

Patent Valuation with a Fuzzy Binomial Model

Xiaolu Wang

Institute for Advanced Management Systems Research

Åbo Akademi University

Turku, Finland

Email: xiaolu.wang@abo.fi

Abstract—Patent valuation has always been a difficult task due to the considerable uncertainty involved in both the patent application procedure and the patent exploitation. This paper proposes, under the fuzzy framework, a multiperiod binomial approach for patent valuation of the latter case. We depart from the model introduced in S. Muzzioli et al. [1] by utilizing binomial tree to simulate the movement of patent monopoly benefits through time and representing the volatility of the patent monopoly benefits with fuzzy numbers, which is triggered by the fact that, in most cases, the Intellectual Property Rights (IPR) managers may not have good information on the parameters, and that the conventional probability theory is incapable of capturing this type of imprecision. Our model also reveals the values of the managerial flexibilities embedded in the patent exploitation.

Keywords—Patent valuation; fuzzy sets; American call option

I. INTRODUCTION

Patented inventions have played a very important role in human life and, in fact, pervaded every aspect of it, from telephone (patent held by Alexander Graham Bell) and first computer (patent held by Herman Hollerith), to insulin (patent held by the University of Toronto) and holography (patent held by Gábor Dénes). Having been granted the right to negate the use of the underlying inventions by others, which is also known as the underlying intellectual asset, the patent holders are obliged to disclose information on their inventions publicly for exploitation and further improvement by other researchers and inventors, which “enriches the total body of technical knowledge in the world, and therefore promotes further creativity and innovation in others” [2]. In addition to the monopolistic status, another principal incentive to file for patents is the interdependency between patent protection and research and development (R&D). On the one hand, powerful patent protection stimulates and facilitates economically efficient R&D, and the investments in R&D, on the other hand, require legal protection for its outputs. E. Eicher et al. [3] describes this interdependency between expenditures in IPR protection and research investments as follows “a lower degree of protection reduces the returns from research and hence the incentives to do R&D; similarly, a low level of R&D will reduce the return to investment in IPR protection”. Therefore, it is of great interest to not only the patent applicants, but also those who manage granted patents and are to make crucial decisions about their management to know the values of both the underlying inventions and their patents (if granted) accurately and adequately. It is worth emphasizing that projects of commercializing the underlying inventions and patents protecting

those inventions are two distinct entities [4], and in our study, we focus on the latter alone. Additionally, the patent valuation model to be presented is considered from the microeconomic perspective, that is to say, it is designed to evaluate individual patents, patent families and patent portfolios instead of the benefits they bring to the economy as a whole.

Patent valuation, however, is often a difficult task, primarily because both patent application process and patent exploitation are subject to various choices and decisions at a number of successive stages and thus involve a particularly high degree of flexibility. For example, patents can be licensed, sold, abandoned at any time in the post-grant phase or simply kept in force for own use by paying the annual renewal fee. Even the patent applicants may confront the decisions as to whether to file a foreign application and whether to continue with the ongoing patent application at several decision points. It therefore makes perfect sense to employ real option valuation methods in hope of capturing the embedded flexibilities and thus acquiring a more accurate estimate of the value of the underlying patents. There has been much research on patent valuation that is based on real options-thinking and can be categorized into different subsets according to the methodologies that have been used. A. Pakes [5] considers the post-grant phase of a patent as *a series of call options* on the monopoly benefits of the coming year with the strike price of each option being the next renewal fee. He also claims that the IPR manager will renew the patent only if the monopoly benefits over the coming year plus the value of the option to renew the patent at the end of the year exceeds the renewal fee. H. Weeds [6] takes into consideration the possibilities of sleeping patents and studies a two-stage investment project, with research carried out and then a patent filed for the new invention in the first stage and irreversible market entry in the second, and models the two-stage problem as *real sequential compound options*. In more specific terms, exercising the option to conduct research and file a patent for any resultant invention buys the inventor a second option to commercialize the innovation. In addition, extensive literature on the management of patenting decisions has combined game theory and real option theory to study the effect of *strategic interactions* between different firms, as in R. Meng [7]. This is promoted by the fact that, in many real-world asset markets, the firms under consideration are not necessarily “monopolists or entities in a perfectly competitive market” [7], and therefore “exercise strategies cannot be determined separately, but must be formed as part of a strategic equilibrium” [8].

Another problem which could arise when we deal with patent valuation is that the volatility of the underlying asset, which is assumed to be constant in the Discounted Cash Flow (DCF) analysis and Black-Scholes formula, is likely to vary drastically throughout both the patent application process and the life of the patent since “the fact that the patent has survived thus far makes it increasingly likely that it will be successful and profitable” [4]. This therefore provides the incentives for using fuzzy set theory to better formalize these uncertainties as “fuzzy sets are sets that allow gradation of belonging and serve as a representation of uncertain or imprecise knowledge” [9]. M. Collan et al. [10] introduced patent valuation with the fuzzy pay-off method [9], which is based on real option-thinking and specifically suitable for asset valuation involving uncertainty on the future cash-flow and estimation inaccuracy. E. Agliardi [11] also combined a real option to litigate with fuzzy valuation due to the unpredictable outcomes of trials.

Our study is an extension of the ideas of applying the conventional multiperiod binomial model to patent valuation, which is also juiced up with the work of S. Muzzioli et al. [1]. The original idea was triggered by the observation that the managerial flexibilities embedded in patents resemble the possibilities financial options grant their holders, and more specifically, that a patent asset is like an American call option with the underlying asset being the cash flow generated from the patent, which can be priced under the risk-neutral valuation paradigm by adopting numerical methods of finance such as the binomial model. In Section (II), we briefly introduce the multiperiod binomial pricing model for American (call) options and continue shortly on fuzzy sets and fuzzy numbers. Section (III) is devoted to the proposed patent valuation method. The last section concludes and suggests the possible future work.

II. PRELIMINARIES

A. The Binomial Model for the Pricing of an American Option

An American option is a financial instrument which gives its holder the right, but not the obligation, to trade in a certain amount of underlying assets at a predetermined price at or before a specified date. A call (resp. put) option gives the right to buy (resp. sell) the underlying assets. The Cox-Ross-Rubinstein (or CRR) model presented in Cox et al. [12] is a simple discrete-time binomial model for option pricing, which allows preference-free or arbitrage pricing of every contingent claim¹, and only requires the following inputs: the strike price K , the present value of the underlying asset S_0 , the number of time steps to maturity n , the risk-free interest rate r and the upward and downward jumping factors of the underlying asset price u and d , respectively. Cox et al. [12] also makes the following fundamental assumptions:

- 1) The asset price satisfies a multiplicative binomial process over discrete periods.

- 2) There are no transaction costs, margin requirements or taxes.
- 3) The risk-free interest rate r is constant.
- 4) In order to rule out arbitrage opportunities, it is assumed that $d < 1 + r < u$.
- 5) The model is complete.

In addition, we assume for simplicity that no dividends will be paid during the entire life cycle of the option.

First we look at the single-period binomial model, which is $n = 1$. In a complete market, the fair price of every contingent claim H is equal to the initial value of the unique replicating portfolio V_0 , which can be calculated explicitly as the expected value of the discounted claim $\bar{H}$ under the risk-neutral probability measure Q defined by $(q, 1 - q)$ such that $Q(S_1 = uS_0) = q$ and $Q(S_1 = dS_0) = 1 - q$. Therefore, for a single-period binomial model, we have

$$V_0 = \mathbb{E}_Q[\bar{H}].$$

The discounted asset price also has constant expectation under the risk-neutral probability measure Q so as to fulfill the no-arbitrage assumption, which implies²

$$S_0 = \mathbb{E}_Q[\bar{S}_1]$$

that is

$$1 = q \frac{u}{1+r} + (1-q) \frac{d}{1+r}$$

leading to

$$q = \frac{(1+r) - d}{u - d}, \quad (1)$$

$$1 - q = \frac{u - (1+r)}{u - d}. \quad (2)$$

The upward and downward jumping factors of the underlying asset price are approximated by (see Cox et al. [12])

$$u = e^{\sigma\sqrt{\Delta t}}, \quad (3)$$

$$d = e^{-\sigma\sqrt{\Delta t}}, \quad (4)$$

where σ is the volatility of the underlying asset.

For the multiperiod binomial model, the replicating portfolio at each stage is formed exactly in the same way as in the single-period model. Therefore, without loss of generality, the values of the American (call) option at the terminal and intermediate stages can be determined as follows:

$$\begin{aligned} V_n^i &= (S_n^i - K)^+, \\ S_n^i &= S_0 u^i d^{n-i}, \quad i = 0, \dots, n, \\ V_m^j &= \max \left(S_m^j - K, \frac{1}{1+r} \left(q V_{m+1}^{j+1} + (1-q) V_{m+1}^j \right) \right), \\ S_m^j &= S_0 u^j d^{m-j}, \quad m = 0, \dots, n-1, \quad j = 0, \dots, m. \end{aligned}$$

²We compute the risk-neutral probabilities based on the risk neutrality assumption which gives the same result as the replicating portfolio approach (or the arbitrage pricing theory approach), which is presented in Elliott and Kopp [13] and Shreve [14].

¹“A contingent claim represents the potential liability inherent in a derivative security”. (Elliott and Kopp [13])

B. Fuzzy Sets and Fuzzy Numbers

Fuzzy sets serve as a approach of representing vague concepts expressed in natural language [15]. It provides us with a tool of describing the ambiguous notions, such as “the weather today is sunny” and “someone is tall”. Mathematically, a fuzzy set A in a given universal set X can be defined as a set of ordered pairs, with the first element of each pair from X and the second element being its grade of membership standing for the degree to which the individual in the universe corresponds to the concept represented by the fuzzy set A , and with exactly one ordered pair assigned to each element of X . This implies the following membership function of A

$$\mu_A = X \rightarrow [0, 1],$$

where 0 and 1 represent non-membership and complete membership respectively, while the values in between represent the intermediate degrees of membership.

The γ -cut is an important concept associated with fuzzy sets, which is defined as follows:

Definition 1. A γ -cut (or γ -level set) of a fuzzy set A of X is a non-fuzzy set denoted by $[A]^\gamma$ and defined by

$$[A]^\gamma = \begin{cases} \{x \in X \mid A(x) \geq \gamma\}, & \text{if } \gamma > 0 \\ cl(suppA), & \text{if } \gamma = 0 \end{cases}$$

where $cl(suppA)$ denotes the closure of the support of A , which has the following form

$$supp(A) = \{x \in X \mid A(x) > 0\}.$$

Before we can bring up the definition of fuzzy numbers, the following concepts are needed first,

Definition 2. [16] A fuzzy set A of X is called convex if $[A]^\gamma$ is a convex subset of X for all $\gamma \in [0, 1]$.

Definition 3. [16] A fuzzy set A of a classical set X is called normal if there exists an $x \in X$ such that $A(x) = 1$. Otherwise A is subnormal.

Essentially fuzzy numbers are nothing more than a special type of fuzzy sets which are defined on the real line $\mathbb{R}$, and their membership functions have the form

$$\mu_A = \mathbb{R} \rightarrow [0, 1].$$

Therefore they are generally used to “capture our intuitive conceptions of approximate numbers or intervals” [15]. For example a triangular fuzzy number (with center a) and a trapezoidal fuzzy number (with core $[a, b]$) can be seen as fuzzy quantities “ x is approximately equal to a given real number a ” and “ x is approximately in a given interval $[a, b]$ of real numbers”, respectively.

Definition 4. [16] A fuzzy number A is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support.

Remark 1. The terms membership function and fuzzy set and their notations can be interchangeably used since each fuzzy set is uniquely defined by its membership function [15].

Definition 5. A fuzzy set A is called a triangular fuzzy number with center (or peak) a , left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form

$$A(x) = \begin{cases} 0, & \text{for } x < a - \alpha \text{ and } x > a + \beta \\ 1 - \frac{a-x}{\alpha}, & \text{for } a - \alpha \leq x \leq a \\ 1 - \frac{x-a}{\beta}, & \text{for } a < x \leq a + \beta \end{cases}$$

and can be denoted by $A = (\alpha, a, \beta)$. It is easy to show that

$$\begin{aligned} [A]^\gamma &= [a - (1 - \gamma)\alpha, a + (1 - \gamma)\beta] \\ &= [a - \alpha + \gamma\alpha, a + \beta - \gamma\beta] \end{aligned}$$

for all $\gamma \in [0, 1]$, and therefore the support of A is $(a - \alpha, a + \beta)$.

Definition 6. A fuzzy set A is called a trapezoidal fuzzy number with core (or tolerance interval) $[a, b]$, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form

$$A(x) = \begin{cases} 0, & \text{for } x < a - \alpha \text{ and } x > a + \beta \\ 1 - \frac{a-x}{\alpha}, & \text{for } a - \alpha \leq x < a \\ 1, & \text{for } a \leq x \leq b \\ 1 - \frac{x-b}{\beta}, & \text{for } b < x \leq b + \beta \end{cases}$$

and can be denoted by $A = (a, b, \alpha, \beta)$. It is easy to verify that

$$\begin{aligned} [A]^\gamma &= [a - (1 - \gamma)\alpha, b + (1 - \gamma)\beta] \\ &= [a - \alpha + \gamma\alpha, b + \beta - \gamma\beta] \end{aligned}$$

for all $\gamma \in [0, 1]$, and therefore the support of A is $(a - \alpha, b + \beta)$.

III. THE FUZZY BINOMIAL MODEL FOR PATENT VALUATION

Before proceeding to the discussion of the proposed model, it is worth emphasizing the concept of patents first. A patent can be defined as a set of exclusive rights, usually granted by national governments, to make exclusive use of a new, non-obvious invention for a predetermined (limited) period of time in return for publication of the underlying invention. Generally speaking, most patent applicants and patentees face the following four basic decision types at different stages [4]

- 1) Whether to file a patent application.
- 2) Whether to continue with the ongoing patent application at a few decision points.
- 3) Whether to maintain any patent granted in effect or abandon it.
- 4) Whether and how to exploit the patent after it is granted.

Here we only consider decision type 4, that is to say, we want to maintain the patent granted by paying renewal fees annually and look to make extra profits by licensing, besides monopoly benefits due to holding the patent.

In this study, a fuzzy binomial model is adopted to simulate the movement of patent monopoly benefits through time. As mentioned above, the opportunity of licensing for extra revenues is also taken into consideration on a regular basis,

which is usually once a year. That is, the patent would be evaluated annually and licensed to other investors if the monopoly benefits are high enough. This, however, would have an effect upon the existing competitive edge to some extent and would bring down the monopoly profits accordingly. Therefore, this flexibility takes the form of enabling the patent to be licensed in exchange for licensing fees and royalties on incurring a reduction in the monopoly profits, which is very similar to an embedded American option.

To capture the side effect on the monopoly benefits due to patent licensing, we introduce a triangular fuzzy discount factor³ $\delta = (\alpha_\delta, a_\delta, \beta_\delta)$, which, by the conception of fuzzy numbers presented in Section (II-B), represents “the set of real numbers close to a_δ ” [15]. This is to be determined according to experts’ estimation and has the following form

$$\begin{aligned}\delta &= [\delta_{\min}, \delta_{\max}] \\ &= [a_\delta - \alpha_\delta, a_\delta + \beta_\delta],\end{aligned}$$

where $0 < a_\delta - \alpha_\delta < a_\delta + \beta_\delta < 1$.

A triangular fuzzy number $\sigma = (\alpha_\sigma, a_\sigma, \beta_\sigma)$ is also utilized to represent the volatility of the monopoly benefits, which, for simplicity of the notations, is defined in terms of its γ -cut by

$$\begin{aligned}\sigma &= [a_\sigma - (1 - \gamma)\alpha_\sigma, a_\sigma + (1 - \gamma)\beta_\sigma] \\ &= [a_\sigma - \alpha_\sigma + \gamma\alpha_\sigma, a_\sigma + \beta_\sigma - \gamma\beta_\sigma],\end{aligned}\quad (5)$$

where $\gamma \in [0, 1]$. According to (3) and (4), the fuzzy⁴ upward and downward jumping factors can be determined as follows:

$$\begin{aligned}u &= [u_{\min}, u_{\max}] \\ &= [e^{a_\sigma - \alpha_\sigma + \gamma\alpha_\sigma}, e^{a_\sigma + \beta_\sigma - \gamma\beta_\sigma}] \\ d &= [d_{\min}, d_{\max}] \\ &= [e^{-(a_\sigma + \beta_\sigma - \gamma\beta_\sigma)}, e^{-(a_\sigma - \alpha_\sigma + \gamma\alpha_\sigma)}]\end{aligned}$$

where $\Delta t = 1$ in our case since the evaluation is performed annually. Assumption 4. in Section (II-A) implies that

$$d_{\max} < 1 + r < u_{\min}$$

which is

$$-(a_\sigma - \alpha_\sigma) < \ln(1 + r) < a_\sigma - \alpha_\sigma$$

when γ is set to be 0.

Next we adopt the approach presented in S. Muzzioli et al. [1] to determine the fuzzy risk-neutral probabilities. By using (1) and (2), we have

$$\frac{\partial q}{\partial u} = \frac{\partial \left(\frac{(1+r)-d}{u-d} \right)}{\partial u} = \frac{d - (1+r)}{(u-d)^2} < 0$$

and

$$\frac{\partial q}{\partial d} = \frac{\partial \left(\frac{(1+r)-d}{u-d} \right)}{\partial d} = \frac{(1+r) - u}{(u-d)^2} < 0,$$

which implies that the maximum (resp. minimum) of q is obtained at $u_{\min}$ and $d_{\min}$ (resp. $u_{\max}$ and $d_{\max}$). Analogously, it is easy to verify

$$\frac{\partial (1-q)}{\partial u} = \frac{\partial \left(\frac{u-(1+r)}{u-d} \right)}{\partial u} = \frac{(1+r) - d}{(u-d)^2} > 0$$

and

$$\frac{\partial (1-q)}{\partial d} = \frac{\partial \left(\frac{u-(1+r)}{u-d} \right)}{\partial d} = \frac{u - (1+r)}{(u-d)^2} > 0,$$

that is, the maximum (resp. minimum) of $1 - q$ is obtained at $u_{\max}$ and $d_{\max}$ (resp. $u_{\min}$ and $d_{\min}$). Therefore, the fuzzy risk-neutral probabilities are as follows:

$$\begin{aligned}q &= [q_{\min}, q_{\max}] \\ &= \left[\frac{(1+r) - d_{\max}}{u_{\max} - d_{\max}}, \frac{(1+r) - d_{\min}}{u_{\min} - d_{\min}} \right], \\ 1 - q &= [(1-q)_{\min}, (1-q)_{\max}] \\ &= \left[\frac{u_{\min} - (1+r)}{u_{\min} - d_{\min}}, \frac{u_{\max} - (1+r)}{u_{\max} - d_{\max}} \right],\end{aligned}$$

and they satisfy the following equalities

$$q_{\min} + (1-q)_{\max} = 1 \text{ and } q_{\max} + (1-q)_{\min} = 1.$$

By applying the fuzzy risk-neutral probabilities and the fuzzy upward and downward jumping factors, our model becomes⁵

$$\begin{aligned}V_n^i &= \max(S_n^i - R, \delta S_n^i - R + L), \\ S_n^i &= S_0 [u_{\min}^i, u_{\max}^i] \otimes [d_{\min}^{n-i}, d_{\max}^{n-i}], \quad i = 0, \dots, n, \\ V_m^j &= \max \left(\delta S_m^j - R + L, \frac{1}{1+r} \left([q_{\min}, q_{\max}] V_{m+1}^{j+1} \oplus \right. \right. \\ &\quad \left. \left. [(1-q)_{\min}, (1-q)_{\max}] V_{m+1}^j \right) \right), \\ S_m^j &= S_0 [u_{\min}^j, u_{\max}^j] \otimes [d_{\min}^{m-j}, d_{\max}^{m-j}], \\ &\quad m = 0, \dots, n-1, j = 0, \dots, m,\end{aligned}$$

⁵We define the arithmetic operations on fuzzy number and the ordering of fuzzy numbers as follows: let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two fuzzy numbers, we define

$$\begin{aligned}A \oplus B &= [a_1 + b_1, a_2 + b_2], \\ A \ominus B &= [a_1 - b_2, a_2 - b_1], \\ A \otimes B &= [a_1 b_1, a_2 b_2],\end{aligned}$$

and, provided that $0 \notin [b_1, b_2]$,

$$A \oslash B = [a_1/b_1, a_2/b_2];$$

let $c \in \mathbb{R}$, then we define

$$\begin{aligned}\max(A, B) &= [\max(a_1, b_1), \max(a_2, b_2)], \\ \max(A, c) &= [\max(a_1, c), \max(a_2, c)].\end{aligned}$$

³Triangular fuzzy numbers are adopted for simplicity.

⁴See G. J. Klir et al. [15] for the characteristics of arithmetic operations on fuzzy numbers. From this point forward, all fuzzy numbers are represented by the bounded intervals, which should not be confused with the γ -cut.

where

- V = the fuzzy patent values
- S_0 = the present value of the monopoly benefits from the underlying patent
- R = annual patent renewal fees
- L = patent licensing fees and royalties⁶
- δ = the fuzzy discount rate of the monopoly profits due to patent licensing
- n = time periods to patent maturity.

Note that, by adjusting γ in (5), this valuation model is capable of providing the patentees varying weighted intervals of the fair price of the patent. If $\gamma = 1$, the bounded interval collapses to one crisp number which is the “most likely” [17] patent value.

IV. DISCUSSION AND CONCLUSIONS

In this paper, we extend the ideas of applying the conventional multiperiod binomial model to patent valuation by taking into consideration the uncertainty on the volatility of the patent monopoly benefits, which is formalized in our study through fuzzy set-based methodology. We started from determining the fuzzy upward and downward jumping factors according to Cox et al. [12], which leads to the fuzzy risk-neutral probabilities calculated by following the method presented in S. Muzzioli et al. [1]. Additionally, we take into consideration the side effect of patent licensing through introducing a fuzzy discount factor on the monopoly benefits. Together with the derived fuzzy risk-neutral probabilities, the patent value is determined in the similar way as in the American option pricing method.

Our model allows direct input of estimates of the volatility parameter, instead of assuming an interval of possible values for the upward and downward jumping factors respectively as in S. Muzzioli et al. [1] and [17]. However, the single option to license the underlying patent is a limitation of this model, since the option of direct commercialization, a combination of commercialization and licensing, outright sale or abandonment [4] may also exist in practical cases. For example, S.-H. Liao et al. [18] have proposed an approach to evaluate investment project under uncertainty, which has taken into account the managerial flexibility to expand or contract its scale of operation at a specific time.

In our model, we only consider the case when the underlying patent has been granted and we want to exploit it while keeping the patent in force. However, the decision type 1 and 2 mentioned in Section (III) are also of great interest to IPR managers as the patent application procedure is not just “a simple investment project with initial costs and near certain future returns” [4] but a very complex process, each stage of which involves cost, actual profits or potential future profits with great uncertainties and decision-making points of whether to continue the application or not. Therefore, patent

application valuation is intuitively the natural candidate for our future work.

REFERENCES

- [1] S. Muzzioli and H. Reynaerts, “American option pricing with imprecise risk-neutral probabilities,” *International Journal of Approximate Reasoning*, vol. 49, 2008, pp. 140–147.
- [2] (2011, Jan.) Frequently Asked Questions: What Role do Patents Play in Everyday Life? [Online]. Available: http://www.wipo.int/patentscope/en/patents_fa.html#protection
- [3] T. Eicher and C. Garca-Pealosa, “Endogenous strength of intellectual property rights: Implications for economic development and growth,” *European Economic Review*, vol. 52, 2008, pp. 237–258.
- [4] R. Pitkethly, “The valuation of patents: a review of patent valuation methods with consideration of option based methods and the potential for further research,” Judge Institute Working Paper WP 21/97, The Judge Institute of Management Studies, Trumpington Street, Cambridge CB2 1AG.
- [5] A. Pakes, “Patents as options: some estimates of the value of holding European patent stocks,” *Econometrica*, vol. 54, no. 4, 1986, pp. 755–784.
- [6] H. Weeds, “Sleeping patents and compulsory licensing: an options analysis,” *Warwick economic research papers*, no. 577, 1999, 34 pages.
- [7] R. Meng, “A patent race in a real options setting: investment strategy, valuation, CAPM beta, and return volatility,” *Journal of Economic Dynamics & Control*, vol. 32, 2008, pp. 3192–3217.
- [8] S. R. Grenadier, “Option exercise game: an application to the equilibrium investment strategies of firms,” *Rev. Financ. Stud.*, vol. 15, 2002, pp. 691–721.
- [9] M. Collan, R. Fullér and J. Mezei, “A fuzzy pay-off method for real option valuation,” *Journal of Applied Mathematics and Decision Sciences*, 2009, Article ID 238196, 14 pages.
- [10] M. Collan and M. Heikkilä, “Valuation of patents with the pay-off method,” 5th International Forum on Knowledge Asset Dynamics, July 2010, Matera, Italy.
- [11] E. Agliardi, “The fuzzy value of patent litigation under imprecise information,” 12th International Conference on Real Options: Theory and Applications, July 2008, Rio de Janeiro, Brazil.
- [12] J. Cox, S. Ross and M. Rubinstein, “Option pricing: a simplified approach,” *Journal of Financial Economics*, vol. 7, 1979, pp. 229–263.
- [13] R. Elliott and P. Kopp, *Mathematics of Financial Markets*, Springer Verlag, 1999.
- [14] S. E. Shreve, *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*, Springer Verlag, 2004.
- [15] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall PTR, 1995.
- [16] R. Fullér, *Fuzzy Reasoning and Fuzzy Optimization*, Turku Centre for Computer Science, TUCS General Publication No 9, September 1998.
- [17] S. Muzzioli and C. Torricelli, “A multiperiod binomial model for pricing options in a vague world,” *Journal of Economic Dynamics & Control*, vol. 28, 2004, pp. 861 – 887.
- [18] S.-H. Liao and S.-H. Ho, “Investment project valuation based on a fuzzy binomial approach,” *Information Sciences*, vol. 180, 2010, pp. 2124–2133.

⁶Both patent licensing fees and royalties are assumed to be constant for simplicity.