

A Polynomial Observer Design for a Wider Class of Polynomial Fuzzy Systems

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Abstract—This paper presents a polynomial fuzzy observer design for a wider class of polynomial fuzzy systems via a sum of squares (SOS, for brevity) approach. The proposed SOS-based framework provides a number of innovations and improvements over the existing LMI-based approaches to Takagi-Sugeno (T-S) fuzzy controller and observer designs. First, we briefly summarize previous results for a class of polynomial fuzzy systems that is more general representation of the well-known T-S fuzzy system. Next, we propose a polynomial fuzzy observer to estimate states in a wider class of polynomial fuzzy systems and derive SOS conditions to design polynomial fuzzy controllers and observers. A remarkable feature of the SOS design conditions is that they realize the so-called separation principle, that is, that a polynomial fuzzy controller and observer for this class can be separately designed without lack of guaranteeing the stability of the overall control system in addition to converging state estimation error (via the observer) to zero. The design conditions in the proposed approach can be represented in terms of SOS and are symbolically and numerically solved via the recent developed SOSTOOLS and a semidefinite program (SDP) solver, respectively. To illustrate the validity and applicability of the proposed approach, a design example is provided. The example demonstrates advantages of the SOS-based approach for the existing LMI approaches to T-S fuzzy observer designs.

Keywords: polynomial fuzzy controller, polynomial fuzzy observer, polynomial fuzzy system, sum of squares

I. INTRODUCTION

The Takagi-Sugeno (T-S) fuzzy model-based control methodology [1], [2] has received a great deal of attention after LMI-based designs have been discussed in [3]-[4]. The fuzzy model-based control methodology provides a natural, simple and effective design approach to complement other nonlinear control techniques (e.g., [5]) that require special and rather involved knowledge.

Recently, the authors have first presented a sum of squares (SOS, for brevity) approach [6]-[11] to polynomial fuzzy control system designs. This is a completely different approach from the existing LMI approaches [2], [12]-[24]. Our SOS approach [6]-[11] provided more extensive results for the existing LMI approaches to T-S fuzzy model and control. However, to the best of our knowledge, there existed no literature on SOS-based observer designs for polynomial fuzzy systems before our previous result has been given in [25].

This paper presents SOS-based observer designs for a wider class than the class of the polynomial fuzzy systems discussed in [25]. The proposed SOS-based framework for polynomial fuzzy systems provides a number of innovations and improvements over the existing LMI approaches to T-S fuzzy observer-based control, e.g., [2], [12], [13]. First, it is known that nonlinear systems with polynomial terms can not be generally converted to globally exact T-S fuzzy models. Only local or semi-global T-S fuzzy models can be constructed for such nonlinear systems [2]. Thus, resulting control design conditions guarantee global stabilization and global state-estimation convergence only for local or semi-global models, but not always guarantee global stabilization and global state-estimation convergence for original nonlinear systems. On the other hand, it is possible to convert even nonlinear systems with polynomial terms to globally exact polynomial fuzzy models. Hence all the conditions derived here guarantee global stabilization and global state-estimation convergence for original nonlinear systems that are perfectly equivalent to polynomial fuzzy models. Secondly, even if local or semi-global T-S fuzzy models are permitted to use in practical sense, variables in polynomial terms appear in premise (part) variables of T-S fuzzy models. In polynomial fuzzy models, variables in polynomial terms do not appear in their premise parts and remain in system polynomial matrices A_i and B_i in consequence parts of polynomial fuzzy models. The difference is quite large from fuzzy observer design points of view. In general, fuzzy observer designs are not permitted to have premise variables depending on the states to be estimated. Therefore, T-S fuzzy observer designs can not be generally applied to nonlinear systems with polynomial terms. Conversely, the polynomial fuzzy observer designs proposed in this paper can be applied to even such systems. We will see these facts in the design examples later.

This paper presents an SOS-based observer design for a wider class of polynomial fuzzy systems. First, we present the previous results in [25], i.e., an observer-based design for the polynomial fuzzy systems with the polynomial matrices A_i and B_i being independent of the states x to be estimated (shortly name it as Class I). Secondly, in this paper, we newly provide an observer-based design for a wider class of polynomial fuzzy systems with the polynomial matrices

A_i that are permitted to be dependent of the states x to be estimated (shortly name it as Class II). It should be emphasized that the design conditions derived in this paper realize the so-called separation design even for the wider class. All the design conditions discussed here are represented in terms of SOS.

It is well known that stability conditions for the T-S fuzzy system reduce to LMIs, e.g., [2]. Hence, the stability conditions can be solved numerically and efficiently by interior point algorithms, e.g., by LMI solvers. On the other hand, some kinds of control design conditions [6]-[11] for polynomial fuzzy systems reduce to SOS problems. Clearly, it is difficult to directly solve the problems by LMI solvers and the problems can be solved via the SOSTOOLS [26] and an SDP solver. Thus, SOS can be regarded as an extensive representation of LMIs. The computational method used in this paper relies on the SOS decomposition of multivariate polynomials. A multivariate polynomial $f(x(t))$ (where $x(t) \in R^n$) is an SOS if there exist polynomials $f_1(x(t)), \dots, f_k(x(t))$ such that $f(x(t)) = \sum_{i=1}^k f_i^2(x(t))$. It is clear that $f(x(t))$ being an SOS naturally implies $f(x(t)) \geq 0$ for all $x(t) \in R^n$. For more details of SOS, see [26].

The rest of the paper is organized as follows. Section II recalls a polynomial fuzzy system defined in [6]-[11] and previous result [25]. Section III discusses an SOS-based polynomial fuzzy controller and observer design for Class II. In addition, Section IV entails a design example to demonstrate the viability of our SOS design approach.

In this paper, to save the space, we employ the following short notations with respect to matrix representation.

$$\mathcal{L}\{M\} = M^T + M,$$

$$E_1 = \text{diag}[\epsilon_{11} \ \epsilon_{12} \ \dots \ \epsilon_{1s}],$$

$$E_{2i}(x) = \text{diag}[\epsilon_{2i1}(x) \ \epsilon_{2i2}(x) \ \dots \ \epsilon_{2is}(x)],$$

where M is an arbitrary square matrix. ϵ_{1k} ($k = 1, 2, \dots, s$) are positive values and $\epsilon_{2ik}(x)$ ($i = 1, 2, \dots, r$, $k = 1, 2, \dots, s$) are nonnegative polynomials such that $\epsilon_{2ik}(x) > 0$ for $x \neq 0$. ϵ_{1k} and $\epsilon_{2ik}(x)$ (E_1 and $E_{2i}(x)$) will be used as slack variables (matrices) to keep positivity of SOS conditions derived in this paper. s is the matrix size of E_1 and $E_{2i}(x)$ that are assumed to have appropriate dimensions. r is the number of fuzzy model rules.

II. PREVIOUS RESULTS

A. Polynomial fuzzy model

This subsection II-A recalls a polynomial fuzzy system defined in [6]-[11]. Consider the following nonlinear system:

$$\dot{x}(t) = f(x(t), u(t)), \quad (1)$$

where f is a smooth nonlinear function such that $f(0, 0) = 0$. $x(t) = [x_1(t) \ x_2(t) \ \dots \ x_n(t)]^T$ is the state vector and $u(t) = [u_1(t) \ u_2(t) \ \dots \ u_m(t)]^T$ is the input vector. In [6] and [9], we proposed a new type of fuzzy model with polynomial model consequence, i.e., fuzzy model whose consequent parts are represented by polynomials. Using the sector nonlinearity concept [2], we exactly represent (1) with the following polynomial fuzzy model (2). The main difference between the T-S fuzzy model [27] and the polynomial fuzzy model

is consequent part representation. The fuzzy model (2) has a polynomial model consequence.

Model Rule i :

$$\begin{aligned} &\text{If } z_1(t) \text{ is } M_{i1} \text{ and } \dots \text{ and } z_p(t) \text{ is } M_{ip} \\ &\text{then } \dot{x}(t) = A_i(x(t))x(t) + B_i(x(t))u(t), \end{aligned} \quad (2)$$

where $i = 1, 2, \dots, r$. r denotes the number of *Model Rules*. $z_j(t)$ ($j = 1, 2, \dots, p$) is the premise variable. Note that $z_j(t)$ is assumed to be independent of the states x to be estimated. In other words, each $z_j(t)$ is a measurable time-varying quantity that may be measurable external variables, outputs and/or time. The membership function associated with the i th *Model Rule* and the j th premise variable component is denoted by M_{ij} . $A_i(x(t)) \in R^{n \times n}$ and $B_i(x(t)) \in R^{n \times m}$ are polynomial matrices in $x(t)$. Therefore, $A_i(x(t))x(t) + B_i(x(t))u(t)$ is a polynomial vector. Thus, the polynomial fuzzy model (2) has a polynomial in each consequent part.

The defuzzification process of the model (2) can be represented as

$$\dot{x}(t) = \sum_{i=1}^r h_i(z(t)) \{A_i(x(t))x(t) + B_i(x(t))u(t)\}, \quad (3)$$

where

$$h_i(z(t)) = \frac{\prod_{j=1}^p M_{ij}(z_j(t))}{\sum_{k=1}^r \prod_{j=1}^p M_{kj}(z_j(t))}.$$

It should be noted from the properties of membership functions that $h_i(z(t)) \geq 0$ for all i and $\sum_{i=1}^r h_i(z(t)) = 1$. Thus, the overall fuzzy model is achieved by fuzzy blending of the polynomial system models.

It is in general difficult to separately design a polynomial controller and a polynomial observer for (3) since $A_i(x(t))$ and $B_i(x(t))$ are dependent of the states $x(t)$ to be estimated. Hence, as a first step, we introduce the following representation of polynomial fuzzy systems.

$$\dot{x}(t) = \sum_{i=1}^r h_i(z(t)) \{A_i(\rho_A(t))x(t) + B_i(\rho_B(t))u(t)\}, \quad (4)$$

where (4) reduces to (3) when $\rho_A(t) = \rho_B(t) = x(t)$. In general, we can consider three types of polynomial observer-based control according to three classes of polynomial fuzzy systems:

Class I: $\rho_A(t) = \zeta(t)$ and $\rho_B(t) = \zeta(t)$.

Class II: $\rho_A(t) = x(t)$ and $\rho_B(t) = \zeta(t)$.

Class III: $\rho_A(t) = \rho_B(t) = x(t)$.

$\zeta(t)$ is a measurable time-varying vector that may be measurable external variables, outputs and/or time. In other words, $\zeta(t)$ is assumed to be independent of the states $x(t)$ to be estimated. As we can see, Class III is the most complicated class. We have already presented a polynomial fuzzy observer design for Class I in [25]. This paper focuses on a polynomial fuzzy observer design for a wider class, i.e., for Class II. A polynomial fuzzy observer-based design for Class III will be presented in [28].

From now, to lighten the notation, we will drop the notation with respect to time t . For instance, we will employ $\mathbf{x}$ and $\hat{\mathbf{x}}$ instead of $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$, respectively, where $\hat{\mathbf{x}}(t)$ denotes the state estimated by a polynomial fuzzy observer as will be discussed later. Thus, we drop the notation with respect to time t , but it should be kept in mind that $\mathbf{x}$ and $\hat{\mathbf{x}}$ means $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$, respectively.

Next, we define the outputs for the polynomial fuzzy model as

$$\mathbf{y} = \sum_{i=1}^r h_i(\mathbf{z}) \mathbf{C}_i \mathbf{x}, \quad (5)$$

where $\mathbf{y} \in \mathbb{R}^q$ is the output.

B. Previous Result (Class I Observer Design [25])

This subsection presents the previous result discussed in [25].

Consider the following polynomial fuzzy system. The system matrices $\mathbf{A}_i$ and $\mathbf{B}_i$ depend on the vector $\boldsymbol{\zeta}$,

$$\begin{cases} \dot{\mathbf{x}} = \sum_{i=1}^r h_i(\mathbf{z}) \{ \mathbf{A}_i(\boldsymbol{\zeta}) \mathbf{x} + \mathbf{B}_i(\boldsymbol{\zeta}) \mathbf{u} \} \\ \mathbf{y} = \sum_{i=1}^r h_i(\mathbf{z}) \mathbf{C}_i \mathbf{x}, \end{cases} \quad (6)$$

where $\mathbf{y} \in \mathbb{R}^q$ denotes the output.

We design a polynomial fuzzy observer to estimate the states of (6).

$$\begin{cases} \dot{\hat{\mathbf{x}}} = \sum_{i=1}^r h_i(\mathbf{z}) \{ \mathbf{A}_i(\boldsymbol{\zeta}) \hat{\mathbf{x}} + \mathbf{B}_i(\boldsymbol{\zeta}) \mathbf{u} \\ \quad + \mathbf{L}_i(\boldsymbol{\zeta}) (\mathbf{y} - \hat{\mathbf{y}}) \} \\ \hat{\mathbf{y}} = \sum_{i=1}^r h_i(\mathbf{z}) \mathbf{C}_i \hat{\mathbf{x}}, \end{cases} \quad (7)$$

where $\hat{\mathbf{x}} \in \mathbb{R}^n$ is the state vector estimated by the fuzzy observer and $\hat{\mathbf{y}} \in \mathbb{R}^q$ is estimated output calculated from $\hat{\mathbf{y}} = \sum_{i=1}^r h_i(\mathbf{z}) \mathbf{C}_i \hat{\mathbf{x}}$.

To stabilize the system (6) and (7), we design a polynomial fuzzy controller with the state-feedback estimated by the polynomial fuzzy observer.

$$\mathbf{u} = - \sum_{i=1}^r h_i(\mathbf{z}) \mathbf{F}_i(\boldsymbol{\zeta}) \hat{\mathbf{x}} \quad (8)$$

Theorem 1 provides SOS conditions to separately design the polynomial fuzzy controller (8) and the polynomial fuzzy observer (7).

Theorem 1. [25] *If there exist positive definite matrices $\mathbf{X}_1 \in \mathbb{R}^{n \times n}$, $\mathbf{X}_2 \in \mathbb{R}^{n \times n}$ and polynomial matrices $\mathbf{M}_i(\boldsymbol{\zeta}) \in \mathbb{R}^{p \times n}$, $\mathbf{N}_i(\boldsymbol{\zeta}) \in \mathbb{R}^{n \times q}$ such that (9)~(14) are satisfied, the polynomial*

fuzzy controller (8) stabilizes the system (6) and the estimation error via the polynomial observer (7) tends to zero.

$$\mathbf{v}_1^T (\mathbf{X}_1 - \mathbf{E}_1) \mathbf{v}_1 \text{ is SOS} \quad (9)$$

$$\mathbf{v}_2^T (\mathbf{X}_2 - \mathbf{E}_2) \mathbf{v}_2 \text{ is SOS} \quad (10)$$

$$- \mathbf{v}_3^T \left(\mathcal{L} \{ \mathbf{A}_i(\boldsymbol{\zeta}) \mathbf{X}_1 - \mathbf{B}_i(\boldsymbol{\zeta}) \mathbf{M}_i(\boldsymbol{\zeta}) \} + \mathbf{E}_{3i}(\boldsymbol{\zeta}) \right) \mathbf{v}_3 \text{ is SOS} \quad (11)$$

$$- \mathbf{v}_4^T \left(\mathcal{L} \{ \mathbf{X}_2 \mathbf{A}_i(\boldsymbol{\zeta}) - \mathbf{N}_i(\boldsymbol{\zeta}) \mathbf{C}_i \} + \mathbf{E}_{4i}(\boldsymbol{\zeta}) \right) \mathbf{v}_4 \text{ is SOS} \quad (12)$$

$$- \mathbf{v}_5^T \left(\mathcal{L} \{ \mathbf{A}_i(\boldsymbol{\zeta}) \mathbf{X}_1 - \mathbf{B}_i(\boldsymbol{\zeta}) \mathbf{M}_j(\boldsymbol{\zeta}) \} + \mathcal{L} \{ \mathbf{A}_j(\boldsymbol{\zeta}) \mathbf{X}_1 - \mathbf{B}_j(\boldsymbol{\zeta}) \mathbf{M}_i(\boldsymbol{\zeta}) \} \right) \mathbf{v}_5 \text{ is SOS} \quad (13)$$

$$- \mathbf{v}_6^T \left(\mathcal{L} \{ \mathbf{X}_2 \mathbf{A}_i(\boldsymbol{\zeta}) - \mathbf{N}_i(\boldsymbol{\zeta}) \mathbf{C}_j \} + \mathcal{L} \{ \mathbf{X}_2 \mathbf{A}_j(\boldsymbol{\zeta}) - \mathbf{N}_j(\boldsymbol{\zeta}) \mathbf{C}_i \} \right) \mathbf{v}_6 \text{ is SOS} \quad (14)$$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ and $\mathbf{v}_6 \in \mathbb{R}^n$ denote vectors that are independent of $\mathbf{x}, \hat{\mathbf{x}}$ and $\boldsymbol{\zeta}$. From the solutions $\mathbf{X}_1$ and $\mathbf{M}_i(\boldsymbol{\zeta})$, we obtain polynomial feedback gains $\mathbf{F}_i(\boldsymbol{\zeta})$ as $\mathbf{F}_i(\boldsymbol{\zeta}) = \mathbf{M}_i(\boldsymbol{\zeta}) \mathbf{X}_1^{-1}$. From the solutions $\mathbf{X}_2$ and $\mathbf{N}_i(\boldsymbol{\zeta})$, we obtain polynomial observer gains $\mathbf{L}_i(\boldsymbol{\zeta})$ as $\mathbf{L}_i(\boldsymbol{\zeta}) = \mathbf{X}_2^{-1} \mathbf{N}_i(\boldsymbol{\zeta})$ as well.

Remark 1. The conditions (9), (11) and (13) are for SOS conditions of polynomial fuzzy controller design. The conditions (10), (12) and (14) are for SOS conditions of polynomial fuzzy observer design. Thus, Theorem 1 provides SOS design conditions to separately design polynomial fuzzy controllers and observers.

Remark 2. If $\mathbf{A}_i(\boldsymbol{\zeta}), \mathbf{B}_i(\boldsymbol{\zeta}), \mathbf{L}_i(\boldsymbol{\zeta})$ and $\mathbf{F}_i(\boldsymbol{\zeta})$ reduce to constant matrices in (6), (7) and (8), they reduce to the ordinary T-S fuzzy model, the T-S fuzzy controller and observer, respectively. In addition, Theorem 1 reduces to the existing LMI design conditions, e.g., [13], for the T-S fuzzy controller and observer. Hence, Theorem 1 provides more general results.

Remark 3. Currently, sum of squares programs (SOSPs) are solved by reformulating them as semidefinite programs (SDPs). SOSTOOLS automates the conversion from SOS to SDP and the SDP can be solved by a SDP solver [26]. At present, SOSTOOLS uses other free MATLAB add-ons such as SeDuMi [29] or SDPT3 [30] as the SDP solver. In this paper, we numerically find $\mathbf{X}_1, \mathbf{X}_2, \mathbf{M}_i(\boldsymbol{\zeta})$ and $\mathbf{N}_i(\boldsymbol{\zeta})$ satisfying the SOS conditions in Theorem 1 via SeDuMi in addition to SOSTOOLS. For more details of how to solve the SDPs using SeDuMi, see [26] and [29].

Remark 4. To obtain more reliable solutions for SOS conditions, we perform the following double checking throughout this paper. We first carefully check whether the command ‘sossolve’ find a solution without any error messages, i.e., $\text{pinf}=0$, $\text{dinf}=0$ and $\text{numerr}=0$, or not. If any error messages exist, we judge ‘infeasible’. After getting the feasible solutions using the command ‘sossolve’, the ‘findsos’ command is employed to check the feasibility of SOS conditions by substituting solutions into SOS conditions. We also carefully check whether the command ‘findsos’ provides a feasibility solution or not. If the command ‘findsos’ returns an infeasible result, we also judge ‘infeasible’. This double checking is important to have reliable solutions in the use of SOSTOOLS [26] and SeDuMi [29].

Remark 5. The conditions $\epsilon_{1k} > 0$, $\epsilon_{2k} > 0$, $\epsilon_{3ik}(\zeta) > 0$ and $\epsilon_{4ik}(\zeta) > 0$ for $\zeta \neq 0$ can be accommodated by sum of squares optimization in a similar way as in [31].

III. POLYNOMIAL CONTROLLER AND OBSERVER DESIGN FOR CLASS II

In Section II-B, we discussed an observer design for the polynomial fuzzy system (6) with $A_i(\zeta)$ and $B_i(\zeta)$ matrices. This section presents a more complicated class, i.e., A_i depends on the state x instead of the vector ζ . Although the separation design for Class II is difficult, we derive SOS conditions to achieve it in this section. The reason will be mentioned in Remark 6.

Consider the following polynomial fuzzy system.

$$\begin{cases} \dot{x} = \sum_{i=1}^r h_i(z) \{A_i(x)x + B_i(\zeta)u\} \\ y = \sum_{i=1}^r h_i(z) C_i x \end{cases} \quad (15)$$

We design a polynomial fuzzy observer to estimate the states of (15).

$$\begin{cases} \dot{\hat{x}} = \sum_{i=1}^r h_i(z) \{A_i(\hat{x})\hat{x} + B_i(\zeta)u \\ \quad + L_i(\hat{x})(y - \hat{y})\} \\ \hat{y} = \sum_{i=1}^r h_i(z) C_i \hat{x} \end{cases} \quad (16)$$

To stabilize the system, we design a polynomial fuzzy controller with the state-feedback estimated by the polynomial observer.

$$u = - \sum_{i=1}^r h_i(z) F_i(\hat{x}) \hat{x} \quad (17)$$

The difference between (17) and (8) is that (17) has the polynomial feedback gains in $\hat{x}$ instead of those in ζ in (8) Theorem 2 provides SOS conditions to separately design the polynomial fuzzy controller (17) and the polynomial fuzzy observer (16).

Theorem 2. If there exist positive definite matrices $X_1 \in \mathbb{R}^{n \times n}$, $X_2 \in \mathbb{R}^{n \times n}$ and polynomial matrices $M_i(\hat{x}) \in \mathbb{R}^{p \times n}$, $N_i(\hat{x}) \in \mathbb{R}^{n \times q}$ satisfying (18)~(23), the polynomial fuzzy controller (17) stabilizes the system (15) and the estimation error via the polynomial fuzzy observer (16) tends to zero.

$$v_1^T (X_1 - E_1) v_1 \text{ is SOS} \quad (18)$$

$$v_2^T (X_2 - E_2) v_2 \text{ is SOS} \quad (19)$$

$$-v_3^T \left(\mathcal{L}\{A_i(\hat{x})X_1 - B_i(\zeta)M_i(\hat{x})\} + E_{3i}(\zeta, \hat{x}) \right) v_3 \text{ is SOS} \quad (20)$$

$$-v_4^T \left(\mathcal{L}\{X_2 \bar{A}_i(x, \hat{x}) - N_i(\hat{x})C_i\} + E_{4i}(x, \hat{x}) \right) v_4 \text{ is SOS} \quad (21)$$

$$-v_5^T \left(\mathcal{L}\{A_i(\hat{x})X_1 - B_i(\zeta)M_j(\hat{x})\} + \mathcal{L}\{A_j(\hat{x})X_1 - B_j(\zeta)M_i(\hat{x})\} \right) v_5 \text{ is SOS } i < j \leq r \quad (22)$$

$$-v_6^T \left(\mathcal{L}\{X_2 \bar{A}_i(x, \hat{x}) - N_i(\hat{x})C_j\} + \mathcal{L}\{X_2 \bar{A}_j(x, \hat{x}) - N_j(\hat{x})C_i\} \right) v_6 \text{ is SOS } i < j \leq r \quad (23)$$

where $\bar{A}_i(x, \hat{x})e = A_i(x)x - A_i(\hat{x})\hat{x}$. $v_1, v_2, v_3, v_4, v_5, v_6 \in \mathbb{R}^n$ denote vectors that are independent of $x, \hat{x}$ and ζ . From the solutions X_1 and $M_i(\hat{x})$, we obtain polynomial feedback gains $F_i(\hat{x})$ as $F_i(\hat{x}) = M_i(\hat{x})X_1^{-1}$. From the solutions X_2 and $N_i(\hat{x})$, we obtain polynomial observer gains $L_i(\hat{x})$ as $L_i(\hat{x}) = X_2^{-1}N_i(\hat{x})$ as well.

Proof: Consider the estimation error, $e = x - \hat{x}$, by the observer. Then, the error system with respect to e can be represented as

$$\begin{aligned} \dot{e} &= \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) \{A_i(x)x - A_i(\hat{x})\hat{x} - L_i(\hat{x})C_j e\} \\ &= \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) \{\bar{A}_i(x, \hat{x}) - L_i(\hat{x})C_j\} e, \end{aligned}$$

where $\bar{A}_i(x, \hat{x})e = A_i(x)x - A_i(\hat{x})\hat{x}$. The augmented system with the augmented vector $x_v = [\hat{x}^T \quad e^T]^T$ is given as

$$\begin{aligned} \dot{x}_v &= \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) \\ &\times \begin{bmatrix} A_i(\hat{x}) - B_i(\zeta)F_j(\hat{x}) & L_i(\hat{x})C_j \\ 0 & \bar{A}_i(x, \hat{x}) - L_i(\hat{x})C_j \end{bmatrix} x_v \\ &= \sum_{i=1}^r h_i^2(z) G_{ii}(x, \zeta, \hat{x}) x_v \\ &+ \sum_{i=1}^r \sum_{i < j} h_i(z) h_j(z) (G_{ij}(x, \zeta, \hat{x}) + G_{ji}(x, \zeta, \hat{x})) x_v \end{aligned} \quad (24)$$

where

$$\begin{aligned} G_{ij}(x, \zeta, \hat{x}) &= \begin{bmatrix} G_{11ij}(\zeta, \hat{x}) & G_{12ij}(\hat{x}) \\ \mathbf{0} & G_{22ij}(x, \hat{x}) \end{bmatrix}, \\ G_{11ij}(\zeta, \hat{x}) &= A_i(\hat{x}) - B_i(\zeta)F_j(\hat{x}), \\ G_{12ij}(\hat{x}) &= L_i(\hat{x})C_j, \\ G_{22ij}(x, \hat{x}) &= \bar{A}_i(x, \hat{x}) - L_i(\hat{x})C_j. \end{aligned}$$

Now, consider a candidate of Lyapunov function.

$$V(x_v) = x_v^T \tilde{X} x_v, \quad (25)$$

where

$$\tilde{X} = \begin{bmatrix} \alpha X_1^{-1} & \mathbf{0} \\ \mathbf{0} & X_2 \end{bmatrix}, \quad (26)$$

α is a positive value, $X_1^{-1} \in \mathbb{R}^{n \times n}$ and $X_2 \in \mathbb{R}^{n \times n}$ are positive definite matrices. Note that $V(x_v) > 0$ at $x_v \neq \mathbf{0}$. It is clear from Lyapunov theory that the overall control system (24) is stable if it is proved that $\dot{V}(x_v) < 0$ at $x_v \neq \mathbf{0}$.

The time derivative of $V(x_v)$ along the trajectory of the system is obtained as

$$\begin{aligned} \dot{V}(x_v) &= \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) x_v^T \mathcal{L}\{\tilde{X} G_{ij}(x, \zeta, \hat{x})\} x_v \\ &= \sum_{i=1}^r h_i^2(z) x_v^T \mathcal{L}\{\tilde{X} G_{ii}(x, \zeta, \hat{x})\} x_v \\ &\quad + \sum_{i=1}^r \sum_{i < j}^r h_i(z) h_j(z) \times \\ &\quad x_v^T \mathcal{L}\{\tilde{X} (G_{ij}(x, \zeta, \hat{x}) + G_{ji}(x, \zeta, \hat{x}))\} x_v. \end{aligned}$$

If the following conditions are satisfied, $\dot{V}(x_v) < 0$ at $x_v \neq \mathbf{0}$.

$$\mathcal{L}\{\tilde{X} G_{ii}(x, \zeta, \hat{x})\} < \mathbf{0} \quad (27)$$

$$\mathcal{L}\{\tilde{X} (G_{ij}(x, \zeta, \hat{x}) + G_{ji}(x, \zeta, \hat{x}))\} \leq \mathbf{0} \quad i < j \leq r \quad (28)$$

As well as in the proof of Theorem 1, (27) can be separately rewritten as

$$\mathcal{L}\{X_1^{-1}(A_i(\hat{x}) - B_i(\zeta)F_i(\hat{x}))\} < \mathbf{0}, \quad (29)$$

$$\mathcal{L}\{X_2(\bar{A}_i(x, \hat{x}) - L_i(\hat{x})C_i)\} < \mathbf{0}. \quad (30)$$

Multiplying both side of (29) by X_1 and defining a new variable $M_i(\hat{x}) = F_i(\hat{x})X_1$, we obtain the following conditions.

$$\mathcal{L}\{A_i(\hat{x})X_1 - B_i(\zeta)M_i(\hat{x})\} < \mathbf{0} \quad (31)$$

Defining another new variable $N_i(\hat{x}) = X_2 L_i(\hat{x})$, the inequality (30) can be described as

$$\mathcal{L}\{X_2 \bar{A}_i(x, \hat{x}) - N_i(\hat{x})C_i\} < \mathbf{0}. \quad (32)$$

In the same way as above, (28) can be also represented as

$$\begin{aligned} &\mathcal{L}\{A_i(\hat{x})X_1 - B_i(\zeta)M_j(\hat{x})\} \\ &+ \mathcal{L}\{A_j(\hat{x})X_1 - B_j(\zeta)M_i(\hat{x})\} \leq \mathbf{0}, \end{aligned} \quad (33)$$

$$\begin{aligned} &\mathcal{L}\{X_2 \bar{A}_i(x, \hat{x}) - N_i(\hat{x})C_j\} \\ &+ \mathcal{L}\{X_2 \bar{A}_j(x, \hat{x}) - N_j(\hat{x})C_i\} \leq \mathbf{0} \end{aligned} \quad (34)$$

for $i < j \leq r$. It is clear from the inequality conditions (31)-(34) that $\dot{V}(x_v) < 0$ at $x_v \neq \mathbf{0}$ if the SOS conditions (18)~(23) hold. ■

Remark 6. As we can see, Theorems 1 and 2 show that the so-called separation principle is realized, i.e., that the fuzzy polynomial controller and observer can be separately designed without lack of guaranteeing the stability of the overall control system in addition to converging state estimation error (via the observer) to zero. This is a very important point in our fuzzy polynomial controller and observer design. In particular, in Theorem 2, a key feature of realizing the separation design is that, by introducing the transformation $\bar{A}(x, \hat{x})e = A(x)x - A(\hat{x})\hat{x}$, the (2,1) element in $G_{ij}(x, \zeta, \hat{x})$ becomes zero element (matrix). This transformation idea leads to the successful separation design.

IV. DESIGN EXAMPLE

Consider the following nonlinear system, where x_1 is measurable and $y = x_1$.

$$\begin{cases} \dot{x}_1 = \sin x_1 - 0.3x_2 + (x_1^2 + 1)u \\ \dot{x}_2 = -1.5x_1 - 2x_2 - x_2^3 \end{cases} \quad (35)$$

Fig. 1 shows system behaviors without inputs. It can be seen from Fig. 1 that the system is unstable.

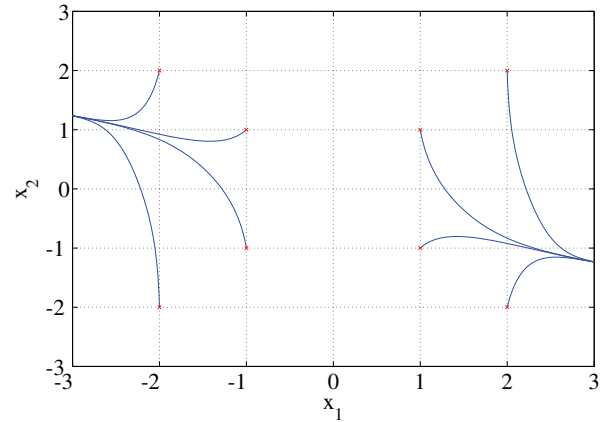


Fig. 1. System behaviors without inputs.

This system has polynomial terms $(x_1^2 + 1)u$ and x_2^3 . To obtain a T-S fuzzy model, we need to assume the ranges of x_1 and x_2 . Thus, the T-S fuzzy model is a local model. This means that the T-S fuzzy model stabilization and state-estimation convergence are not guaranteed for outside the ranges. The polynomial fuzzy model constructed in this example can exactly and globally represent the dynamics of the original system. Even if a local or semi-global T-S fuzzy model is permitted to use in practical sense, the premise variable vector z contain x_2 to be estimated. Hence, the previous LMI conditions mentioned above can not be applied to the nonlinear system. On the other hand, the premise variable vector z in

polynomial fuzzy model does not contain x_2 and x_2 appears in polynomial system matrices A_i in consequent parts of polynomial fuzzy models. Since the Class II design permits to have unmeasurable states in A_i matrices, it is possible to design a polynomial fuzzy observer in this example.

The dynamics of the nonlinear system can be exactly represented as the polynomial fuzzy system (15), where $r = 2$, $z = \zeta = y$,

$$\begin{aligned} A_1(x) &= \begin{bmatrix} 1 & -0.3 \\ -1.5 & -2 - x_2^2 \end{bmatrix}, \\ A_2(x) &= \begin{bmatrix} -0.2172 & -0.3 \\ -1.5 & -2 - x_2^2 \end{bmatrix}, \\ B_1(\zeta) &= B_2(\zeta) = \begin{bmatrix} y^2 + 1 \\ 0 \end{bmatrix}, \\ C_1 &= C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ h_1(z) &= \frac{\sin y + 0.2172y}{1.2172y}, \\ h_2(z) &= \frac{y - \sin y}{1.2172y}. \end{aligned}$$

In this example, note that

$$\begin{aligned} \bar{A}_1(x, \hat{x})e &= A_1(x)x - A_1(\hat{x})\hat{x} \\ &= \begin{bmatrix} 1 & -0.3 \\ -1.5 & -2 - x_2^2 - x_2\hat{x}_2 - \hat{x}_2^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \bar{A}_2(x, \hat{x})e &= A_2(x)x - A_2(\hat{x})\hat{x} \\ &= \begin{bmatrix} -0.2172 & -0.3 \\ -1.5 & -2 - x_2^2 - x_2\hat{x}_2 - \hat{x}_2^2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \end{aligned}$$

By solving the SOS conditions in Theorem 2, we have X_i , $M_i(\hat{x})$ and $N_i(\hat{x})$, where the orders of $M_i(\hat{x})$ and $N_i(\hat{x})$ are two.

$$\begin{aligned} X_1 &= \begin{bmatrix} 0.41696 & 0.34597e^{-6} \\ 0.34597e^{-6} & 0.50738 \end{bmatrix} \\ X_2 &= \begin{bmatrix} 0.95521 & 0.11778e^{-5} \\ 0.11778e^{-5} & 0.61866 \end{bmatrix} \\ M_1(\zeta) &= \begin{bmatrix} 0.90492 \\ 0.25275e^{-5} - 0.19247e^{-11}\hat{x}_2^2 \end{bmatrix} \\ M_2(\zeta) &= \begin{bmatrix} 0.57747 \\ 0.22252e^{-5} - 0.19251e^{-11}\hat{x}_2^2 \end{bmatrix} \\ N_1(\zeta) &= \begin{bmatrix} 1.6776 \\ -0.90461 - 0.32618e^{-5}\hat{x}_2^2 \end{bmatrix} \\ N_2(\zeta) &= \begin{bmatrix} 0.61447 \\ -0.87404 - 0.32619e^{-5}\hat{x}_2^2 \end{bmatrix} \end{aligned}$$

From the solutions X_i , $M_i(\hat{x})$ and $N_i(\hat{x})$, we obtain the

following polynomial feedback and observer gains,

$$\begin{aligned} F_1(\hat{x}) &= \begin{bmatrix} 2.17028 + 0.31476e^{-17}\hat{x}_2^2 \\ 0.35016e^{-5} - 0.37934e^{-11}\hat{x}_2^2 \end{bmatrix} \\ F_2(\hat{x}) &= \begin{bmatrix} 1.38495 + 0.31482e^{-17}\hat{x}_2^2 \\ 0.34413e^{-5} - 0.37942e^{-11}\hat{x}_2^2 \end{bmatrix} \\ L_1(y, \hat{x}) &= \begin{bmatrix} 1.75626 + 0.650097e^{-11}\hat{x}_2^2 \\ -1.46221 - 0.52724e^{-5}\hat{x}_2^2 \end{bmatrix} \\ L_2(y, \hat{x}) &= \begin{bmatrix} 0.64328 + 0.65012e^{-11}\hat{x}_2^2 \\ -1.41280 - 0.52725e^{-5}\hat{x}_2^2 \end{bmatrix} \end{aligned}$$

Fig. 2 shows the control and estimation result by the designed polynomial fuzzy controller and observer, where the initial states are $x(0) = [1 \ 1]$ and $\hat{x}(0) = [0 \ 0]$. Fig. 3 shows control trajectories for same initial states as in Fig. 1. It can be seen from the figures that the designed controller stabilizes the nonlinear system and the estimation error via the polynomial fuzzy observer tends to zero.

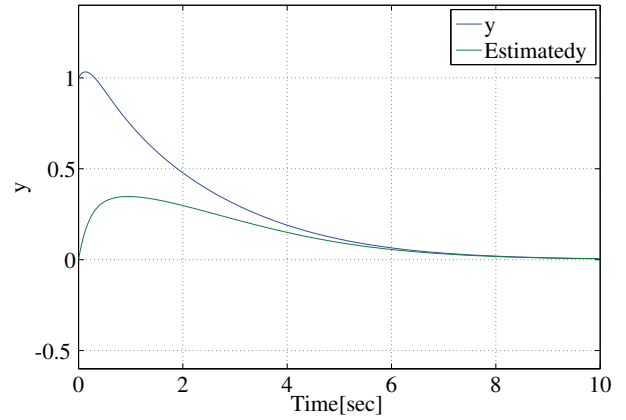


Fig. 2. Control and estimation result.

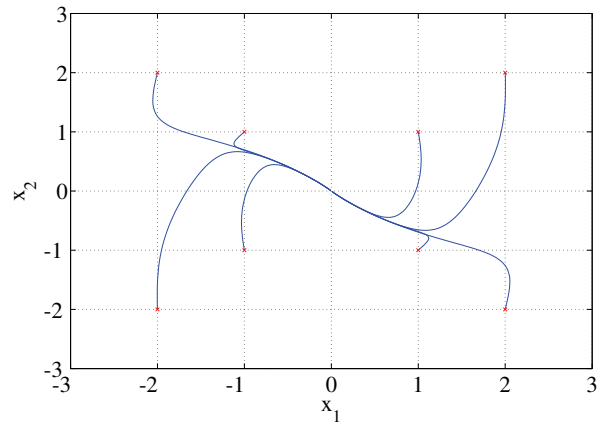


Fig. 3. Control trajectories for same initial states as in Fig. 1.

Remark 7. Since $A_1(x)$ and $A_2(x)$ have unmeasurable x_2 in this design example, the Class I SOS-based observer design (Theorem 1) can not be applied to this design example. The previous LMI conditions mentioned above can not be also applied to the nonlinear system. On the other hand, since the Class II design (Theorem 2) permits to have unmeasurable states in A_i matrices, it is possible to design a polynomial fuzzy observer in this example.

V. CONCLUSIONS

This paper has presented a sum of squares (SOS) approach to polynomial fuzzy controllers and observers for a wider class of polynomial fuzzy systems. To illustrate the validity and applicability of the proposed approach, a design example has been provided. The example has demonstrated advantages of the SOS-based approaches for the existing LMI approaches to T-S fuzzy observer designs.

Our next subjects are to derive SOS observer design conditions even for Class III [28] and to apply our observer designs to helicopter control [11].

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