the minimum t -norm and the maximum t -conorm.

Even more recently, Greenfield and John [9] presented an optimized grid method for computing the join and meet, one that is much more computationally efficient than a direct approach to using (5) and (8).

A limitation of the geometric and grid methods is they do not obtain closed-form formulas for the join and meet operations. Such formulas can be very useful when a type-2 fuzzy logic system is designed by optimizing an objective function with respect to MF parameters, and explicit derivatives of that function have to be computed so that optimal values of those parameters can be found.

When secondary MFs are triangular (an interesting compromise between interval secondary MFs and general secondary MFs, and one that is also considered by Coupland and John [7]) then Starczewski [22, 24] has shown that "extended t -norms³ of triangular fuzzy truth values may be approximated by triangular fuzzy truth values as well." One of the most interesting aspects of Starczewski's [24] approach is it "... reduces calculations of extended t -norms (a similar approach can be rearranged for s -norms) to computing only the three characteristic functions: principal, upper and lower. Arbitrary traditional t -norms (or s -norms) can be used to calculate these functions. A tremendously useful feature of this approach is that the resultant MF preserves triangular shapes of the two arguments, and this way the approximate t -norms can be expanded to multi-argument form. Moreover, for

³ An extended t -norm is a t -norm obtained by applying the Extension Principle to a T1 t -norm. It was the basis for Mizumoto and Tanaka's [20] works.

each triangular fuzzy membership grade only three parameters have to be stored and processed by [a] FLS, instead of tabularized functions as in the general approach.” Another very useful feature of this approach is that formulas are given for the operations, so that explicit derivative formulas can be obtained if a triangular T2 FLS is designed (optimized) using a method that requires such derivatives (e.g., steepest descent).

Recently, Liu [14] presented a horizontal-slice⁴ α -plane representation for a general T2 FS. This was further elaborated upon in [19, 17]. Related works are in [27], [29] and [30]. Using this new representation, described next, it is possible to compute the union and intersection of general T2 FSs one α -plane at a time.

Definition 1. An α -plane for the general T2 FS $\tilde{A}$, denoted $\tilde{A}_\alpha$, is the union of all primary memberships of $\tilde{A}$ whose secondary grades are greater than or equal to α ($0 \leq \alpha \leq 1$), i.e. [14]

$$\begin{aligned} \tilde{A}_\alpha &= \{(x, u), \mu_{\tilde{A}}(x, u) \geq \alpha \mid \forall x \in X, \forall u \in U = [0, 1]\} \\ &= \int_{\forall x \in X} \int_{\forall u \in U = [0, 1]} \{(x, u) \mid f_x(u) \geq \alpha\} \quad \blacksquare \end{aligned} \quad (9)$$

Let⁵ $S_{\tilde{A}_\alpha}(x)$ denote an α -cut of the secondary MF $\mu_{\tilde{A}}(u \mid x)$, i.e.,

$$S_{\tilde{A}_\alpha}(x) = [\underline{\mu}_{\tilde{A}_\alpha}(x), \bar{\mu}_{\tilde{A}_\alpha}(x)] \quad (10)$$

where $\underline{\mu}_{\tilde{A}_\alpha}(x)$ is the lower MF of $\tilde{A}_\alpha$ at x and $\bar{\mu}_{\tilde{A}_\alpha}(x)$ is the upper MF of $\tilde{A}_\alpha$ at x , i.e.:

$$\underline{\mu}_{\tilde{A}_\alpha}(x) = \inf \{u \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) \geq \alpha\} \quad (11)$$

$$\bar{\mu}_{\tilde{A}_\alpha}(x) = \sup \{u \mid u \in [0, 1], \mu_{\tilde{A}}(x, u) \geq \alpha\} \quad (12)$$

Then, another way to express $\tilde{A}_\alpha$ is:

$$\tilde{A}_\alpha = \int_{\forall x \in X} S_{\tilde{A}_\alpha}(x) / x = \int_{\forall x \in X} \left(\int_{\forall u \in [\underline{\mu}_{\tilde{A}_\alpha}(x), \bar{\mu}_{\tilde{A}_\alpha}(x)]} u \right) / x \quad (13)$$

Definition 2. The α -plane Representation for $\tilde{A}$ is [14]:

$$\tilde{A} = \bigcup_{\alpha \in [0, 1]} \alpha / \tilde{A}_\alpha \quad \blacksquare \quad (14)$$

In (14), $\alpha / \tilde{A}_\alpha$ denotes $\tilde{A}_\alpha$ raised to level α , is called⁶ an α -level T2 FS, and is an IT2 FS whose secondary grades

equal α instead of 1 (as would be the case for the usual IT2 FS). This suggests, and is indeed the case, that operations involving T2 FSs can be performed using readily available operations for IT2 FSs. Note that $\alpha / \tilde{A}_\alpha$ is called a *zSlice* by Wagner and Hagrais [29, 30].

Let $(\tilde{A} \cup \tilde{B})_\alpha$ and $(\tilde{A} \cap \tilde{B})_\alpha$ be the α -planes of $\tilde{A} \cup \tilde{B}$ and $\tilde{A} \cap \tilde{B}$, respectively. For the minimum t-norm [i.e., the star in (4) and (7)], it is true that [14], [19]:

$$\tilde{A} \cup \tilde{B} = \bigcup_{\alpha \in [0, 1]} \alpha / (\tilde{A} \cup \tilde{B})_\alpha = \bigcup_{\alpha \in [0, 1]} \alpha / \tilde{A}_\alpha \cup \tilde{B}_\alpha \quad (15)$$

$$\tilde{A} \cap \tilde{B} = \bigcup_{\alpha \in [0, 1]} \alpha / (\tilde{A} \cap \tilde{B})_\alpha = \bigcup_{\alpha \in [0, 1]} \alpha / \tilde{A}_\alpha \cap \tilde{B}_\alpha \quad (16)$$

To-date, comparable results for other star t-norms, such as the product, are not available; however, any conjunction can be used in the meet, e.g., minimum or product.

To compute, e.g., $\tilde{A}_\alpha \cup \tilde{B}_\alpha$, for each value of α and $\forall x \in X$:

1. Compute $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$, the results of which are two planes that are bounded by their respective lower and upper MFs, $\underline{\mu}_{\tilde{A}_\alpha}(x)$, $\bar{\mu}_{\tilde{A}_\alpha}(x)$, $\underline{\mu}_{\tilde{B}_\alpha}(x)$ and $\bar{\mu}_{\tilde{B}_\alpha}(x)$
2. Compute $\tilde{A}_\alpha \cup \tilde{B}_\alpha$, as:

$$\tilde{A}_\alpha \cup \tilde{B}_\alpha = [\underline{\mu}_{\tilde{A}_\alpha \cup \tilde{B}_\alpha}(x), \bar{\mu}_{\tilde{A}_\alpha \cup \tilde{B}_\alpha}(x)] \quad (17)$$

where

$$\underline{\mu}_{\tilde{A}_\alpha \cup \tilde{B}_\alpha}(x) = \underline{\mu}_{\tilde{A}_\alpha}(x) \vee \underline{\mu}_{\tilde{B}_\alpha}(x) \quad (18)$$

$$\bar{\mu}_{\tilde{A}_\alpha \cup \tilde{B}_\alpha}(x) = \bar{\mu}_{\tilde{A}_\alpha}(x) \vee \bar{\mu}_{\tilde{B}_\alpha}(x) \quad (19)$$

To compute $\tilde{A}_\alpha \cap \tilde{B}_\alpha$, simply replace $\tilde{A}_\alpha \cup \tilde{B}_\alpha$ in (17) – (19) by $\tilde{A}_\alpha \cap \tilde{B}_\alpha$, and $\vee$ in (18) and (19) by $\wedge$.

Because (15) and (16) compute the union and intersection of T2 FSs one horizontal slice at a time (i.e., one α -plane at a time), we shall refer to them as either *horizontal-slice* or α -plane computations.

In this paper we connect results that can be obtained from using (5) and (15) for the union of two T2 FSs, and (8) and (16) for the intersection of two T2 FSs, i.e. we connect the vertical-slice and horizontal-slice computations. In order to do this, we first study the geometry of the join and meet when (5) and (15) are used. Our purpose for doing these things is to *provide more insight into computations of the join and meet* and to *provide a new derivation of (15) and (16) directly from (5) and (8)*.

⁴ Because a horizontal slice is analogous to an α -cut of a T1 FS, it is called the α -plane representation for a T2 FS.

⁵ [19] uses the notations $S_{\tilde{A}}(x \mid \alpha)$, $s_L(x \mid \alpha)$ and $s_R(x \mid \alpha)$ for $S_{\tilde{A}_\alpha}(x)$, $\underline{\mu}_{\tilde{A}_\alpha}(x)$ and $\bar{\mu}_{\tilde{A}_\alpha}(x)$, respectively. Our notation is more consistent with the notation that has been proposed in [1], which is why we use it.

⁶ In [19] $\alpha / \tilde{A}_\alpha$ is called the *footprint of uncertainty (FOU)* of the α -plane. As explained in [17], this is a misnomer, because an FOU lies on the 2D support of the MF of $\tilde{A}$, which corresponds to the $\tilde{A}_0$ -plane.

II. GEOMETRY OF THE JOIN AND MEET: VERTICAL-SLICE COMPUTATIONS

It is straightforward to re-express (5) and (8) recursively, so that their computations are done two type-2 fuzzy sets at a time; hence, in order to study the geometry of join and meet, we focus on (5) and (8) at *any* $x \in X$ for two T2 FSs, $\tilde{A}$ and $\tilde{B}$, as defined in (1) and (2), respectively. For two T2 FSs, letting $F_1 \equiv F$ and $F_2 \equiv G$, and using the minimum⁷ t-norm, minimum for the conjunction operation and maximum for the disjunction operation, (5) and (8) simplify⁸ to:

$$\mu_{F \sqcup G}(\theta) = \begin{cases} \min[f(\theta), g(\theta)] & \theta < v_0 \\ g(\theta) & v_0 \leq \theta \leq v_1 \\ \max[f(\theta), g(\theta)] & \theta > v_1 \end{cases} \quad (20)$$

$$\mu_{F \sqcap G}(\theta) = \begin{cases} \max[f(\theta), g(\theta)] & \theta < v_0 \\ f(\theta) & v_0 \leq \theta \leq v_1 \\ \min[f(\theta), g(\theta)] & \theta > v_1 \end{cases} \quad (21)$$

A. Geometry of the Join

To illustrate the geometry of the join, we chose triangle MFs for both $f(\theta)$ and $g(\theta)$, and then used (20) to compute $\mu_{F \sqcup G}(\theta)$. Note, however, that *our results are valid for all convex secondary MFs*.

It is easy to use (20) to obtain $\mu_{F \sqcup G}(\theta)$ for the (e.g.) nine different relative locations of $f(\theta)$ and $g(\theta)$ that are summarized in Table I. Additional cases can be obtained by interchanging $f(\theta)$ and $g(\theta)$, and this is left to the reader. Observe from these nine cases, that:

1. When $v_1 > v_0$, and $f(\theta)$ and $g(\theta)$ either do not intersect [cases (a) and (e)], or have exactly one intersection [cases (b) and (f)], then $\mu_{F \sqcup G}(\theta) = g(\theta)$, the right-most secondary MF.
2. When $v_1 = v_0 = 0$, and $g(\theta) \subseteq f(\theta)$ [case (d)], then $\mu_{F \sqcup G}(\theta) = f(\theta)$, the right-most secondary MF; if $f(\theta) \subseteq g(\theta)$ re-label $f(\theta)$ as $g(\theta)$ and vice-versa.
3. When $v_1 = v_0 \neq 0$, and $g(\theta) \subseteq f(\theta)$ [case (h)], then $\mu_{F \sqcup G}(\theta)$ involves both $f(\theta)$ and $g(\theta)$.
4. When $v_1 = v_0 \neq 0$, and the left leg of $f(\theta)$ is to the left of the left leg of $g(\theta)$, and the right leg of $f(\theta)$ is

to the left of the right leg of $g(\theta)$ [case (i)], then $\mu_{F \sqcup G}(\theta) = g(\theta)$, the right-most secondary MF.

5. When $v_1 > v_0$ and $f(\theta)$ and $g(\theta)$ intersect at exactly two points [cases (c) and (g)], then $\mu_{F \sqcup G}(\theta)$ involves both $f(\theta)$ and $g(\theta)$. This is also true if $f(\theta)$ and $g(\theta)$ intersect at three points (see Fig. 2(a)).

Situations 1, 2 and 4 are interesting because solutions for the join are completely given and require no further computation. While this is interesting from an educational point of view, it may not be possible to easily carry out the tests for these cases in order to use these results in a computer implementation of these cases.

Situations 3 and 5 are also interesting because they let us focus on the situations when a computer solution for $\mu_{F \sqcup G}(\theta)$ will be required. It is only these situations that will be examined in Section III.

B. Geometry of the Meet

Table II summarizes $\mu_{F \sqcap G}(\theta)$ (computed from (21)) for triangle MFs for both $f(\theta)$ and $g(\theta)$, and for the same nine different relative locations of these MFs as in Table I. Observe from these nine cases, that:

1. When $v_1 > v_0$, and $f(\theta)$ and $g(\theta)$ either do not intersect [cases (a) and (e)], or have exactly one intersection [cases (b) and (f)], then $\mu_{F \sqcap G}(\theta) = f(\theta)$, the left-most secondary MF.
2. When $v_1 = v_0 = 0$, and $g(\theta) \subseteq f(\theta)$ [case (d)], then $\mu_{F \sqcap G}(\theta) = g(\theta)$, the left-most secondary MF; if $f(\theta) \subseteq g(\theta)$ re-label $f(\theta)$ as $g(\theta)$ and vice-versa.
3. When $v_1 = v_0 \neq 0$, and $g(\theta) \subseteq f(\theta)$ [case (h)], then $\mu_{F \sqcap G}(\theta)$ involves both $f(\theta)$ and $g(\theta)$.
4. When $v_1 = v_0 \neq 0$, and the left leg of $f(\theta)$ is to the left of the left leg of $g(\theta)$, and the right leg of $f(\theta)$ is to the left of the right leg of $g(\theta)$ [case (i)], then $\mu_{F \sqcap G}(\theta) = f(\theta)$, the left-most secondary MF.
5. When $v_1 > v_0$ and $f(\theta)$ and $g(\theta)$ intersect at exactly two points [cases (c) and (g)], then $\mu_{F \sqcap G}(\theta)$ involves both $f(\theta)$ and $g(\theta)$. This is also true if $f(\theta)$ and $g(\theta)$ intersect at three points (see Fig. 2(b)).

As for the join, Situations 1, 2 and 4 are interesting because solutions for the meet are completely given and require no further computation; however, it may not be possible to easily carry out the tests for these cases in order to use these results in a computer implementation of these cases.

Situations 3 and 5 are also interesting because they let us focus on the situations when a computer solution for $\mu_{F \sqcap G}(\theta)$

⁷ It is straightforward to use the product t-norm in (5), something we leave to the readers [see also [26], in which (20) and (21) are re-derived; they are also extended to the product t-norm, but (21) is not extended to both the product t-norm and conjunction, a very difficult case that is considered in [12] but for which simple formula like (21) does not exist].

⁸ In the rest of this paper, in order to be consistent with the notation in [15, Appendix A], we let $v_1 \rightarrow v_0$ and $v_2 \rightarrow v_1$. Additionally, f_x and g_x are shortened to f and g , and $\theta|x$ is shortened to θ , because we are focusing on (5) and (8) at *any* $x \in X$.

TABLE I

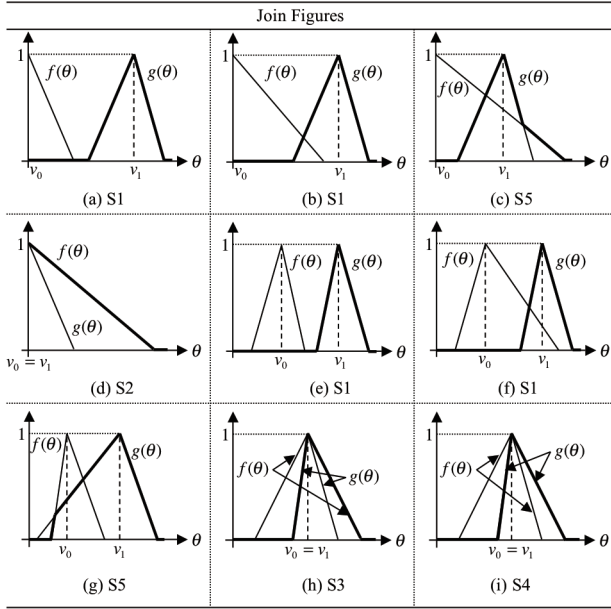
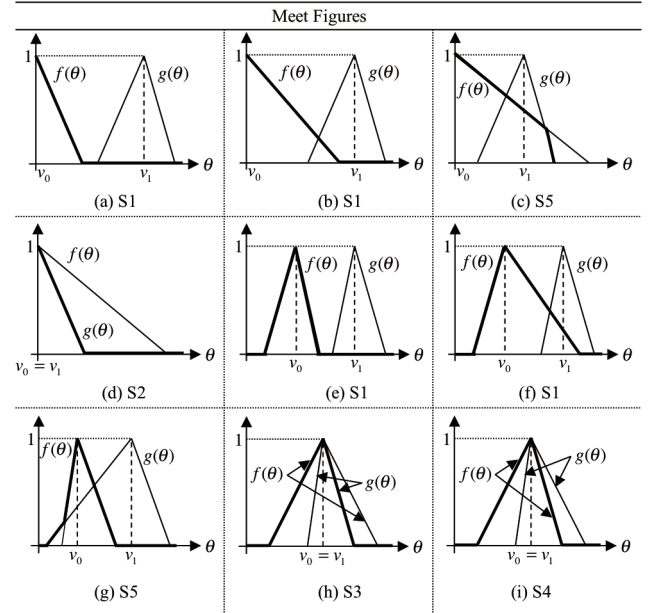
GEOMETRY OF THE JOIN: THE HEAVY LINES DENOTE $\mu_{F \cup G}(\theta)$ 

TABLE II

GEOMETRY OF THE MEET: THE HEAVY LINES DENOTE $\mu_{F \cap G}(\theta)$ 

will be required. It is only these situations that will be examined in Section III.

C. Simple Rules

Based on the results in Tables I and II and the rules extracted from their figures that are given in Sections A and B, it is possible to arrive at the following simple rules:

1. If $v_1 > v_0$ and $f(\theta)$ and $g(\theta)$ do not intersect or only have one intersection, or, if $v_1 = v_0 = 0$ and $f(\theta) \subseteq g(\theta)$ [if $f(\theta) \subseteq g(\theta)$ re-label $f(\theta)$ as $g(\theta)$ and vice-versa], or if $v_1 = v_0 \neq 0$, and the left leg of $f(\theta)$ is to the left of the left leg of $g(\theta)$, and the right leg of $f(\theta)$ is to the left of the right leg of $g(\theta)$, then:

$$\begin{cases} \mu_{F \cup G}(\theta) = g(\theta) \\ \mu_{F \cap G}(\theta) = f(\theta) \end{cases} \quad (22)$$

2. If $v_1 = v_0 \neq 0$, and $g(\theta) \subseteq f(\theta)$, then, although $\mu_{F \cup G}(\theta)$ and $\mu_{F \cap G}(\theta)$ involve both $f(\theta)$ and $g(\theta)$, no computer solutions are needed to compute them.
3. If $v_1 > v_0$ and $f(\theta)$ and $g(\theta)$ intersect at exactly two or more points, that $\mu_{F \cup G}(\theta)$ and $\mu_{F \cap G}(\theta)$ involve both $f(\theta)$ and $g(\theta)$, and require computer solutions.

Although these rules are obtained above for triangle secondary MFs, it is straightforward to show that they are applicable for all convex T1 FSs (Klir and Yuan (1995)).

III. CONNECTING VERTICAL-SLICE CALCULATIONS TO HORIZONTAL-SLICE (α -PLANE) COMPUTATIONS

In Section II.C, Rule 3 establishes when it is necessary to pursue the computations of $\mu_{F \cup G}(\theta)$ and $\mu_{F \cap G}(\theta)$ in more detail. In this section we study these detailed computations from different perspectives so as to connect the vertical-slice and horizontal slice computations. This is done geometrically, first for the join and then for the meet, both for the two T1 secondary MFs that intersect each other at three points, as depicted in Fig. 1. We leave it to the reader to verify that our conclusions are also applicable to two convex secondary T1 MFs that intersect at exactly two points, or for two convex secondary T1 MFs for which $v_1 = v_0 \neq 0$, and $g(\theta) \subseteq f(\theta)$ (Rule 2).

A. Join Computations

Using (20), the join of the two secondary MFs in Fig. 1 is depicted in Fig. 2a using heavy lines.

Next, we show how $\mu_{F \cup G}(\theta)$ can be obtained using the α -cuts of $f(\theta)$ and $g(\theta)$ raised to level α , denoted $f(\theta|\alpha)$ and $g(\theta|\alpha)$, respectively. From Fig. 3, for $\alpha_0 = 0$, observe the following:

$$\begin{cases} f(\theta|\alpha_0) = [a_0, b_0] \\ g(\theta|\alpha_0) = [c_0, d_0] \end{cases} \quad (23)$$

$$\mu_{F \cup G}(\theta|\alpha_0) = [a_0, b_0] = [\max(a_0, c_0), \max(b_0, d_0)] \quad (24)$$

For α_1 , observe that:

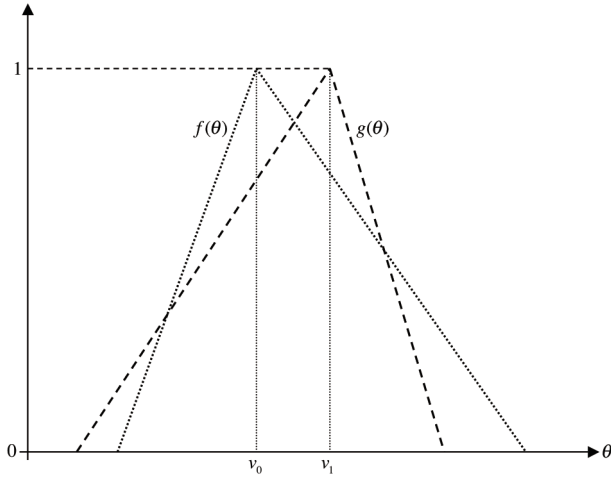


Figure 1. Two overlapping T1 secondary MFs.

$$\begin{cases} f(\theta | \alpha_1) = [a_1, b_1] \\ g(\theta | \alpha_1) = [c_1, d_1] \end{cases} \quad (25)$$

$$\mu_{F \sqcup G}(\theta | \alpha_1) = [a_1, b_1] = [\max(a_1, c_1), \max(b_1, d_1)] \quad (26)$$

Proceeding in a similar manner for the remaining α_i that are depicted on Fig. 3, it is straightforward to show that:

$$\begin{cases} f(\theta | \alpha_i) = [a_i, b_i] \\ g(\theta | \alpha_i) = [c_i, d_i] \end{cases}, \forall \alpha_i \quad (27)$$

$$\mu_{F \sqcup G}(\theta | \alpha_i) = [\max(a_i, c_i), \max(b_i, d_i)], \forall \alpha_i \quad (28)$$

The results in (27) and (28) are the same as the results that would be obtained from computing $\tilde{A}_\alpha \cup \tilde{B}_\alpha$, when (17)-(19) are used.

B. Meet Computations

Using (21), the meet of the two secondary MFs in Fig. 1 is depicted in Fig. 2b using heavy lines.

Next, we show how $\mu_{F \sqcap G}(\theta)$ can be obtained using $f(\theta | \alpha)$ and $g(\theta | \alpha)$. $f(\theta | \alpha_i)$ and $g(\theta | \alpha_i)$, for $i = 0$ and 1 are given in (23) and (25). From Fig. 4, for $\alpha_0 = 0$, observe that

$$\mu_{F \sqcap G}(\theta | \alpha_0) = [c_0, d_0] = [\min(a_0, c_0), \min(b_0, d_0)] \quad (29)$$

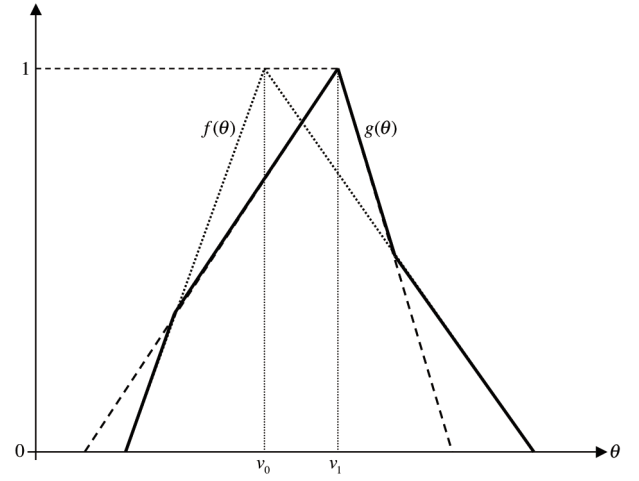
and, for α_1 , observe that:

$$\mu_{F \sqcap G}(\theta | \alpha_1) = [c_1, d_1] = [\min(a_1, c_1), \min(b_1, d_1)] \quad (30)$$

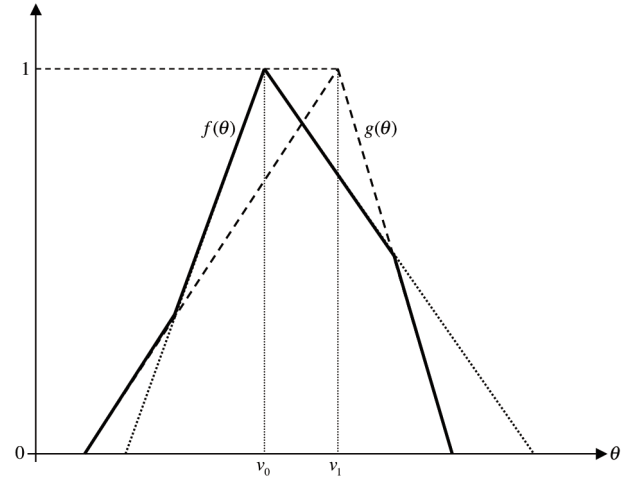
Proceeding in a similar manner for the remaining α_i that are depicted on Fig. 4, it is straightforward to show that:

$$\mu_{F \sqcap G}(\theta | \alpha_i) = [\min(a_i, c_i), \min(b_i, d_i)], \forall \alpha_i \quad (31)$$

The result in (31) is the same as the result that would be obtained from computing $\tilde{A}_\alpha \cap \tilde{B}_\alpha$, when the well-known



(a)



(b)

Figure 2. (a) Join and (b) meet of the two overlapping secondary MFs in Figure 1.

results for computing the intersection of IT2 FSs are used [described just below (19)].

C. Discussion

The results in (28) and (31) are, of course, not new. What is new is that they have been derived directly from the well-known vertical-slice computational results instead of from the α -plane representation. This provides an alternative way to obtain these results, which may be useful when teaching about general T2 FSs when time is limited.

IV. CONCLUSIONS

The union and intersection of general type-2 fuzzy sets (T2 FSs) are fundamental computations for such FSs. In the past, algorithms were developed for the union and intersection computations using vertical-slice and horizontal-slice representations of T2 FSs. The vertical-slice representation of a T2 FS traces its origins back to Zadeh [32] and requires computing the join or meet, whereas the horizontal-slice representation of a T2 FS is very recent, traces its origins to Liu [14], and requires computing the join and meet only for

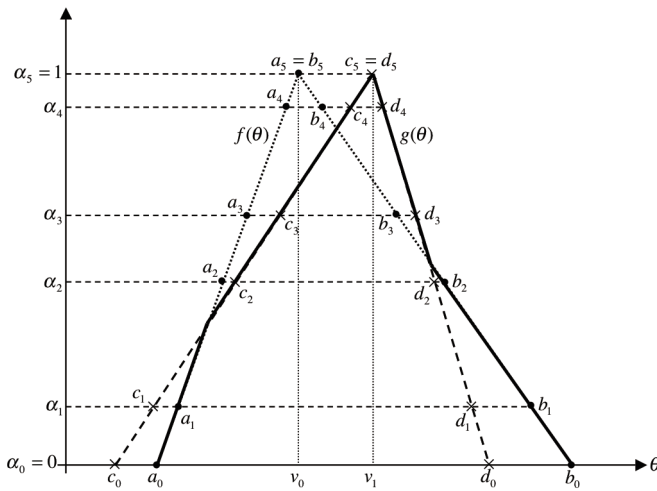


Figure 3. Join of the two overlapping secondary MFs in Fig. 1 and some α -cuts raised to level- α ⁹.

interval T2 FSs that are raised to level alpha, for which closed-form formulas are available [15]. In this paper, by studying the join and meet geometrically for general T2 FSs, we have shown that for many situations closed-form formulas exist for them. We have also shown that the formulas for computing the union and intersection of general T2 FSs that were derived from the horizontal-slice representation of a T2 FS [19] can also be obtained directly from geometrical interpretations of formulas that were derived by Karnik and Mendel [12] for the vertical-slice representation of a T2 FS.

How to turn the simple rules that are given in Section II.C into practical algorithms for computing the join and meet remains to be done. It was not done in this paper, because the purpose of this paper was to provide an alternative way to obtain these results, which may be useful when teaching about general T2 FSs when time is limited. We believe that this purpose has been accomplished.

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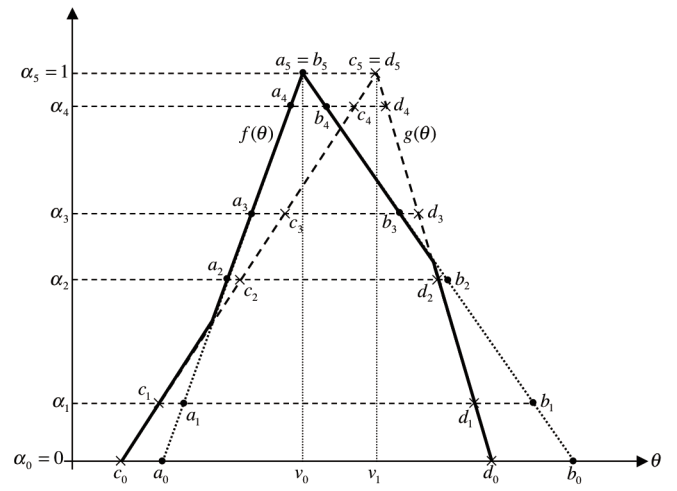


Figure 4. Meet of the two overlapping secondary MFs in Fig. 1 and some α -cuts raised to level- α . See footnote 9.

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⁹ Note that the heavy lines for the join (or for the meet in Figure 4) only exist as $\Delta\alpha \rightarrow 0$ and that for non-zero $\Delta\alpha$ they can be obtained by using piecewise linear approximations that connect the join (meet) from one α -level to the next.