

Production Planning with Uncertain Demands

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Abstract—The paper deals with a single-item production planning problem with uncertain demands modeled by fuzzy intervals whose membership functions are possibility distributions for the values of the uncertain demands. Optimization criteria, in the setting of possibility theory, that lead to choose robust production plans under fuzzy demands are given. Algorithms for determining optimal robust production plans with respect to the proposed criteria are provided and some computational experiments are presented.

Index Terms—Production Planning, Uncertain Demand, Fuzzy Optimization, Possibility Theory.

I. INTRODUCTION

Nowadays, companies do not compete as independent entities but as a part of collaborative supply chains. The uncertainty on demands creates a risk in the supply chain as backordering, obsolete inventory due to the bullwhip effect [1]. To reduce this risk two different approaches exist: the collaboration between the customer and the supplier or the integration of the uncertainty into a planning process. In this paper, we cope with the approaches. Within these supply chains, collaborative processes are usually characterized by a set of point-to-point (customer/supplier) relationships with partial information sharing [2], [3]. More precisely, a procurement plan are built and propagated through the supply chain, moreover in stronger relationship between the customer and the supplier, the supplier can share her/him delivering capacity constraints to the customer. Another way to reduce a risk of the supply chain is to integrate the uncertainty in a planning process. Industries used the *Material requirements planning method* (MRP) [4] to plan a production. The use of *possibility theory* [5] to model the uncertainty in a production planning is due to the difficulties to access to available historical data allowing to compute a probability distribution. Different models (based on possibility theory) have been used to represent imperfections on demands to compute the gross requirements (in MRP context): [6] takes into account the imprecision on quantities with trapezoidal fuzzy intervals; [7] integrates the uncertainty on the order and the imprecision on quantities by a union of fuzzy intervals; and [8] adds to the previous model the imprecision on the date by a trapezoidal fuzzy interval. The gross requirements, in case of integration of imperfection, become fuzzy gross requirements (or a set of fuzzy gross requirements [8]) where the required quantities by period are fuzzy quantities. In the context of

collaborative supply chain the customer sends a procurement plan to the supplier but this procurement plan can not be fuzzy, also the problem is to compute from the fuzzy gross requirement a procurement plan. In this communication, we are interested in computing a robust procurement plan (with and without delivering capacity of the supplier sharing). The delivering capacity are composed of two bounds: the lower one being the minimal accepted quantity that is sent to the customer and the upper bound which is due to a production capacity of the supplier. Moreover the customer accepts to have backordering but it is more penalized that inventory. This problem is equivalent to the *problem of production planning* with backordering, more precisely to a certain version of the *lot sizing problem* (see, e.g., [9], [10]), where: the procured quantities are production quantities, a *production plan*; delivering constraints are production constraints, *capacity limits on production plans*; and the gross requirements are *demands*. Thus, the problem consists in finding a production plan that fullfils capacity limits and minimizes the total cost of storage and backordering subject to the conditions of satisfying each demand. It is efficiently solvable when the demands are precisely known (see, e.g., [11], [12], [13]). However, the demands are seldom precisely known in advance and the uncertainty must be taken into account.

In this paper, we wish to investigate the above problem with uncertain demands modeled by fuzzy intervals. The membership function of a fuzzy interval is a *possibility distribution* describing, for each value of the demand, the extent to which it is a possible value. In other words, it means that the value of this demand belongs to a λ -cut of the fuzzy interval with the degree of necessity (confidence) $1 - \lambda$. To evaluate a production plan, we assign to it, a *degree of necessity* that its cost does not exceed a given threshold and a degree of necessity that costs of the plan fall within a given fuzzy goal. In order to find “robust solutions” under fuzzy demands, we apply two criteria. The first one consists in choosing a production plan which maximizes the degree of necessity (certainty) that its cost does not exceed a given threshold. This criterion is similar in spirit to *threshold model* (see, e.g., [14]) in *stochastic optimization* in which we seek a solution maximizing the probability that its random value does not exceed a given threshold. The second criterion is weaker than the first one and consists in choosing

a plan with the maximum degree of necessity that costs of the plan fall within a given fuzzy goal. A similar criterion has been proposed in [15] for discrete optimization problems with fuzzy costs. We provide some methods for finding a robust production plan with respect to the proposed criteria under fuzzy-valued demands which heavily rely on methods for finding a robust production plan, called *optimal robust production plan*, in the problem of production planning under interval-valued demands with the *min-max* criterion. Namely, it turns out that the considered fuzzy problems can be reduced to examining a family of the interval problems with the min-max criterion.

II. THE DETERMINISTIC PROBLEM

We are given T periods. For period t , $t = 1, \dots, T$, let d_t be the demand in period t , $d_t \geq 0$, x_t the production amount in period t , $x_t \geq 0$, l_t, u_t the production capacity limits on x_t . Let $\mathbb{X} \subseteq \mathbb{R}_{\geq 0}^T$ be the set of feasible production amounts. Two cases are distinguished, the case *with no capacity limits* $\mathbb{X} = \{(x_1, \dots, x_T) : x_t \geq 0, t = 1, \dots, T\}$ and the one *with capacity limits* $\mathbb{X} = \{(x_1, \dots, x_T) : l_t \leq x_t \leq u_t, t = 1, \dots, T\}$. Set $\mathbf{D}_t = \sum_{i=1}^t d_i$ and $\mathbf{X}_t = \sum_{i=1}^t x_i$, $\mathbf{D}_t$ and $\mathbf{X}_t$ stand for the cumulative demand up to period t and the production level up to period t , respectively. Obviously, $\mathbf{X}_{t-1} \leq \mathbf{X}_t$ and $\mathbf{D}_{t-1} \leq \mathbf{D}_t$, $t = 2, \dots, T$.

We make the assumption: the costs of carrying one unit of inventory from period t to period $t+1$ for every $t = 1, \dots, T$ are equal, we denote it by c_I and the costs of backordering one unit from period $t+1$ to period t for every $t = 1, \dots, T$ are equal, we denote it by c_B . Furthermore, we assume that $c_I < c_B$. The nonnegative real function $L(u, v)$ represents either the cost of storing inventory from period t to period $t+1$ or the cost of backordering quantity from period $t+1$ to period t , namely $L(\mathbf{X}_t, \mathbf{D}_t) = c_I(\mathbf{X}_t - \mathbf{D}_t)$ if $\mathbf{X}_t \geq \mathbf{D}_t$; $c_B(\mathbf{D}_t - \mathbf{X}_t)$ otherwise. The Function has the form $L(\mathbf{X}_t, \mathbf{D}_t) = \max\{c_I(\mathbf{X}_t - \mathbf{D}_t), c_B(\mathbf{D}_t - \mathbf{X}_t)\}$.

The production planning problem consists in finding a feasible production plan $\mathbf{x} = (x_1, \dots, x_T)$, $\mathbf{x} \in \mathbb{X}$, that minimizes the total cost of storage and backordering subject to the conditions of satisfying each demand, namely

$$\min_{\mathbf{x} \in \mathbb{X}} F(\mathbf{x}) = \min_{\mathbf{x} \in \mathbb{X}} \sum_{t=1}^T L(\mathbf{X}_t, \mathbf{D}_t). \quad (1)$$

Without loss of generality, we can assume that an initial inventory I_0 and an initial backorder B_0 are equal to zero. Otherwise, one can append period 0 and assign $x_0 = I_0$ and $d_0 = 0$ with zero inventory cost if $I_0 > 0$ or assign $x_0 = 0$ and $d_0 = B_0$ with zero backorder cost if $B_0 > 0$.

In the case with no capacity limits problem (1) has a trivial optimal solution equal to $\mathbf{d}$. In the case with capacity limits, (1) can be formulated as the minimum cost flow problem (see, e.g., [16]):

$$\begin{aligned} \min \quad & \sum_{t=1}^T (c_I I_t + c_B B_t) \\ \text{s.t.} \quad & B_t - I_t = \sum_{j=1}^t (d_j - x_j), \quad t = 1, \dots, T, \\ & l_t \leq x_t \leq u_t, \quad t = 1, \dots, T, \\ & B_t, I_t \geq 0, \quad t = 1, \dots, T. \end{aligned} \quad (2)$$

Problem (2) can efficiently solved, for instance, by an algorithm presented in [13] that takes into account a special structure of the underlying network.

III. ROBUST PROBLEM

Assume that demands d_t , $t = 1, \dots, T$, in problem (1), are only known to belong to intervals $D_t = [d_t^-, d_t^+]$, $d_t^- \geq 0$. This means that we neither know the exact demands, nor can we set them precisely. We assume that the demands are unrelated to one another and there is no probability distribution in D_t , $t = 1, \dots, T$. A vector $S = (d_1, \dots, d_T)$, $d_t \in D_t$, that represents an assignment of demands d_t to periods t , $t = 1, \dots, T$, is called a *scenario*. Thus every scenario expresses a realization of the demands. We denote by Γ the set of all the scenarios, i.e. $\Gamma = [d_1^-, d_1^+] \times \dots \times [d_T^-, d_T^+]$. Among the scenarios of Γ *extreme scenarios* can be distinguished, that is the ones, which belong to $\{d_1^-, d_1^+\} \times \dots \times \{d_T^-, d_T^+\}$, the set of extreme scenarios is denoted by Γ_{ext} . We denote by S^+ (resp. S^-) the extreme scenario in which all the demands are set to their upper (resp. lower) bounds. The demand and the cumulative demand in period t under scenario S are denoted by $d_t(S) \in D_t$ and $\mathbf{D}_t(S)$, respectively, $\mathbf{D}_t(S) = \sum_{i=1}^t d_i(S)$. Clearly, for every $S \in \Gamma$ it holds $\mathbf{D}_{t-1}(S) \leq \mathbf{D}_t(S)$, $t = 2, \dots, T$, and $\mathbf{D}_t(S) \in [\mathbf{D}_t(S^-), \mathbf{D}_t(S^+)]$. The function $L(\mathbf{X}_t, \mathbf{D}_t(S)) = \max\{c_I(\mathbf{X}_t - \mathbf{D}_t(S)), c_B(\mathbf{D}_t(S) - \mathbf{X}_t)\}$, represents either the cost of storing inventory from period t to period $t+1$ or the cost of backordering quantity from period $t+1$ to period t under scenario S . Now $F(\mathbf{x}, S)$ denotes the total cost of a production plan $\mathbf{x} \in \mathbb{X}$ under scenario S , i.e. $F(\mathbf{x}, S) = \sum_{t=1}^T L(\mathbf{X}_t, \mathbf{D}_t(S))$.

In order to choose a robust production plan, one of robust criteria, called the *min-max* can be adopted. In the *min-max* version of problem (1), we seek a feasible production plan with the minimum the worst total cost over all scenarios, that is

$$\begin{aligned} \text{ROB : } \mathbf{x}^* &= \arg \min_{\mathbf{x} \in \mathbb{X}} A(\mathbf{x}) = \min_{\mathbf{x} \in \mathbb{X}} \max_{S \in \Gamma} F(\mathbf{x}, S) \\ &= \min_{\mathbf{x} \in \mathbb{X}} \max_{S \in \Gamma} \sum_{t=1}^T L(\mathbf{X}_t, \mathbf{D}_t(S)). \end{aligned}$$

In other words, we wish to find among all production plans the one that minimizes the maximum production plan cost over all scenarios, that minimizes $A(\mathbf{x})$, $A(\mathbf{x})$ is the *maximal cost of production plan* $\mathbf{x}$. An optimal solution $\mathbf{x}^*$ to the problem ROB is called *optimal robust production plan*. Let $\mathbf{x} \in \mathbb{X}$ be a given production plan. A scenario $S^w \in \Gamma$ that maximizes cost $F(\mathbf{x}, S)$ in problem ROB for $\mathbf{x}$ is called *the worst case scenario*.

Let us first study the problem of computing a worst case scenario for a given production plan $\mathbf{x}^* \in \mathbb{X}$, that is:

$$A(\mathbf{x}^*) = F(\mathbf{x}^*, S^w) = \max_{S \in \Gamma} F(\mathbf{x}^*, S). \quad (3)$$

Since $F(\mathbf{x}^*, S)$ is a continuous function on a bounded closed set Γ , it attains maximum for some $S \in \Gamma$. Furthermore, determining S can be restricted to the vertices of Γ , that is to the set of extreme scenarios Γ_{ext} , since $F(\mathbf{x}^*, S)$ is convex on Γ for any fixed production plan $\mathbf{x}^* \in \mathbb{X}$. The problem (3)

can be formulated as a mixed integer programming problem (MIP):

$$\begin{aligned}
\max \quad & \sum_{t=1}^T (c_I I_t + c_B B_t) \\
\text{s.t.} \quad & B_t - I_t = \sum_{j=1}^t (s_j - x_j^*), \quad t = 1, \dots, T, \\
& s_t \in [d_t^-, d_t^+], \quad t = 1, \dots, T, \\
& B_t, I_t \geq 0, \quad t = 1, \dots, T, \quad (4) \\
& I_t \leq (1 - \delta_t) \sum_{j=1}^t (x_j^* - d_j^-), \quad t = 1, \dots, T, \\
& B_t \leq \delta_t \sum_{j=1}^t (d_j^+ - x_j^*), \quad t = 1, \dots, T, \\
& \delta_t \in \{0, 1\}, \quad t = 1, \dots, T.
\end{aligned}$$

Let s_t^w, B_t^w, I_t^w and $\delta_t, t = 1, \dots, T$, be an optimal solution to problem (4). Then $S^w = (s_1^w, \dots, s_T^w)$ is a worst case scenario for $\mathbf{x}^*$ (a pessimistic realization of uncertain demands) and I_t^w is storing inventory amount from period t to period $t+1$ and B_t^w is backordering amount from period $t+1$ to period t under the worst case scenario S^w . The last two constraints model (4) and the binary variables δ_t ensure that storing inventory from period t to period $t+1$ and backordering from period $t+1$ to period t is not performed simultaneously (either $I_t > 0$ or $B_t > 0$). If $\delta_t = 1$ then backordering is performed $B_t > 0$; otherwise storing inventory is performed $I_t > 0$.

In order to assert *necessity* that the cost of a given plan $\mathbf{x}^* \in \mathbb{X}$ does not exceed a given threshold g under interval demands, i.e. evaluating whether $F(\mathbf{x}^*, S) \leq g$ for every scenario $S \in \Gamma$, we only need to determine worst case scenario S^w by solving (4) and evaluate $F(\mathbf{x}^*, S^w) \leq g$. Moreover, by means of worst case scenario S^w , for a given solution $\mathbf{x}^* \in \mathbb{X}$, one can determine the upper bound on its possible values of costs, namely $F(\mathbf{x}^*, S) \leq F(\mathbf{x}^*, S^w)$ for every scenario $S \in \Gamma$.

Let us consider the problem ROB with no capacity limits, i.e. the problem with the set $\mathbb{X} = \{(x_1, \dots, x_T) : x_t \geq 0, t = 1, \dots, T\}$. Note that function $F(\mathbf{x}, S)$ is continuous on $\mathbb{X}$ and Γ, Γ is a closed bounded set, and so $A(\mathbf{x})$ is well defined continuous function on $\mathbb{X}$ (see, e.g., [17, Theorem 1.4]). One can show that an optimal robust production plan $\hat{\mathbf{x}} = (\hat{x}_1, \dots, \hat{x}_T)$, i.e. $\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathbb{X}} A(\mathbf{x})$, can be computed by the following formulae:

$$\begin{aligned}
\hat{\mathbf{X}}_1 &:= \frac{c_B \mathbf{D}_1(S^+) + c_I \mathbf{D}_1(S^-)}{c_B + c_I}, \quad \hat{x}_1 := \hat{\mathbf{X}}_1 \\
\hat{\mathbf{X}}_t &:= \frac{c_B \mathbf{D}_t(S^+) + c_I \mathbf{D}_t(S^-)}{c_B + c_I}, \quad \hat{x}_t := \hat{\mathbf{X}}_t - \hat{\mathbf{X}}_{t-1}, \quad t \geq 2. \quad (5)
\end{aligned}$$

An algorithm for determining optimal robust production plan $\hat{\mathbf{x}}$ according to (5) can be implemented in $O(n)$ time. Note that if an initial backorder B_0 or an initial inventory I_0 are not equal to zero then one can modify the interval demand D_1 as follows: $D_1 := [d_1^- + B_0, d_1^+ + B_0]$ if $B_0 > 0$ or $D_1 := [d_1^- - I_0, d_1^+ - I_0]$ if $0 < I_0 \leq d_1^-$, and apply formulae (5) to determine an optimal robust production plan for problem ROB with the modified demand. If $I_0 > d_1^-$ then one appends period 0, as it has been described in Section II, and applies an algorithm (Algorithm 1 - presented later in this section) for the case with capacity limits, $l_t = 0, u_t = M, t = 1, \dots, T$, where M is a large number.

Let us turn to the problem ROB with capacity limits, i.e. the problem with the set $\mathbb{X} = \{(x_1, \dots, x_T) : l_t \leq x_t \leq u_t, t =$

$1, \dots, T\}$. Notice Γ is a bounded closed set. Function $F(\mathbf{x}, S)$ is continuous on $\mathbb{X}$ and Γ and hence $A(\mathbf{x})$ is continuous function on $\mathbb{X}$ (see, e.g., [17, Theorem 1.4]). From this and the fact $\mathbb{X}$ is a bounded closed set it follows that $A(\mathbf{x})$ attains its minimum on $\mathbb{X}$.

We now construct an iterative algorithm for solving problem ROB based on an iterative relaxation scheme for min-max problems proposed in [18]. Similar methods were developed for min-max regret linear programming problems with an interval objective function [19], [20]. Let us consider the problem (RX-ROB) being a relaxation of problem ROB that consists in replacing a given scenario set Γ with a discrete scenario set $\Gamma_{\text{dis}} = \{S^1, \dots, S^K\}$:

$$\begin{aligned}
\text{RX-ROB:} \quad & a^* = \min a \\
\text{s.t.} \quad & a \geq F(\mathbf{x}, S^k) \quad \forall S^k \in \Gamma_{\text{dis}}, \quad (6) \\
& \mathbf{x} \in \mathbb{X},
\end{aligned}$$

where $S^k = (s_t^k)_{t=1}^T$. The constraint $a \geq F(\mathbf{x}, S^k)$, called *scenario cut*, is associated with exactly one scenario $S^k \in \Gamma_{\text{dis}}$. Obviously, the maximal cost a^* of an optimal solution $\mathbf{x}^*$ of problem RX-ROB over discrete scenario set Γ_{dis} is a lower bound on the maximal cost of an optimal robust production plan $\mathbf{x}^r$ for problem ROB, i.e. $a^* \leq A(\mathbf{x}^r)$. Note that the scenario cut, $a \geq F(\mathbf{x}, S^k)$, associated with S^k is not a linear constraint. One can linearize the cut by replacing it in RX-ROB with the following $T+1$ constraints and $2T$ new decision variables:

$$\begin{aligned}
a &\geq \sum_{t=1}^T (c_I I_t^{S^k} + c_B B_t^{S^k}), \\
B_t^{S^k} - I_t^{S^k} &= \sum_{j=1}^t (s_j^k - x_j), \quad t = 1, \dots, T, \\
B_t^{S^k}, I_t^{S^k} &\geq 0, \quad t = 1, \dots, T.
\end{aligned}$$

Our algorithm (Algorithm 1) starts with zero lower bound on the maximal cost of an optimal robust production plan $\mathbf{x}^r$, $LB = 0$, a candidate $\mathbf{x}^* \in \mathbb{X}$ for an optimal solution for ROB and empty discrete scenario set, $\Gamma_{\text{dis}} = \emptyset$. At each iteration, a worst case scenario S^w for $\mathbf{x}^*$ is computed by applying the method (4) - this is the most time consuming part. Clearly, $A(\mathbf{x}^*) = F(\mathbf{x}^*, S^w)$ is an upper bound on $A(\mathbf{x}^r)$, $A(\mathbf{x}^r) \leq A(\mathbf{x}^*)$. If a termination criterion is fulfilled (usually $A(\mathbf{x}^*) \leq LB + \epsilon, \epsilon > 0$ is a given tolerance) then algorithm stops with production plan $\mathbf{x}^*$, which is an approximation of an optimal robust production plan. Otherwise the worst case scenario $S^w = (s_t^w)_{t=1}^T$ is added to Γ_{dis} , the corresponding to S^w scenario cut is appended to problem RX-ROB. Next the updated linear programming problem RX-ROB is solved to obtain a better candidate $\mathbf{x}^*$ for an optimal solution for ROB and new lower bound $LB = a^*$. Since set Γ_{dis} is updated during the course of the algorithm, the computed values lower bounds are nondecreasing sequence of their values. Then new iteration is started.

In order to choose a good initial production plan $\mathbf{x}^* \in \mathbb{X}$ in Algorithm 1, we suggest to solve the classical production planning problem (1) with capacity limits (the model (2)) under the midpoint demand scenario S^m , i.e. $d_t(S^m) = (d_t^- + d_t^+)/2, t = 1, \dots, T$ and take an optimal production plan under the midpoint scenario as an initial production plan.

Algorithm 1: Finding an optimal robust production plan.

Input: Interval demands $D_t = [d_t^-, d_t^+]$, $t = 1, \dots, T$, costs c_I , c_B , initial production plan $\mathbf{x}^* \in \mathbb{X}$, a convergence tolerance parameter $\epsilon > 0$.

Output: A production plan $\hat{\mathbf{x}}^*$, an approximation of an optimal robust production plan, and its worst case scenario S^w .

Step 0. $k := 0$, $LB := 0$, $\Gamma_{\text{dis}} := \emptyset$.

Step 1. $\mathbf{x}^k := \mathbf{x}^*$.

Step 2. Compute a worst case scenario S^w for $\mathbf{x}^k$ by applying the method (4).

Step 3. If $F(\mathbf{x}^k, S^w) \leq LB + \epsilon$ then output $\mathbf{x}^k, S^w$ and STOP.

Step 4. $k := k + 1$.

Step 5. $S^k := S^w$, $\Gamma_{\text{dis}} := \Gamma_{\text{dis}} \cup \{S^k\}$ and append scenario cut $a \geq F(\mathbf{x}, S^k)$ to problem RX-ROB.

Step 6. Compute an optimal solution $(\mathbf{x}^*, a^*)$ for RX-ROB, $LB := a^*$, and go to Step 1.

Theorem 1: Algorithm 1 terminates in a finite number of steps for any given $\epsilon > 0$.

Proof: The proof is almost the same as those given in [21, Theorem 2.5] and [18, Theorem 3]. ■

Note that if an initial inventory I_0 or an initial backorder B_0 are not equal to zero then one appends period 0, as it has been described in Section II, and applies Algorithm 1 for $T + 1$ periods.

In order to check the efficiency of Algorithm 1, we performed some computational tests. For every $T = 100, 200, \dots, 1000$, ten instances of the problem ROB with capacity limits were generated. In every instance, $c_I = 1$, $c_B = 10$, the demands and the production capacities were randomly generated intervals $[X, Y]$, where X is an integer-valued random variable uniformly distributed in $\{0, 1, \dots, 99\}$ and Y is an integer-valued random variable uniformly distributed in $\{100, 101, \dots, 199\}$. To solve the generated instances, we used IBM ILOG CPLEX 12.2 library and a computer equipped with Intel Core 2 Duo 2.5 GHz. In Table I, minimal, average and maximal computation times in seconds, required to find optimal robust production plans, are presented. All computations finished in a few iterations and about 98% of the total running time was spent on computing worst case scenarios by MIP model (4). As we can see from the obtained results, Algorithm 1 allows us to solve quite large problems having up to 1000 periods in reasonable time.

IV. FUZZY PROBLEM

In this section, we apply a more elaborate approach to model uncertain demands. Namely, the uncertain demands, in problem (1), are modeled by fuzzy intervals $\tilde{D}_t$, $t = 1, \dots, T$. A *fuzzy interval* $\tilde{A}$ is a fuzzy set in $\mathbb{R}$ whose membership function $\mu_{\tilde{A}}$ is normal, quasi concave and upper semicontinuous. Usually, it is assumed that the support of a fuzzy interval is bounded. The main property of a fuzzy interval is the fact

TABLE I
MINIMAL, AVERAGE AND MAXIMAL COMPUTATION TIMES IN SECONDS

T	100	200	300	400	500
MIN	0.22	0.44	1.23	3.91	5.73
AVG	0.42	2.29	7.88	13.28	41.96
MAX	0.90	4.76	25.04	37.38	96.62
T	600	700	800	900	1000
MIN	9.22	43.22	7.70	35.13	33.94
AVG	104.07	203.86	155.97	335.56	581.95
MAX	396.30	358.20	398.78	580.79	1152.54

that all its λ -cuts, that is the sets $\tilde{A}^{[\lambda]} = \{x : \mu_{\tilde{A}}(x) \geq \lambda\}$, $\lambda \in (0, 1]$, are closed intervals. We will assume that $\tilde{A}^{[0]}$ is the smallest closed set containing the support of $\tilde{A}$. So, every fuzzy interval $\tilde{A}$ can be represented as a family of closed intervals $\tilde{A}^{[\lambda]} = [a^{-[\lambda]}, a^{+[\lambda]}]$, parametrized by the value of $\lambda \in [0, 1]$. In many applications, the class of *trapezoidal fuzzy intervals* is used. A trapezoidal fuzzy interval, denoted by a quadruple $(a^-, a^+, \alpha, \beta)$, can be represented as a family of intervals $[a^- - \alpha(1 - \lambda), a^+ + \beta(1 - \lambda)]$ for $\lambda \in [0, 1]$. Notice that this representation contains classical intervals ($\alpha = \beta = 0$) and real numbers (additionally $a^- = a^+$) as special cases.

Here, a membership function of $\tilde{D}_t$ is regarded as a *possibility distribution* for the values of the unknown demand d_t (see book [5], that is entirely devoted to possibility theory). The *possibility degree* of the assignment $d_t = s$ is $\Pi(d_t = s) = \pi_{d_t}(s) = \mu_{\tilde{D}_t}(s)$. Let $S = (s_t)_{t=1}^T$ be a scenario that represents a state of the world where $d_t = s_t$, for $t = 1, \dots, T$. It is assumed that the demands are unrelated one to each other. Hence, the possibility distributions associated with the demands induce the following possibility distribution over all scenarios in $S \in \mathbb{R}^T$ (see [22]):

$$\begin{aligned} \pi(S) &= \Pi((d_1 = s_1) \wedge \dots \wedge (d_T = s_T)) \\ &= \min_{t=1, \dots, T} \Pi(d_t = s_t) = \min_{t=1, \dots, T} \mu_{\tilde{D}_t}(s_t). \end{aligned} \quad (7)$$

The value of $\pi(S)$ stands for the possibility of the event that scenario $S \in \mathbb{R}^T$ will occur. We have thus extended scenario set Γ given by the intervals (see Section III) to the fuzzy case and now $\tilde{\Gamma}$ is a fuzzy set of scenarios with membership function $\mu_{\tilde{\Gamma}}(S) = \pi(S)$, $S \in \mathbb{R}^T$. We see at once that the λ -cuts of $\tilde{\Gamma}$ for every $\lambda \in (0, 1]$ fulfill the following equality:

$$\tilde{\Gamma}^{[\lambda]} = \{S : \pi(S) \geq \lambda\} = [d_1^{-[\lambda]}, d_1^{+[\lambda]}] \times \dots \times [d_T^{-[\lambda]}, d_T^{+[\lambda]}],$$

which is from (7) and the definition of λ -cut. We also define $\tilde{\Gamma}^{[0]} = [d_1^{-[0]}, d_1^{+[0]}] \times \dots \times [d_T^{-[0]}, d_T^{+[0]}]$. Notice that $\tilde{\Gamma}^\lambda$, $\lambda \in [0, 1]$, is the classical scenario set containing all scenarios whose possibility of occurrence is not less than λ .

In order to choose a reasonable production plan under fuzziness, we first show how to evaluate a given production plan $\mathbf{x} \in \mathbb{X}$. Notice that a cost of a production plan is unknown quantity, denoted by $f_{\mathbf{x}}$, since demands are unknown and modeled by fuzzy intervals in the setting of possibility theory. Thus, the unknown cost $f_{\mathbf{x}}$ falls within fuzzy interval $\tilde{F}_{\mathbf{x}}$, called *fuzzy cost*, whose membership function $\mu_{\tilde{F}_{\mathbf{x}}}$ is a possibility distribution for the values of fuzzy variable $f_{\mathbf{x}}$, $\pi_{\tilde{F}_{\mathbf{x}}} = \mu_{\tilde{F}_{\mathbf{x}}}$,

defined as follows:

$$\mu_{\tilde{F}_x}(v) = \Pi(f_x = v) = \sup_{\{S: F(x, S) = v\}} \pi(S), \quad v \in \mathbb{R}. \quad (8)$$

Making use of (8), we can define the *degree of necessity* that a cost of a given plan $x \in \mathbb{X}$ does not exceed a given threshold g :

$$\begin{aligned} N(f_x \leq g) &= 1 - \Pi(f_x > g) = 1 - \sup_{v > g} \pi_{\tilde{F}_x}(v) \\ &= 1 - \sup_{\{S: F(x, S) > g\}} \pi(S). \end{aligned} \quad (9)$$

It is easily seen that $N(f_x \leq g) = 1 - \lambda$ means that for all scenarios S such that $\pi(S) > \lambda$, costs of plan x under these scenarios do not exceed g .

We now propose two criteria of choosing a robust solution in the fuzzy-valued problem (1). We have a threshold g , and we would like to find a production plan which maximizes the degree of certainty (necessity) that its cost does not exceed threshold g . Thus, we would like to solve the following problem:

$$\max_{x \in \mathbb{X}} N(f_x \leq g). \quad (10)$$

There are no doubts that an optimal production plan computed according to (10) is a robust one, since with the highest degree of certainty costs of the plan over scenarios will not exceed threshold g . Problem (10) is equivalent to the following mathematical programming problem:

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & f_x^{+[\lambda]} \leq g, \\ & \lambda \in [0, 1], \\ & x \in \mathbb{X}. \end{aligned} \quad (11)$$

If λ^* is the optimal objective and x^* is an optimal solution for problem (11) then $N(f_{x^*} \leq g) = 1 - \lambda^*$ and x^* is an optimal production plan for (10). If (11) is infeasible then $N(f_x \leq g) = 0$ for all $x \in \mathbb{X}$.

We now present a more general criterion of choosing a robust production plan than (10). Namely, suppose that a decision maker knows her/his preferences about a cost of a production plan f_x and expresses it by a *fuzzy goal* $\tilde{G}$, which is a fuzzy interval with a bounded support and a nonincreasing upper semicontinuous membership function $\mu_{\tilde{G}} : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$, such that $\mu_{\tilde{G}}(v) = 1$ for $v \in [0, g]$. The value of $\mu_{\tilde{G}}(f_x)$ is the extent to which cost f_x satisfies the decision maker. Now the requirement “ $f_x \leq g$ ” is replaced softer one, i.e. “ $f_x \in \tilde{G}$ ”. So, using (8), the necessity that event “ $f_x \in \tilde{G}$ ” holds can be expressed as follows:

$$\begin{aligned} N(f_x \in \tilde{G}) &= 1 - \Pi(f_x \notin \tilde{G}) \\ &= 1 - \sup_{v \in \mathbb{R}} \min\{\pi_{f_x}(v), 1 - \mu_{\tilde{G}}(v)\} \\ &= 1 - \sup_S \min\{\pi(S), 1 - \mu_{\tilde{G}}(F(x, S))\}. \end{aligned} \quad (12)$$

Thus, if $N(f_x \in \tilde{G}) = 1 - \lambda$ means that for all scenarios S such that $\pi(S) > \lambda$, the degree that costs of plan x fall within fuzzy goal $\tilde{G}$, is not less than $1 - \lambda$. Note that $N(f_x \in \tilde{G})$ is more general and weaker than $N(f_x \leq g)$. If $\mu_{\tilde{G}}(v) = 0$ for

$v \in (g, \infty)$ then they are the same. Moreover, $N(f_x \leq g) \leq N(f_x \in \tilde{G})$.

Let us give the second criterion of choosing a robust plan. We have a fuzzy goal $\tilde{G}$, and we wish to find a production plan which maximizes the necessity degree that costs of the plan fall within fuzzy goal $\tilde{G}$. Thus we need to solve the following optimization problem,

$$\max_{x \in \mathbb{X}} N(f_x \in \tilde{G}). \quad (13)$$

We check at once that problem (13) is equivalent to the following mathematical programming problem:

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & f_x^{+[\lambda]} \leq g^{+[1-\lambda]}, \\ & \lambda \in [0, 1], \\ & x \in \mathbb{X}. \end{aligned} \quad (14)$$

If (x^*, λ^*) is an optimal solution for problem (14), then $N(f_{x^*} \in \tilde{G}) = 1 - \lambda^*$. If (14) is infeasible then $N(f_x \in \tilde{G}) = 0$ for all $x \in \mathbb{X}$.

An algorithm for solving problem (14) (resp. (11)) is based on the standard binary search technique in $[0, 1]$ (the interval of possible values of λ) which follows from the fact that $f_x^{+[\lambda]}$ and $g^{+[1-\lambda]}$ (resp. g) are nonincreasing and nondecreasing functions of λ , respectively. We call the algorithm the *binary search based algorithm*. To find an optimal (x^*, λ^*) , $x^* \in \mathbb{X}$, $\lambda^* \in [0, 1]$, with a given error tolerance $\xi > 0$, we seek at each iteration, for a fixed λ , a plan $x \in \mathbb{X}$ satisfying $f_x^{+[\lambda]} \leq g^{+[1-\lambda]}$ (resp. $f_x^{+[\lambda]} \leq g$), which boils down to seeking an optimal robust production plan x^r and its worst case scenario under scenario set $\tilde{\Gamma}^{[\lambda]}$, i.e. to solving problem ROB (see Section III). Note that $f_x^{+[\lambda]} = A(x)$ and also that $f_x^{+[\lambda]} \leq g^{+[1-\lambda]}$ (resp. $f_x^{+[\lambda]} \leq g$) holds for some $x \in \mathbb{X}$ if and only if it holds for an optimal robust production plan under $\tilde{\Gamma}^{[\lambda]}$. Thus, at each iteration, we use either formulae (5) for the case without capacity limits or Algorithm 1 for the case with capacity limits. If the length of determined interval of possible values of λ is less than ξ , then an optimal robust production plan x^r for a fixed λ , (x^r, λ) , is an approximation of an optimal solution for problem (14) (resp. (11)) with precision ξ . The running time of the above algorithm is $O(I(T) \log \xi^{-1})$ time, where $\xi > 0$ is a given error tolerance and $I(|T|)$ is time required for finding an optimal robust production plan and its worst case scenario under interval scenario set $\tilde{\Gamma}^{[\lambda]}$ (the running time of either (5) or Algorithm 1).

In order to evaluate the efficiency of the binary search based algorithm, we show some results of computational experiments. For every $T = 100, 200, \dots, 1000$, ten instances of the problem (13) with capacity limits were generated. In every instance, $c_I = 1$, $c_B = 10$, the production capacities were randomly generated intervals $[X, Y]$, where X is an integer-valued random variable uniformly distributed in $\{0, 1, \dots, 99\}$ and Y is an integer-valued random variable uniformly distributed in $\{100, 101, \dots, 199\}$, the demands are triangular fuzzy intervals with the supports equal to $[0, 199]$ and the modal values equal to Z , where Z is an integer-valued random

TABLE II
AVERAGE COMPUTATION TIMES IN SECONDS.

T	0.00	0.25	0.50	0.75	1.00
100	0.75	0.76	0.74	0.73	0.70
200	3.47	3.47	3.40	3.38	3.37
300	14.03	13.96	13.95	13.93	13.93
400	25.64	25.60	25.64	25.60	25.61
500	34.70	34.68	34.64	34.66	34.56
600	86.74	86.73	86.90	86.78	86.87
700	141.40	141.41	141.05	141.08	141.31
800	210.70	210.81	210.68	210.79	207.91
900	338.86	338.49	338.65	338.86	338.36
1000	668.77	672.50	672.18	672.61	671.49

variable uniformly distributed in $\{0, 1, \dots, 199\}$, the fuzzy goal $\tilde{G}$ was modeled as a trapezoidal fuzzy interval $(0, g, 0, \beta)$, where g was chosen as the maximal cost of an optimal robust production plan for the problem ROB without capacity limits (an ideal supplier) and under the interval demands being the supports of the generated triangular fuzzy demands. The values of β were equal to $0.00 \cdot g, 0.25 \cdot g, 0.50 \cdot g, 0.75 \cdot g$ and $1.00 \cdot g$ and the error tolerance $\xi = 0.01$. We used IBM ILOG CPLEX 12.2 library and a computer equipped with Intel Core 2 Duo 2.5 GHz to solve the generated instances. In Table II average computation times in seconds are presented. As we can see from the obtained results, the binary search based algorithm, which calls Algorithm 1 at each iteration, can solve efficiently the problem (13) with capacity limits having up to 1000 periods.

V. CONCLUSION

In this paper, we have investigated a certain version of production planning problem with uncertain demands modeled by fuzzy intervals, whose membership functions are possibility distributions for the values of the unknown demands. We have introduced, in this setting, the degree of necessity that the cost of a plan does not exceed a given threshold and the degree of necessity that costs of a plan fall within a given fuzzy goal, which allows us to evaluate a given production plan. For finding robust production plans under fuzzy demands, we have proposed two criteria: the first one consists in choosing a production plan which maximizes the degree of necessity that its cost does not exceed a given threshold, the second criterion is softer than the first one and consists in choosing a plan with the maximum degree of necessity that costs of the plan fall within a given fuzzy goal. We have constructed the algorithms for determining optimal robust production plans with respect to the criteria and confirmed their efficiency experimentally. The criteria are a generalization, to the fuzzy case, of the known from literature the min-max criterion. Consequently, we have shown in the paper that there exists a link between interval uncertainty with the min-max criterion and possibilistic uncertainty with the necessity based criteria. Therefore, we have discussed first the interval-valued case and then we have extended the methods introduced for the interval-valued problem to the fuzzy-valued one.

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