

Optimal Static Output Feedback Control of Fuzzy-Model-Based Control Systems

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Abstract—By integrating the stabilizability condition, the orthogonal-functions approach (OFA), and the hybrid Taguchi-genetic algorithm (HTGA), an integrative method is presented in this paper to design the stable and quadratic-optimal static output feedback parallel-distributed-compensation (PDC) controller such that (i) the Takagi-Sugeno (TS) fuzzy-model-based control system can be stabilized, and (ii) a quadratic integral performance index for the TS-fuzzy-model-based control system can be minimized. In this paper, the stabilizability condition is proposed in terms of linear matrix inequalities (LMIs). A design example of stable and quadratic-optimal static output feedback PDC controller for a nonlinear inverted pendulum system controlled by a separately excited direct-current (DC) motor is given to demonstrate the applicability of the proposed new integrative approach.

Keywords—static output feedback PDC controller; Takagi-Sugeno fuzzy model; orthogonal-functions approach; hybrid Taguchi-genetic algorithm; linear matrix inequalities

I. INTRODUCTION

Recently, it has been shown that the fuzzy-model-based representation proposed by Takagi and Sugeno [1], known as the TS fuzzy model, is a successful approach for dealing with the nonlinear control systems, and there are many successful applications of the TS-fuzzy-model-based approach to the nonlinear control systems [2].

Despite the success of applying the TS-fuzzy-model-based approach to the nonlinear control systems, it has become evident that many research issues remain to be addressed. In fact, in many cases it is very difficult, if not impossible, to obtain a full order output feedback controller of a nonlinear control system. This is due to the inaccessible measurement or the expensive measure. Therefore, recently, some researches [3]–[5] have proposed the linear-matrix-inequality-based (LMI-based) approach to design the static output feedback parallel-distributed-compensation (PDC) controllers for the TS-fuzzy-model-based control systems. Therefore, summing up the above statements and reasons, we can see that it is worth while to present an efficiently numerical optimization approach accompanied with the stabilizability condition to design the stable and quadratic-optimal static output feedback

PDC controllers for the TS-fuzzy-model-based control systems, where the performance index subject to the constraint of stabilizability is considered to be directly minimized.

The purpose of this paper is to propose a numerical optimization method accompanied with the stabilizability condition to design the stable and quadratic-optimal static output feedback PDC controllers for the optimal control problem of the TS-fuzzy-model-based control systems by integrating the OFA, the hybrid Taguchi-genetic algorithm (HTGA) and the LMI technique, where the LMI technique is used to derive the stabilizability condition for ensuring that the closed-loop TS-fuzzy-model-based control systems can be stabilized. An illustrative example is also given in this paper to demonstrate the applicability of the proposed new integrative method.

II. PROBLEM STATEMENT

The TS-fuzzy-model-based control system for the nonlinear control system can be obtained as the following form:

$$\begin{aligned} \tilde{R}^i : & \text{IF } z_1(t) \text{ is } M_{i1} \text{ and } \dots \text{ and } z_g(t) \text{ is } M_{ig}, \\ \text{THEN } & \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t), \\ y(t) = C_i x(t), \end{cases} \end{aligned} \quad (1)$$

with the initial state vector $x(0)$, where $\tilde{R}^i$ ($i = 1, 2, \dots, N$) denotes the i -th implication, N is the number of fuzzy rules, $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ denotes the n -dimensional state vector, $y(t) = [y_1(t), y_2(t), \dots, y_r(t)]^T$ denotes the r -dimensional output vector, $u(t) = [u_1(t), u_2(t), \dots, u_p(t)]^T$ denotes the p -dimensional input vector, $z_i(t)$ ($i = 1, 2, \dots, g$) are the premise variables, A_i , B_i and C_i ($i = 1, 2, \dots, N$) are, respectively, the $n \times n$, $n \times p$ and $r \times n$ consequent constant matrices, and M_{ij} ($i = 1, 2, \dots, N$ and $j = 1, 2, \dots, g$) are the fuzzy sets.

The resulting TS-fuzzy-model-based dynamic system inferred from (1) is represented as

$$\dot{x}(t) = \sum_{i=1}^N h_i(z(t)) (A_i x(t) + B_i u(t)), \quad (2a)$$

$$y(t) = \sum_{i=1}^N h_i(z(t)) C_i x(t), \quad (2b)$$

in which $z(t) = [z_1(t), z_2(t), \dots, z_g(t)]^T$ denotes the g -dimensional premise vector, $h_i(z(t)) = w_i(z(t)) / \sum_{i=1}^N w_i(z(t))$, $w_i(z(t)) = \prod_{j=1}^g M_{ij}(z_j(t))$, and $M_{ij}(z_j(t))$ are the grades of membership of $z_j(t)$ in the fuzzy sets M_{ij} ($i=1, 2, \dots, N$ and $j=1, 2, \dots, g$). It can be seen that, for all t , $h_i(z(t)) \geq 0$ and $\sum_{i=1}^N h_i(z(t)) = 1$.

The given dynamic system can be stabilized by the following static output feedback PDC controller:

$$\begin{aligned} u(t) &= - \sum_{i=1}^N h_i(z(t)) F_i y(t) \\ &= - \sum_{i=1}^N \sum_{j=1}^N h_i(z(t)) h_j(z(t)) F_i C_j x(t), \end{aligned} \quad (3)$$

where F_i ($i=1, 2, \dots, N$) denote the $p \times r$ local static output feedback gain matrices.

By substituting (3) into (2a), we can get the closed-loop TS-fuzzy-model-based dynamic system as

$$\dot{x}(t) = \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N h_i(z(t)) h_j(z(t)) h_k(z(t)) (A_i - B_i F_j C_k) x(t). \quad (4)$$

In what follows, we present an LMI-based stabilizability criterion to analyze whether the closed-loop TS-fuzzy-model-based dynamic system in (4) can be stabilized by the static output feedback PDC controller or not, where the local static output feedback gain matrices F_j have been specified in advance.

Theorem:

The closed-loop TS-fuzzy-model-based dynamic system in (4) is stable, if, for the specified local static output feedback gains matrices F_j ($j=1, 2, \dots, N$) in (3), there exist a symmetric positive definite matrix P and the symmetric positive semidefinite matrices Q_k ($k=1, 2, \dots, N$) such that the following LMIs are simultaneously satisfied:

$$G_{kii}^T P + P G_{kii} + (\bar{s} - 1) Q_k < 0, \quad (5a)$$

and

$$\left(\frac{G_{kij} + G_{kji}}{2} \right)^T P + P \left(\frac{G_{kij} + G_{kji}}{2} \right) - Q_k \leq 0, \quad (5b)$$

where $G_{ijk} = A_i - B_i F_j C_k$, $i < j$, $\bar{s} > 1$, and $i, j, k = 1, 2, \dots, N$.

Proof: Here we omit it due to space limitation.

The problem considered in this paper is how to specify the local static output feedback gain matrices F_i ($i=1, 2, \dots, N$) of the static output feedback PDC controller in (3) such that the constraint of LMI-based stabilizability condition in (5) for the closed-loop TS-fuzzy-model-based dynamic system in (4) can

be satisfied, and such that the optimal control performance for the TS-fuzzy-model-based dynamic system in (2) can be achieved by minimizing the following quadratic integral performance index:

$$\begin{aligned} J &= \int_0^{qt_f} [y^T(t) Q y(t) + u^T(t) R u(t)] dt \\ &= \sum_{k=0}^{q-1} \int_{kt_f}^{(k+1)t_f} [y^T(t) Q y(t) + u^T(t) R u(t)] dt, \end{aligned} \quad (6)$$

where t_f denotes a small time interval which is chosen for the independent variable t , q is a positive integer specified by the designer, Q is a symmetric positive-semidefinite matrix, and R is a symmetric positive-definite matrix.

III. STABLE AND QUADRATIC-OPTIMAL STATIC OUTPUT FEEDBACK PDC CONTROLLERS DESIGN

Here, consider the time interval $kt_f \leq t \leq (k+1)t_f$, where t_f is chosen for the independent variable t , and let us define

$$t = kt_f + \eta, \quad (7)$$

and

$$x_k = x(kt_f), \quad (8)$$

in which $k=0, 1, 2, \dots, q-1$, and $0 \leq \eta \leq t_f$.

The state vector $x(t)$, within $kt_f \leq t \leq (k+1)t_f$, can be approximated by the truncated orthogonal-functions (OF) representation as

$$x(t) = \sum_{s=0}^{m-1} x_s^{(k)} T_s(t) = \tilde{x}^{(k)} T(t), \quad (9)$$

where m is the number of terms required for the OF, $T(t) = [T_0(t), T_1(t), \dots, T_{m-1}(t)]^T$ denotes the $m \times 1$ OF basis vector, $T_i(t)$ ($i=0, 1, \dots, m-1$) denote the OF, $x_s^{(k)}$ ($s=0, 1, \dots, m-1$) are the $n \times 1$ coefficient vector, and $\tilde{x}^{(k)} = [x_0^{(k)}, x_1^{(k)}, \dots, x_{m-1}^{(k)}]$ is the $n \times m$ coefficient matrix of $x(t)$.

Substituting (3) and the truncated OF representation of $x(t)$ in (9) into the quadratic integral performance index in (6), the quadratic integral performance index J becomes the following algebraic form:

$$\begin{aligned} J &= \sum_{k=0}^{q-1} \text{trace} \left[W (\tilde{x}^{(k)})^T \left(\sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N \sum_{o=1}^N h_i(z_k) h_j(z_k) h_l(z_k) h_o(z_k) \right. \right. \\ &\quad \left. \left. C_i^T (Q + F_j^T R F_j) C_o (\tilde{x}^{(k)}) \right) \right], \end{aligned} \quad (10)$$

where the constant matrix W is the product-integration-matrix of two OF basis vectors [6].

Since, before the consequent output can be inferred within the small time interval $kt_f \leq t \leq (k+1)t_f$, the degree of fulfillment of the antecedent must be computed in advance, so, as the works given by Ho and Chou [7], we can let the value of $h_i(z(t))$, within $kt_f \leq t \leq (k+1)t_f$, be $h_i(z(kt_f))$. Then, integrating (2a) from $t = kt_f$ to $t = t$ within $kt_f \leq t \leq (k+1)t_f$, we obtain

$$x(t) - x(k t_f) = \sum_{i=1}^N h_i(z_k) \left[A_i \int_{k t_f}^t x(t) dt + B_i \int_{k t_f}^t u(t) dt \right], \quad (11)$$

where $h_i(z_k) = h_i(z(k t_f))$ and $k = 0, 1, 2, \dots, q-1$.

Using the following integral property of the OF:

$$\int_{k t_f}^t T(t) dt = HT(t), \quad (12)$$

and applying (3), (8) and (9), equation (11) can be cast into the form

$$\begin{aligned} & \tilde{x}^{(k)} - [x_k, 0, 0, \dots, 0] \\ &= \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N h_i(z_k) h_j(z_k) h_l(z_k) (A_i - B_i F_j C_l) \tilde{x}^{(k)} H, \end{aligned} \quad (13)$$

in which H is the operational matrix of integration for the OF [7].

Equation (13) can be rewritten as

$$\tilde{x}^{(k)} - \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N h_i(z_k) h_j(z_k) h_l(z_k) (A_i - B_i F_j C_l) \tilde{x}^{(k)} H = \tilde{Q}^{(k)}, \quad (14)$$

where $\tilde{Q}^{(k)} = [x_k, 0, 0, \dots, 0]$ is an $n \times m$ matrix.

Making use of the Kronecker product, the explicit form for the coefficient matrix $\tilde{x}^{(k)}$ comes directly from (14) as

$$\hat{x}^{(k)} = \left[I_{mn} - \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N h_i(z_k) h_j(z_k) h_l(z_k) (H^T \otimes (A_i - B_i F_j C_l)) \right]^{-1} \hat{Q}^{(k)}, \quad (15)$$

where I_{mn} denotes the $mn \times mn$ identity matrix, $\hat{x}^{(k)} = [x_0^{(k)T}, x_1^{(k)T}, \dots, x_{m-1}^{(k)T}]^T$, $\hat{Q}^{(k)} = [x_k^T, 0^T, 0^T, \dots, 0^T]^T$, and $\otimes$ denotes the Kronecker product [8]. This implies that $\tilde{x}^{(k)}$ can be obtained from (15).

The value of the performance index of algebraic form in (10) is actually dependent on the set of local static output feedback gain matrices $\{F_1, F_2, \dots, F_N\}$, which means

$$J = G(f_{111}, f_{112}, \dots, f_{Npr}), \quad (16)$$

where f_{ijk} ($i=1, 2, \dots, N$, $j=1, 2, \dots, p$ and $k=1, 2, \dots, r$) denote the elements of the local static output feedback gain matrices F_i . This is equivalent to the static constrained-optimization problem in the following:

$$\text{minimize } J = G(f_{111}, f_{112}, \dots, f_{Npr}) \quad (17)$$

subject to $|f_{ijk}| \leq C_{ijk}$, and subject to the constraint that there exist a symmetric positive definite matrix P and the symmetric positive semidefinite matrices Q_k ($k=1, 2, \dots, N$) to make the LMIs in (5) hold, for $i=1, 2, \dots, N$, $j=1, 2, \dots, p$ and $k=1, 2, \dots, r$, where C_{ijk} are the positive real numbers given from the practical consideration, respectively. Then, the HTGA [9] can be employed to search for the optimal solution of the static constrained-optimization problem in (17).

IV. ILLUSTRATIVE EXAMPLE

A nonlinear inverted pendulum system controlled by a separately excited direct-current (DC) motor [10] is

considered in this example as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ K_1 \sin x_1(t) + K_2 x_3(t) \\ K_3 x_2(t) + K_4 x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K_5 \end{bmatrix} u(t), \quad (18)$$

where $x_1(t) \in [-\pi, \pi]$ is the angle of the pendulum, $x_2(t) = \dot{x}_1(t)$ is the angular velocity, $x_3(t)$ is the current of the motor, $u(t)$ denotes the control input voltage, $K_1 = 9.8$ ($1/\text{sec}^2$), $K_2 = 1$ ($1/\text{A} \cdot \text{sec}^2$), $K_3 = -10$ ($\text{V} \cdot \text{sec}/\text{rad} \cdot \text{mH}$), $K_4 = -10$ (Ω/mH), and $K_5 = 10$ ($1/\text{mH}$). The TS-fuzzy-model-based dynamic system for (18) can be obtained as following:

$$\tilde{R}^1 : \text{IF } z_1(t) \text{ is } M_{11}, \text{ THEN } \begin{cases} \dot{x}(t) = A_1 x(t) + B_1 u(t), \\ y(t) = C_1 x(t), \end{cases} \quad (19a)$$

$$\tilde{R}^2 : \text{IF } z_1(t) \text{ is } M_{21}, \text{ THEN } \begin{cases} \dot{x}(t) = A_2 x(t) + B_2 u(t), \\ y(t) = C_2 x(t), \end{cases} \quad (19b)$$

where

$$\begin{aligned} z_1(t) &= \sin x_1(t)/x_1(t), & x(t) &= [x_1(t), x_2(t), x_3(t)]^T, \\ x(0) &= [\pi/6, 0, 0]^T, & x_1(t) &\in [-\pi, \pi], & A_1(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix}, \end{aligned}$$

$$A_2(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix}, \quad B_1(t) = B_2(t) = [0, 0, 10]^T,$$

$$\begin{aligned} C_1 = C_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ denote the feedback of the angle and} \\ &\text{angular velocity of the pendulum, simultaneously,} \\ M_{11}(z_1(t)) &= \begin{cases} z_1(t), & x_1(t) \neq 0, \\ 1, & \text{otherwise,} \end{cases} \text{ and } M_{21}(z_1(t)) = 1 - M_{11}(z_1(t)). \end{aligned}$$

The quadratic integral performance index is

$$\begin{aligned} J &= \int_0^s [y^T(t) Q y(t) + u^T(t) R u(t)] dt \\ &= \sum_{k=0}^{q-1} \int_{k t_f}^{(k+1) t_f} [y^T(t) Q y(t) + u^T(t) R u(t)] dt, \end{aligned} \quad (20)$$

where $q = 500$, $t_f = 0.01$, $Q = \text{diag}\{1, 1\}$ and $R = 1$.

By using the proposed integrative approach and the LMI toolbox [11], we can obtain a symmetric positive definite matrix P and the symmetric positive semidefinite matrix Q_1 and Q_2 , respectively, as

$$\begin{aligned} P &= \begin{bmatrix} 167.7466 & 7.3397 & 0.7956 \\ 7.3397 & 6.4890 & 0.1333 \\ 0.7956 & 0.1333 & 0.0294 \end{bmatrix}, \\ Q_1 &= \begin{bmatrix} 6.6895 & 0.5061 & 0.5553 \\ 0.5061 & 12.5640 & 0.2238 \\ 0.5553 & 0.2238 & 0.0690 \end{bmatrix} \end{aligned}$$

and

$$Q_2 = \begin{bmatrix} 59.2127 & 18.7103 & 0.5746 \\ 18.7103 & 13.1713 & 0.0558 \\ 0.5746 & 0.0558 & 0.0463 \end{bmatrix}.$$

The responses of the angle of the pendulum $x_1(t)$, and the angular velocity $x_2(t)$ for a nonlinear inverted pendulum system controlled by a separately excited DC motor system with the designed stable and quadratic-optimal static output feedback PDC controller obtained by using the proposed integrative approach are, respectively, shown in Figures 1 and 2. It can be seen that the proposed approach can have the satisfactory control results.

V. CONCLUSIONS

Based on the OFA, an algorithm has been presented in this paper for solving the TS-fuzzy-model-based feedback dynamic equations. Then, the presented algebraic algorithm is integrated with the HTGA to design the stable and quadratic-optimal static output feedback PDC controllers of the TS-fuzzy-model-based control systems such that the control objective of directly minimizing a quadratic integral performance index subject to the constraint of LMI-based stabilizability condition can be achieved, where the quadratic integral performance index is also converted into the algebraic form by using the OFA. The illustrative example regarding the nonlinear inverted pendulum system controlled by a separately excited DC motor has shown that the proposed approach is effective for designing the stable and quadratic optimal static output feedback PDC controllers of the TS-fuzzy-model-based control systems.

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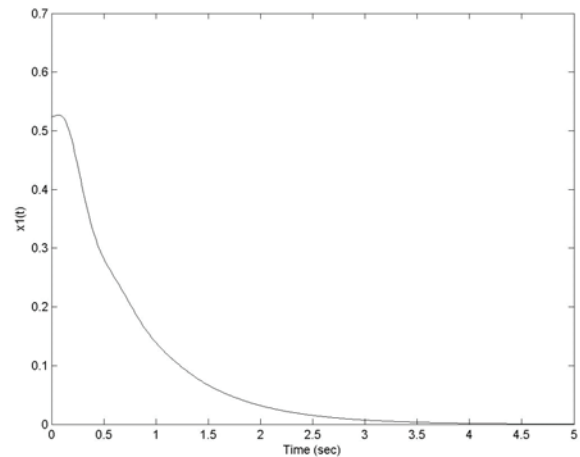


Figure 1. Response of the angle of the pendulum $x_1(t)$ for the nonlinear inverted pendulum system controlled by a separately excited DC motor.

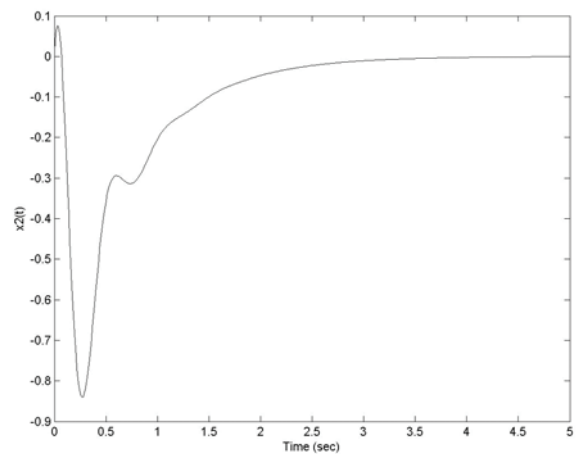


Figure 2. Response of the angular velocity $x_2(t)$ for the nonlinear inverted pendulum system controlled by a separately excited DC motor.