

Generalized Stabilizing Controllers for Fuzzy Systems via Circle Criterion – LMI and SOS

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Abstract—We investigate a generalized control problem via the Circle criterion borrowed from system theory; the generalized controller can be cast into existing canonical controllers depending on parameterizations. We first show a fuzzy version of the Circle criterion and then provide a synthesis result based on the Circle criterion, establishing closed-loop stabilizability for state-feedback controllers. In this paper, we show how to incorporating the membership information by approximating the cone of copositive matrices via SOS that is solvable by SOSTOOLS. Finally the validity and applicability of the approach are illuminated by examples.

Index Terms—Stabilizing controllers; Circle criterion; TS fuzzy models; Linear matrix inequality (LMI); Sum of Squares (SOS); Exact solutions

I. INTRODUCTION

OVER the decades, a large number of important results on stabilization have been derived for TS fuzzy models, which rely on the existence of a common P matrix to a quadratic Lyapunov function [1]. In a broader view, a fuzzy system may have nonlinear functions (such as parameter uncertainties, saturating inputs) involved in the model. How to tackle such problem becomes the focus of this paper. Motivated by the above mentioned nonlinearity involved in a fuzzy model and the literature on system theory [2], [3], [4], [5], [6] and references therein, we investigate a unified approach to synthesize stabilizing controllers for nonlinear fuzzy systems via system criteria. The multivariate circle criterion has been used to graphically analyze the robust stability of a fuzzy feedback linearization regulator [7]. Furthermore the circle criterion and the related Popov criterion have suggested ways to study the stability of Mamdani-type fuzzy control systems with a linear plant [8], [9], [10], [11]. While most existing papers use Mamdani-type fuzzy systems, when applying the circle criterion via graphical analysis, is for stability test, we focus on T-S fuzzy control models. Equipped with the system criteria in order to profit from the advantage of convex optimization, we propose a generalized controller, which unifies all existing controllers, for T-S fuzzy control systems, aiming at synthesizing stabilizing controllers such that the corresponding

closed-loop fuzzy system is asymptotically stable. Although various control laws have been studied in the fuzzy literature, there exist no reports showing, via the generalized controller, the subtle relations among all stabilizing control laws.

Genetically, stability/stabilization of a fuzzy control problem is a copositive problem since checking positivity of the fuzzy matrix-valued polynomial in μ ($-\sum_i \sum_j \mu_i \mu_j M_{ij} > 0$) involves copositive matrices $-M_{ij}$, known to be NP-hardness. To avoid the NP-hardness, a solution to this problem was proposed by Parrilo [12] who showed that one can approximate the copositive cone to a given accuracy by a sufficiently large set of LMIs (therefore, only sufficient condition is shown.) Following the developments [13], [14], [15], [16], [17], we, via SOS, investigate copositive relaxation searching for existence of an SOS decomposition assuring exactness of solution to fuzzy PD-LMIs. Under this relaxation, the search for a bounded degree d , indicating exactness, can be efficiently performed and cast as an SOS program, establishing the use of SOS decomposition to solve the LMI formulations.

The organization of this paper is as follows. Section II re-derives a fuzzy version of the Circle theorems expressed in an LMI formulation. In section III, a state feedback controller falling out of the general setting is addressed. The mechanism involved demonstrates that a unified treatment aiming at solving the absolute stability problems is addressed. An illustrative example is considered in Section V and concluding remarks are made in Section VI.

Notations: The symbol $\star$ is used for the transposed elements in a symmetric matrix, and $M < 0$ means a negative-definite symmetric matrix M . The notations Y_μ and $Y_{\mu\mu}$ stand for $\sum_{i=1}^r \mu_i Y_i$ and $\sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j Y_{ij}$, where $\mu_i \geq 0$ and $\sum_{i=1}^r \mu_i = 1$. The indices $i, j = 1, 2, \dots, r$ (number of fuzzy rules) are used throughout this paper. $!$ denotes factorial for combinatoric expression. Let $\mathcal{K}(h)$, where h is the degree of a polynomial in μ , be the set of r -tuple defined below:

$$\mathcal{K}(h) = \{(k_1 \ k_2 \ \cdots \ k_r) : k_1 + k_2 + \cdots + k_r = h,$$

$$\forall k_i \in I^+(\text{non-negative integers}), i = 1, 2, \dots, r\}.$$

Since the number of rules in the fuzzy rule base is r , the number of elements in $\mathcal{K}(h)$ is given by $J(h) = (r + h - 1)!/(h!(r - 1)!)$. To ease the presentation, we adopt the short

This work was supported in part by the National Science Council of the ROC under grant NSC-98-2221-E-008-110.

hand notations displayed below:

$$\begin{aligned} k &= k_1 k_2 \cdots k_r \\ \mu^k &= \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_r^{k_r} \\ \pi(k) &= (k_1!)(k_2!) \cdots (k_r!) \\ C_{ii}^k(h) &= \frac{(h!)k_i(k_i-1)}{\pi(k)}, \quad C_{ij}^k(h) = \frac{(h!)k_i k_j}{\pi(k)}. \end{aligned}$$

II. THE CIRCLE CRITERION

We first state the Circle criterion and then study the synthesis of a state feedback controller. To facilitate the presentation, we assume that the readers are familiar with the basic fuzzy setup consisting of r fuzzy rules whose consequent parts are characterized by local Takagi-Sugeno type linear models. Then by using a standard fuzzy inference method – singleton fuzzifier, product fuzzy inference and weighted average defuzzifier, the inferred fuzzy model of interest, after normalization, is described by the following nominal (closed-loop) fuzzy system

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{A}_\mu & -\mathcal{B}_\mu \\ \mathcal{C}_\mu & -\mathcal{D}_\mu \end{bmatrix} \begin{bmatrix} x \\ \phi(z, \cdot) \end{bmatrix} \quad (1)$$

where $x \in R^n$, $\phi(z, \cdot) \in R^m$, $z \in R^m$ and $\mu_i \geq 0$ are the state vector, time-varying nonlinearity, controlled output, and grade of membership, respectively. The “.” in the function $\phi(z, \cdot)$ means either the continuous-time t or the discrete-time k . $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, and $\mathcal{D}_i$ are real constant matrices of appropriate dimensions. It is noted that the negative signs in front of $\mathcal{B}_\mu$ and $\mathcal{D}_\mu$ are due to the negative feedback connection in system theory. Although the signs can be absorbed into the matrices, we keep the negative signs to signal the trait of absolute stability. We assume $(\mathcal{A}_i, \mathcal{B}_i)$ is completely controllable and $(\mathcal{A}_i, \mathcal{C}_i)$ is completely observable. To understand the Circle criterion

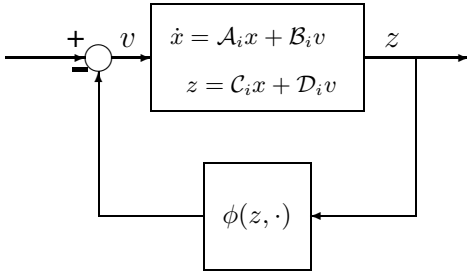


Fig. 1. Fuzzy local subsystem with nonlinear feedback connection.

borrowed from absolute stability, we need to introduce the nonlinear structure of the feedback connection ϕ depicted in Figure 1. The following definition is useful to this aim.

Definition 1 (Nonlinear Connection): Let $K_{\min} = \text{diag}(\alpha_1, \dots, \alpha_m)$ and $K_{\max} = \text{diag}(\beta_1, \dots, \beta_m)$ be two given diagonal matrices, the nonlinearity is characterized by the following sector bounded set $\Phi_c = \{\phi(z, \cdot) : R^m \times R^+ \rightarrow R^m \text{ such that } (\phi - K_{\min}z)'(\phi - K_{\max}z) \leq 0\}$, where R^+ denotes positive real numbers for the continuous-time space over $[0, \infty)$.

The function $\phi(z, \cdot)$ defined above represents a memoryless, possibly time-varying nonlinearity. It is assumed that $\phi(z, \cdot)$ satisfies the Lipschitz condition $\|\phi(z_1, \cdot) - \phi(z_2, \cdot)\| \leq L\|z_1 - z_2\|$ for all $(z_1, \cdot)$ and $(z_2, \cdot) \in R^m \times R^+$, where the positive constant L is called the Lipschitz constant. Moreover, it is sectorly bounded as defined in Definition 1.

Remarks:

- 1) For the scalar case $m = 1$, the sector condition characterizing Φ_c is equivalent to the more familiar SISO condition $\alpha_1 z \leq \phi \leq \beta_1 z$.
- 2) In this paper we interpret the nonlinear connection ϕ as a nonlinear term in local submodels representing unmodel dynamics. However, if ϕ were energy bounded external disturbance ($w \in L_2$), the setting (1) would be a typical H_∞ problem.

Given the definition, we establish the following theorem for the continuous-time systems.

Theorem 1 (Continuous Circle): The nominal system (1) is asymptotically stable if there exists a $\mathcal{P} > 0$ satisfying

$$M_i = \begin{bmatrix} \mathcal{A}'_i \mathcal{P} + \mathcal{P} \mathcal{A}_i & -\mathcal{P} \mathcal{B}_i + \mathcal{C}_i \bar{K}' & \mathcal{C}'_i \\ -\mathcal{B}'_i \mathcal{P} + \bar{K} \mathcal{C}_i & -I - (\bar{K} \mathcal{D}_i + \star) & -\mathcal{D}'_i \\ \mathcal{C}_i & -\mathcal{D}_i & 2K_1 \end{bmatrix} < 0 \quad (2)$$

where $K_1 = (K'_{\min} K_{\max} + \star)^{-1} < 0$ and $\bar{K} = (K_{\min} + K_{\max})/2$.

Proof: Using a quadratic Lyapunov function $V(t) = x' \mathcal{P} x$ to show that the system (1) is asymptotically stable, we derive a stability condition for the following inequality

$$\dot{V} - (\phi - K_{\min}z)'(\phi - K_{\max}z) < 0. \quad (3)$$

We know in passing that the second term in (3) is negative by Definition 1. That is to say satisfaction of (3) implies

$$\dot{V} < (\phi - K_{\min}z)'(\phi - K_{\max}z) \leq 0$$

and the underlying systems is stable absolutely. The problem under study is referred to as the absolute stability problem, also known as the Luré problem. To continue the stability proof, it is straightforward to verify the following identity

$$\begin{aligned} & \dot{V} - (\phi - K_{\min}z)'(\phi - K_{\max}z) \\ &= x'(\mathcal{A}'_\mu \mathcal{P} + \mathcal{P} \mathcal{A}_\mu)x - \phi' \mathcal{B}'_\mu \mathcal{P} x - x' \mathcal{P} \mathcal{B}_\mu \phi \\ & - \phi' \phi + \phi' K_{\max} \mathcal{C}_\mu x \\ & + x' \mathcal{C}'_\mu K'_{\min} \phi - \phi' K_{\max} \mathcal{D}_\mu \phi \\ & - \phi' \mathcal{D}'_\mu K'_{\min} \phi - x' \mathcal{C}'_\mu K'_{\min} K_{\max} \mathcal{C}_\mu x \\ & - \phi' \mathcal{D}'_\mu K'_{\min} K_{\max} \mathcal{D}_\mu \phi \\ & + \phi' \mathcal{D}'_\mu K'_{\min} K_{\max} \mathcal{C}_\mu x + x' \mathcal{C}'_\mu K'_{\min} K_{\max} \mathcal{D}_\mu \phi \\ &= \begin{bmatrix} x \\ \phi \end{bmatrix}' M_\mu \begin{bmatrix} x \\ \phi \end{bmatrix} \end{aligned}$$

where the matrix M_μ is defined below

$$\begin{aligned} M_\mu &= \begin{bmatrix} \mathcal{A}'_\mu \mathcal{P} + \mathcal{P} \mathcal{A}_\mu & -\mathcal{P} \mathcal{B}_\mu + \mathcal{C}_\mu \bar{K}' \\ -\mathcal{B}'_\mu \mathcal{P} + \bar{K} \mathcal{C}_\mu & -I - (\bar{K} \mathcal{D}_\mu + \star) \end{bmatrix} \\ & - \begin{bmatrix} \mathcal{C}'_\mu \\ -\mathcal{D}'_\mu \end{bmatrix} \frac{(K'_{\min} K_{\max} + \star)}{2} \begin{bmatrix} \mathcal{C}_\mu & -\mathcal{D}_\mu \end{bmatrix}. \end{aligned} \quad (4)$$

A sufficient condition ensuring (3) is $M_\mu < 0$. To recast $M_\mu < 0$ into a feasibility problem solvable by convex algorithms, performing the Schur complement on $M_\mu < 0$ and factoring out $\sum_{i=1}^r \mu_i$ yields the sufficient condition LMI (2). ■

III. GENERALIZED STABILIZING CONTROLLERS

Having established the working theorems, we are ready to consider stabilization problems concerning absolute stability. To begin, setting up a T-S fuzzy model

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A_\mu & -B_{1\mu} & B_{2\mu} \\ C_{1\mu} & -D_{11\mu} & D_{12\mu} \\ C_{2\mu} & -D_{21\mu} & 0 \end{bmatrix} \begin{bmatrix} x(\cdot) \\ \phi(z, \cdot) \\ u(\cdot) \end{bmatrix} \quad (5)$$

where $y \in R^q$ is the measured output, u is the control input, and all other variables are as in (1). A measured output variable y is a system variable that can be measured or through which the system influences its environment while a controlled output z is a system variable that can be controlled by some control scheme but not necessarily measurable.

To stabilize the T-S fuzzy model, we consider a generalized controller inspired by [5]

$$\begin{bmatrix} \dot{\hat{x}} \\ u \end{bmatrix} = \begin{bmatrix} F_{\mu\mu} & G_\mu & S_\mu \\ N_\mu & J & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \\ u \end{bmatrix} \quad (6)$$

where F_{ij} , G_i , S_i , N_i and J are generalized parameters of appropriate dimensions. The generalized controller above has a nice merit encompassing the dynamic output feedback controller, the observed-state feedback controller and the state feedback controller.

To describe a state feedback controller, simply neglect the dynamic part of (6) by setting $F_{\mu\mu} = 0$, $G_\mu = 0$, $S_\mu = 0$, $J = 0$, and letting $\hat{x} = x$. Such control depends only on the fuzzy blending of the state x , known as a static (memoryless) state feedback (fuzzy) control or a PDC controller in fuzzy community.

Substituting the controller (6) into the open loop system (5), we have the closed-loop system displayed below.

$$\begin{bmatrix} \dot{\bar{x}} \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{\mu\mu} & -\mathcal{B}_{\mu\mu} \\ \mathcal{C}_{\mu\mu} & -\mathcal{D}_{\mu\mu} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \phi(z, \cdot) \end{bmatrix} \quad (7)$$

where

$$\bar{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

$$\mathcal{A}_{\mu\mu} = \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \begin{bmatrix} A_i + B_{2i} J C_{2j} & B_{2i} N_j \\ G_i C_{2j} + S_i J C_{2j} & F_{ij} + S_i N_j \end{bmatrix}$$

$$\mathcal{B}_{\mu\mu} = \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \begin{bmatrix} B_{1i} + B_{2i} J D_{21j} \\ G_i D_{21j} + S_i J D_{21j} \end{bmatrix}$$

$$\mathcal{C}_{\mu\mu} = \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \begin{bmatrix} C_{1i} + D_{12i} J C_{2j} & D_{12i} N_j \end{bmatrix}$$

$$\mathcal{D}_{\mu\mu} = \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j (D_{11i} + D_{12i} J D_{21j}).$$

Having established the closed-loop system (7) with the generalized controller (6) embedded, our investigation is performed on the basis of this setting and then derive the corresponding results for stabilization analysis and controller synthesis.

A. State Feedback Synthesis - LMI

A corollary characterizing state feedback controller synthesis fall out immediately by simplifying the generalized controller (6) to a state feedback controller $u = \sum_{i=1}^r \mu_i N_i x = N_\mu x$.

Corollary 1 (State Feedback Ctr, LMI): The closed-loop system (7) is asymptotically stable for all $\phi(z, \cdot) \in \Phi_c$ if, given $K_{\min}$ and $K_{\max}$, there exists a feasible solution ($Q > 0, N_j$) satisfying (8) where $K_1 = (K'_{\min} K_{\max} + \star)^{-1} < 0$, $\bar{K} = (K_{\min} + K_{\max})/2$, and $N_j = N_j Q$.

Proof: The proof is straightforward, given Theorem 1. ■

Noting that the PD-LMIs (4) is a matrix-valued polynomial of degree 1 (a.k.a. affine) in μ , and $\sum_{i,j=1}^r \mu_i \mu_j M_{ij}$, where M_{ij} , at the top of next page, is defined in (8), is of degree 2 (a.k.a. quadratic) in μ .

IV. SUM OF SQUARES

Applying Lyapunov-based quadratic stability to the aforementioned fuzzy control system (7), Corollary 1 is shown to bear the following form

$$M_{\mu\mu} < 0 \quad (9)$$

By applying matrix-valued Pólya's Theorem [18], [19], [20], (9) leads to

$$\left(\sum_{i=1}^r \mu_i \right)^d M_{\mu\mu} < 0 \quad (10)$$

The PD-LMIs (10) comprise of infinitely many LMIs, thus being a feasibility problem of infinite dimensions. We need to obtain a set of convex finite-dimensional LMI conditions associated to PD-LMIs (10) so that the relaxation becomes a finite-dimensional problem. The proposed methodology is formally stated below.

Theorem 2 (Exact Solutions for QS): Given $K_{\min}$ and $K_{\max}$. The closed-loop T-S fuzzy control system (7) is quadratically stabilizable by a state feedback PDC control (6) if and only if there exist matrices N_j , $j = 1, 2, \dots, r$; $Q = Q' > 0$ and a sufficient large $d > 0$ such that

$$\sum_{k \in \mathcal{K}(d+2)} H^k \mu^k < 0 \quad (11)$$

where

$$H^k = \sum_{i=1}^r C_{ii}^k(d) M_{ii} + \sum_{i=1}^{r-1} \sum_{j=i+1}^r C_{ij}^k(d) (M_{ij} + M_{ji})$$

and M_{ij} is defined in (8).

Proof: See Theorem 1 in [21] and Lemma 1 in [15]. ■

For Theorem 2, the challenge lies on how the above conditions can be verified. To this end, we seek help from SOS. The details are documented in [13] whilst the corresponding

$$M_{ij} = \begin{bmatrix} A_i Q + B_{2i} \mathbf{N}_j + \star & \star & \star \\ -B'_{1i} + \bar{K}(C_{1i} Q + D_{12i} \mathbf{N}_j) & -I - (\bar{K} D_{11i} + \star) & \star \\ (C_{1i} Q + D_{12i} \mathbf{N}_j) & -D_{11i} & 2K_1 \end{bmatrix} < 0 \quad (8)$$

LMI details are documented in [14]. We emphasize that the homogeneous polynomially parameter-dependent (HPPD) function (11) takes the membership μ into accounts, yielding less conservative results.

A. State Feedback Synthesis - SOS

Therefore, by deliberately working on polynomials consisting of basis monomials, the negated condition in Theorem 2 can be relaxed to the existence of SOS decomposition. Under this relaxation, the search for a bounded degree $2d + 4$ can be efficiently solved. The SOS program is therefore formally stated below.

Corollary 2 (State Feedback Ctr, SOS): Given $K_{\min}$ and $K_{\max}$. There exist a matrix $Q = Q' > 0$ and a PDC controller (6) solving (9) if and only if for each $k \in \mathcal{K}(d+2)$ and a sufficiently large $d > 0$ there exist matrices $\mathbf{N}_j$, $j = 1, 2, \dots, r$, $Q = Q' > 0$ such that

$$\begin{aligned} -v'_1 \left(\sum_{k \in \mathcal{K}(d+2)} (H^k + \phi_1 I) x^{2k} \right) v_1 & \text{ is SOS} \\ v'_2 ((Q - \phi_2 I) x^{2k}) v_2 & \text{ is SOS} \end{aligned}$$

ϕ_1 and ϕ_2 are sufficiently small positive constants and v_1, v_2 are vectors independent of x .

Proof: The proof is readily obtained from Theorem 2 and the copositive statement following Theorem 2. ■

V. AN EXAMPLE

A. Inverted pendulum on a cart

Consider the problem of balancing and swinging up of an inverted pendulum on a cart [6]. The system is approximated by the following two-rule model:

$$\begin{aligned} \text{IF } x_1 \text{ is about } 0 \text{ THEN } \dot{x} &= A_1 x + B_1 v \\ \text{IF } x_1 \text{ is about } \pm \frac{\pi}{2} \text{ THEN } \dot{x} &= A_2 x + B_2 v \end{aligned}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{\frac{4l}{3} - aml} & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ \frac{-\mu a}{\frac{4l}{3} - aml + b_1} \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(\frac{4l}{3} - aml\beta^2)} & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ \frac{-\mu a\beta}{\frac{4l}{3} - aml\beta^2 + b_2} \end{bmatrix} \end{aligned}$$

and $\beta = \cos(88^\circ)$. Membership functions for Rule 1 and Rule 2 are given as follows:

$$p_1 = \cos(x_1), \quad p_2 = 1 - p_1.$$

The control objective is to balance the inverted pendulum for the approximation range $x_1 \in [-\pi/2, \pi/2]$. The saturation

function ϕ is given by

$$\phi = \begin{cases} u & \text{if } |u| \leq 0.1 \\ 0.1 + 0.5(u - 0.1) & \text{if } u > 0.1 \\ -0.1 - 0.5(-u - 0.1) & \text{if } u < -0.1 \end{cases}$$

To demonstrate the superiority of SOS technique as opposed to LMI method, a comparison study for feasibility is conducted. Two varying parameters b_1 and b_2 are inserted into input matrices for feasibility test and the results are obtained for $d = 0$ and shown in Figures 2-3 respectively. It is crystal clear that SOS method can explore more solution space than LMI scheme.

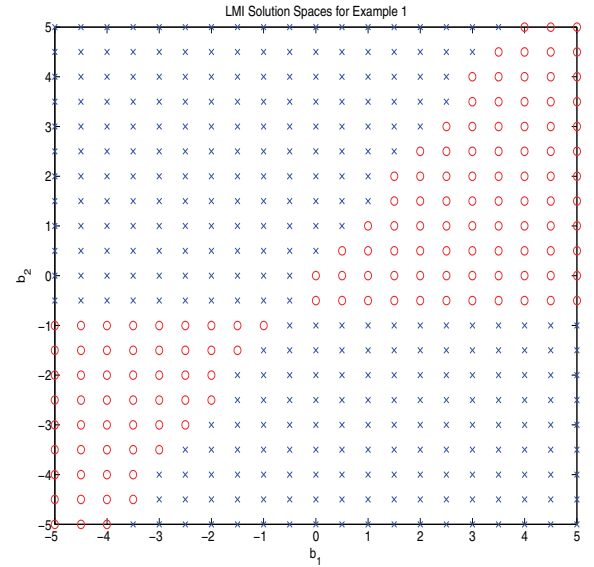


Fig. 2. LMI feasibility test

VI. CONCLUSION

A feedback stabilization problems involving absolute stability is presented. The Circle criterion borrowed from absolute stability is applied to derive LMI and SOS test conditions. Both results are shown, validating that the feasibility via SOS is much less conservative than that of LMIs scheme. A protruding feature of the SOS approach is that NO slack variables are involved and the membership functions are considered.

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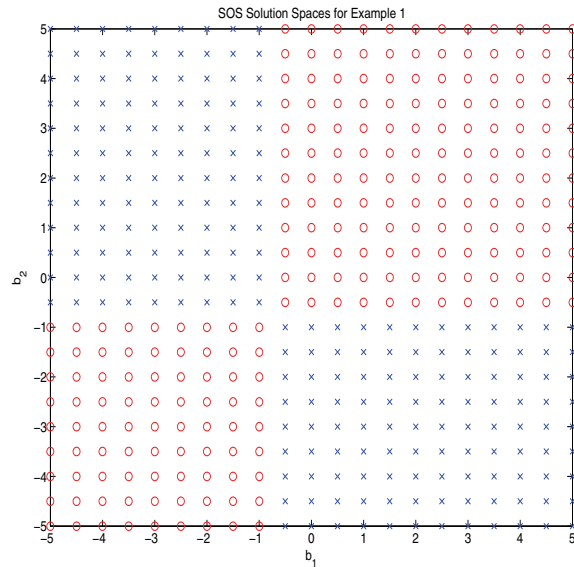


Fig. 3. SOS feasibility test, $d = 0$

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