

Sum of Squares Solutions Assuring Non-quadratic Discrete Stability

Ji-Chang Lo
Mechanical Engineering
National Central University
Jhong-Li, Taiwan
Email: jclo@cc.ncu.edu.tw

Jhen-Shun Kao
Mechanical Engineering
National Central University
Jhong-Li, Taiwan

Abstract—A new stability condition in terms of SOS is studied in this paper for discrete-time fuzzy systems. Based on a parameter-dependent Lyapunov function, we study asymptotically SOS relaxation families, reducing the conservatism that commonly exists in the quadratic stability approaches.

Index Terms—TS fuzzy model; Copositive relaxations; Premise-dependent Lyapunov function; Linear matrix inequality; Sum of Squares.

I. INTRODUCTION

For decades a large number of literature on fuzzy control are TS model-based control where, mostly, the common P approach searching for a single Lyapunov function remains active [1], [2], [3]. Another category emphasizes parameter-dependent functions with multiple P_j matrices as a candidate of Lyapunov function [4], [5], [6]. To remove the time-derivative dependence, progress has been made in obtaining less conservative results using non-quadratic approach (multiple Lyapunov functions) [7], [8], [9], [10], [11] where a fuzzy Lyapunov candidate is used for continuous-time T-S fuzzy models and the resulting stability condition is shown to be more relaxed than the condition derived from the common P approach. Although LMI approaches remain the most favorite tool of choice when a Lyapunov-based stability analysis is used, we, in this paper, propose a copositive relaxation combined with premise-dependent Lyapunov functions to derive necessary and sufficient conditions for the TS fuzzy model stability tests. Inspired by recent advances [10]-[20], [21], [22], we investigate copositive relaxation associated to fuzzy stability and control design from sum of squares viewpoints. The main contribution is that the proposed method takes the firing strength into account and

The paper is organized as follows. Section II rehearses some useful results which forms the foundation for later developments. Sum of squares is also introduced. Section III derives the stability conditions for discrete-time fuzzy systems via a parameter-dependent Lyapunov function. SOS relaxation addressed in Section IV. An example is illustrated in Section V and conclusion is drawn in Section VI.

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Notations: The symbol $\star$ denotes transposed elements in a symmetric matrix. The notation Y_μ stands for $\sum_{i=1}^r \mu_i Y_i$ where $\mu_i \geq 0$ and $\sum_{i=1}^r \mu_i = 1$. $!$ denotes factorial for combinatoric expression. Let $\mathcal{K}(h)$, where h is the degree of a polynomial in μ , be the set of r -tuples defined below:

$$\mathcal{K}(h) = \{(k_1, k_2, \dots, k_r) : k_1 + k_2 + \dots + k_r = h \quad (1)$$

$$\forall k_i \in I^+(\text{non-negative integers}), i = 1, 2, \dots, r\}.$$

Since the number of rules in the fuzzy base is r , the number of elements in $\mathcal{K}(h)$ is given by $J(h) = (r+h-1)!/(h!(r-1)!)$. To ease the presentation, we adopt the short hand notations displayed below:

$$\begin{aligned} k &= (k_1, k_2, \dots, k_r) \\ \mu^k &= \mu_1^{k_1} \mu_2^{k_2} \dots \mu_r^{k_r} \\ e_i &= 0 \dots \underbrace{1}_{i^{th}} \dots 0 \\ k - e_i &= k_1 k_2 \dots (k_i - 1) \dots k_r \\ \pi(k) &= (k_1!)(k_2!) \dots (k_r!) \\ \mathcal{C}_{ii}^k(h) &= \frac{(h!)k_i(k_i-1)}{\pi(k)}, \quad \mathcal{C}_{ij}^k(h) = \frac{(h!)k_i k_j}{\pi(k)}. \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_{iii}^k(h) &= \frac{(h!)k_i(k_i-1)(k_i-2)}{\pi(k)} \\ \mathcal{C}_{iij}^k(h) &= \frac{(h!)k_i(k_i-1)k_j}{\pi(k)}, \quad \mathcal{C}_{ijl}^k(h) = \frac{(h!)k_i k_j k_l}{\pi(k)}. \end{aligned}$$

II. PRELIMINARIES

To begin with, we introduce the following definitions and terminologies which serve as the entry point to this paper.

Definition 1 (SOS functions): [23] A multivariate polynomial $f(x)$, $x \in R^r$, is a sum of squares if there exist polynomials $g_1(x), g_2(x), \dots, g_t(x)$ such that

$$f(x) = \sum_{i=1}^t g_i^2(x).$$

Definition 2 (Monomials): [23] A monomial in x is a function of the form $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_r^{\alpha_r}$ where $\alpha_1, \alpha_2, \dots, \alpha_r$ are nonnegative integers and the degree of the monomial is given

by α which is the sum of all α_i . In short, $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_r$.

Given the aforementioned definition, it is shown in the articles [23], [24] that $f(x)$ is an SOS if and only if there exists a positive semidefinite matrix $Q \geq 0$ such that

$$f(x) = \tilde{x}' Q \tilde{x} \quad (2)$$

where $\tilde{x}$ is a column vector whose entries are all monomials in x . Terminologically, (2) is referred to the Gram matrix method. It is interesting to know that when $\tilde{x} = x$, $f(x)$ in (2) is a quadratic function.

One can search for such a Q by solving a semidefinite program which is done using SOSTOOLS [25], [26] that uses SeDuMi [27] as the semidefinite programming solver. Note that a function that is SOS always implies the underlying function is nonnegative, the converse is not true. However, numerical experiments seem to indicate that the gap between SOS and nonnegativity is small [24]. We note in passing that an extension of scalar polynomial optimization by SOS decomposition to optimization with scalar polynomial objective and nonlinear matrix constraints is stated in [12], [28].

III. STABILITY ANALYSIS

In this section, we derive stability conditions for discrete-time open loop fuzzy systems that are displayed below:

$$z(t+1) = \sum_{i=1}^r \mu_i A_i z = A_\mu z(t) \quad (3)$$

where A_i is a system matrix of each rule i and $\mu_i \geq 0$ is the firing strength of rule i . z is the state vector.

Considering a premise-dependent non-quadratic Lyapunov function

$$V(z(\cdot)) = z^T(\cdot) X_\mu^{-1} z(\cdot) \quad (4)$$

where $X_\mu = \sum_{j=1}^r \mu_j X_j$ and $X_j > 0$ are symmetric, positive definite matrices for all discrete-time index t , we have the following Lemmas.

Lemma 1 (Discrete-time Stability, LMI): The open loop system (3) is stable if there exist matrices $X_j = X_j' > 0$, $j = 1, \dots, r$ satisfying the following HPPD:

$$f(\mu) := \sum_{i=1}^r \sum_{j=1}^r \sum_{l=1}^r \mu_i \mu_j \mu_l^+ M_{ijl} < 0 \quad (5)$$

where

$$M_{ijl} = \begin{bmatrix} -X_j & X_j A_i' \\ A_i X_j & -X_l \end{bmatrix}$$

Proof: Given the non-quadratic Lyapunov function (4), it is straightforward to verify

$$\begin{aligned} \Delta V(t) &= V(t+1) - V(t) \\ &= z'(t+1) X_{\mu^+}^{-1} z(t+1) - z'(t) X_\mu^{-1} z(t) \\ &< 0 \end{aligned}$$

where μ^+ means $\mu(t+1)$. Recalling Schur complement and congruence transformation with $\text{diag}[X_\mu \ I]$, we have an equivalent form

$$\begin{bmatrix} -X_\mu & X_\mu A_\mu' \\ A_\mu X_\mu & -X_{\mu^+} \end{bmatrix} < 0$$

For Lyapunov stability, a sufficient condition is (5). ■

Notice that the matrices M_{ijl} are real symmetric, matrix-valued and linear functions of decision variables (multiple Lyapunov matrices) X_j .

Up to now, we have the following stability condition expressed by matrix-valued homogeneous polynomial function in μ for discrete-time systems.

$$f(\mu) < 0 \quad (6)$$

By applying matrix-valued Pólya's Theorem [29], [30], [31], (6) leads to an equivalent form

$$\begin{aligned} 0 &> \left(\sum_{i=1}^r \mu_i \right)^d f(\mu) \\ &= \sum_{k \in \mathcal{K}(d+3)} H^k \mu^k \end{aligned} \quad (7)$$

where $\mathcal{K}(d+3)$ is defined in 1 and

$$\begin{aligned} H^k &= \sum_{i=1}^r C_{iii}^k(d) M_{iii} \\ &+ \sum_{i=1}^{r-1} \sum_{j=i+1}^r C_{ijj}^k(d) (M_{ijj} + M_{iji} + M_{jii}) \\ &+ \sum_{i=1}^{r-2} \sum_{j=i+1}^{r-1} \sum_{l=j+1}^r C_{ijl}^k(d) (M_{ijl} + M_{jil} \\ &+ M_{ilj} + M_{jli} + M_{lij} + M_{lji}) \end{aligned}$$

For Lemmas 1, the challenge lies on how the above conditions (5) can be verified. We observe that the matrices involved are copositive. The brief review in the preliminaries motivates the following research. Unlike existing LMI-based relaxations, we require Lemma 1 to be sum of squares, asking when (7) has an SOS decomposition.

IV. SOS RELAXATIONS

The sum of squares circumvents the NP-hard problem proving positivity of a polynomial function and provides a computationally tractable technique for the existence of an SOS decomposition [23], [12], [13], [32].

Based on Pólya and Lyapunov theorem, we have established that sufficient conditions for the existence of Parameter-dependent Lyapunov function assuring stability is to verify (5), which is a non-convex problem and NP-hard. Most existing LMI-based relaxations in the literature ignore μ and solve the underlying stability problem as an SDP problem. Unlike those approaches, we take μ into consideration. Particularly *instead of asking*

$$- \sum_{k \in \mathcal{K}(d+3)} H^k \mu^k > 0, \quad \mu \in R_+^r$$

we require them to be sums of squares (SOS) [26]. Having established the rationale, the idea is to find a homogeneous polynomially parameter-dependent (HPPD) function such that

$$-\sum_{k \in \mathcal{K}(d+3)} H^k x^{2k} \text{ is SOS} \quad (8)$$

where $\mu = x^2, x \in R^r$ and v_1 is vector independent of x . We emphasize that the HPPD (8) takes the membership μ into accounts, collects the interactions among the subsystems and yields less conservative results.

Therefore by deliberately working on polynomials consisting of basis monomials, the positive definiteness of the negated condition in Lemma 1 can be relaxed to the existence of SOS decomposition. Under this relaxation, the search for a bounded degree d can be efficiently solved by checking the maximum eigenvalues of Q in 2 [33]. The SOS program is therefore formally stated below.

Theorem 1 (SOS Relaxation): There exist a matrix $X_j = X'_j > 0$ solving (6) if for each $k \in \mathcal{K}(d+3)$ and a sufficiently large $d > 0$ there exist matrices $X_j = X'_j > 0$ such that

$$-\sum_{k \in \mathcal{K}(d+3)} H^k x^{2k} > 0 \quad (9)$$

which is verifiable by SOS displayed below

$$\begin{aligned} -v'_1 \left(\sum_{k \in \mathcal{K}(d+3)} (H^k - \phi_1) x^{2k} \right) v_1 &\text{ is SOS} \\ v'_2 (X_i - \phi_2) v_2, i = 1, \dots, r &\text{ is SOS} \end{aligned}$$

where ϕ_1, ϕ_2 are sufficiently small positive constants NEEDED to guarantee positive definiteness of HPPD function. v_1, v_2 are vectors independent of x .

Proof: The sufficiency is readily seen from HPPD (7) and lemma 1. ■

More details are documented in [22] whilst the corresponding LMI details are documented in [21].

V. AN EXAMPLE

In order to appreciate the efficiency of the proposed method, we consider an example where the T-S fuzzy models are borrowed from existing papers.

A. Discrete fuzzy systems

Consider a discrete-time fuzzy system borrowed from [10] in which the system matrices are given as

$$A_1 = \begin{bmatrix} b & -0.5 \\ a & -0.9 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9 & -a \\ -0.6 & 0.8 \end{bmatrix}$$

where the (a, b) parameters are varied during numerical experiments for feasibility test. Also, compared with the relaxation method proposed in [10], the proposed SOS relaxation generates more feasible regions than that of the cited paper.

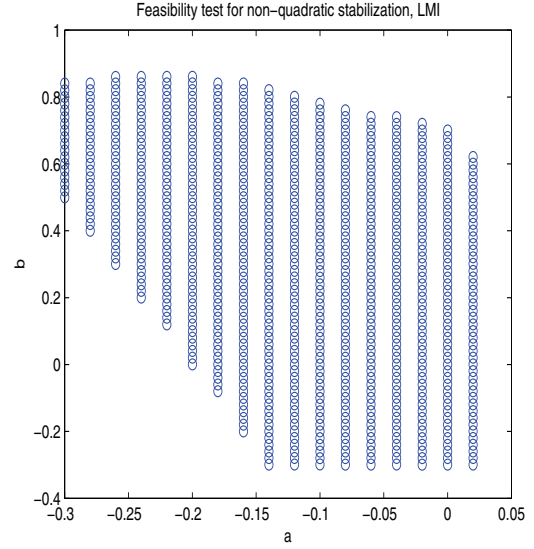


Fig. 1. Feasible regions found by LMI

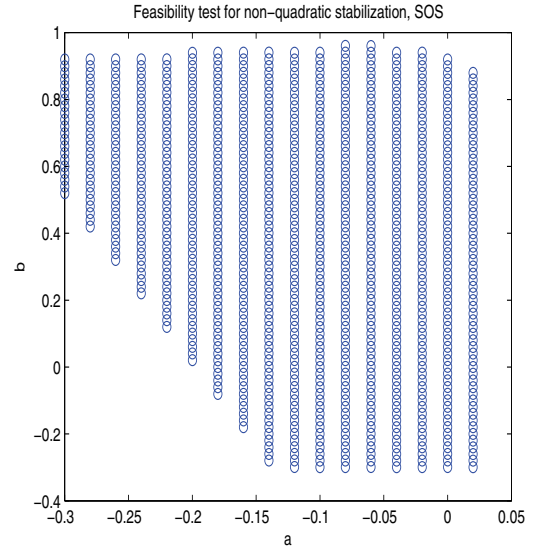


Fig. 2. SOS explores more feasible regions.

VI. CONCLUSION

In this paper, a stability condition based on sum of squares are developed for discrete-time T-S fuzzy systems. The proposed approach utilizes a premise-dependent Lyapunov function to prove the non-quadratic Lyapunov stability of the underlying fuzzy systems, leading to a family of SOS relaxations that assures existence of non-quadratic Lyapunov function. It is shown and demonstrated via an example that the stability can be cast and solved by SOS.

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