

Min-Max and Two-Stage Possibilistic Combinatorial Optimization Problems

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Abstract—This paper deals with a class of combinatorial optimization problems with uncertain costs. The uncertainty is modeled by specifying a scenario set containing a finite number of possible realizations of the costs called scenarios. Additionally, a possibility distribution on the scenario set can be defined. Two robust models, namely the min-max and two-stage, for hedging against uncertainty of the costs in the possibilistic setting are considered. A general framework for solving the problems is proposed. For the linear sum objective a mixed integer programming formulation is shown. For the bottleneck objective, an algorithm is constructed which runs in polynomial time if the deterministic problem, i.e. the one with a single scenario, is polynomially solvable.

Index Terms—Min-Max, Uncertain Cost, Fuzzy Optimization, Possibility Theory.

I. INTRODUCTION

A combinatorial optimization problem discussed in this paper consists of a finite set of elements E and a set Φ of subsets of E , called the set of *feasible solutions*. In the deterministic case, each element in E has a precise cost and we wish to find a feasible solution whose total cost is minimal (the *linear sum* objective) or whose the largest element cost is minimal (the *bottleneck* objective). This formulation encompasses a large variety of problems. It contains, for example, the important class of *network problems*, where E is a set of the edges of a given graph G and Φ consists of some objects in G such as trees, paths or matchings (see, e.g., [1]).

The usual assumption in combinatorial optimization is that the element costs are precisely known. However, in real life this requirement is seldom satisfied and, in many optimization models, a possible uncertainty of costs must be taken into account. In general, this uncertainty can be modeled by specifying a *scenario set* which contains all possible cost realizations, called *scenarios*. This scenario set can be defined in several ways. In the *discrete scenario representation*, which is considered in this paper, we simply list all the possible scenarios. In the *interval scenario representation*, we specify for each cost a closed interval containing all its values. This means that we neither know the exact costs, nor can we set them precisely. The scenario set is then the Cartesian product of all the uncertainty intervals (see, e.g., [2]). For this representation, it is assumed that the costs are unrelated, that

is the value of a cost does not depend on the values of the remaining costs. The interval scenario representation can be generalized by assuming that the scenario set is any convex set (see, e.g., [3]), which allows us to model the relationships between the costs. We can also use a combination of discrete and interval uncertainty representations (see, e.g., [4]).

A criterion of choosing a solution depends on additional information provided with the scenario set. If a probability distribution on the scenario set is specified, then the *stochastic optimization* can be used [5], where the typical goal is to optimize the expected cost of a solution built. If there is no probability distribution on the scenario set, then the *robust optimization* can be applied [2], [6]. In order to choose a solution in this setting, the *min-max* criterion can be used. Under this criterion, we seek a solution which minimizes the largest cost over all scenarios. We can consider a *single-stage model* in which a complete solution must be determined before a cost scenario is observed (see, e.g., [2]) or a *two-stage model*, where only a partial solution must be computed when the costs are precisely known and completed when a cost scenario is observed (see, e.g., [7], [8]). Finally, if a possibility distribution on the scenario set is given, then the *fuzzy optimization* is appropriate [9], [10], [11]. In the fuzzy optimization, the goal is to choose a solution optimizing a criterion based on the *possibility* or *necessity* measures.

In this paper, we consider the discrete scenario uncertainty representation. So, we assume that the scenario set contains a finite number of distinct scenarios (element cost realizations). A decision maker can additionally specify a possibility distribution on the scenario set which describes more or less plausible scenarios. Namely, she/he can assign a number from the interval $[0,1]$ to each scenario, which is the possibility of the event that this scenario will occur. Such a more elaborate description of scenario sets is often required in practical applications, where decision makers may know likely scenarios, but the probability distributions are unknown or there is lack of statistical data validating the choice of the probability distributions. In this paper, we propose the solution concepts which generalize the single and two stage models with the min-max criterion. We thus extend the robust approach to the fuzzy one by using *possibility theory* [12] which offers

a framework of decision making under uncertainty and allows us to extend the robust models in a natural way.

II. COMBINATORIAL OPTIMIZATION PROBLEM WITH DETERMINISTIC COSTS

Let $E = \{e_1, \dots, e_n\}$ be a finite ground set and let $\Phi \subseteq 2^E$ be a set of subsets of E called the set of the *feasible solutions*. A nonnegative cost c_e is given for each element $e \in E$. In a combinatorial optimization problem $\mathcal{P}$ with the linear sum objective, we seek a solution $X \in \Phi$ whose total cost, $f(X) = \sum_{e \in X} c_e$, is minimal. In a combinatorial optimization problem $\mathcal{BP}$ with the bottleneck objective, we wish to find a solution $X \in \Phi$ whose largest element cost, $f(X) = \max_{e \in X} c_e$, is minimal. The formulations $\mathcal{P}$ and $\mathcal{BP}$ encompass a large variety of classical combinatorial optimization problems such as the shortest path, the minimum spanning tree, the minimum cost matching, the 0-1 knapsack *etc.* (see, e.g. [13], [1], [14]). The problem $\mathcal{P}$ is often formulated as the following mathematical programming problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^n c_i x_i \\ & (x_1, \dots, x_n) \in \Phi' \\ & x_i \in \{0, 1\} \quad i = 1, \dots, n, \end{aligned} \quad (1)$$

where x_i is a binary variable equal to 1 if and only if element e_i belongs to the solution computed. Thus Φ' is the set of all characteristic vectors of Φ and the condition $(x_1, \dots, x_n) \in \Phi'$ is specified in the form of constraints involving variables $x_1, \dots, x_n$. It is worth pointing out that many important combinatorial problems are NP-hard (see e.g., [1]). The framework proposed in this paper can also be applied to the NP-hard problems if one can represent them in the form of (1).

III. BASIC NOTIONS OF POSSIBILITY THEORY

Possibility theory [12] is an approach for dealing with incomplete information and is based on two dual measures: *possibility* and *necessity*, which express plausibility and certainty of events. Both measures are built from a *possibility distribution*. Let Z be a real valued quantity whose value is not precisely known. We specify for Z a *possibility distribution* π_Z , which describes the set of more or less plausible, mutually exclusive values of Z . The value of $\pi_Z(z) = \Pi(Z = z)$ is a possibility of the event that Z will take the value of z . In particular, $\pi_Z(z) = 0$ means that $Z = z$ is impossible and $\pi_Z(z) > 0$ means that $Z = z$ is plausible. The possibility distribution encodes a family of probability functions and a degree of possibility can be viewed as an upper bound on a degree of probability [15]. A detailed interpretation of the possibility distribution and some methods of obtaining it from a knowledge possessed about Z are described in [12].

Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function $\mu_{\tilde{A}}$. Then $Z \in \tilde{A}$ is a *fuzzy event* and we can determine the possibility that this event will occur in the following way:

$$\Pi(Z \in \tilde{A}) = \sup_{z \in \mathbb{R}} \min\{\pi_Z(z), \mu_{\tilde{A}}(z)\}. \quad (2)$$

Thus $\Pi(Z \in \tilde{A})$ evaluates the extent to which “ $Z \in \tilde{A}$ ” is possibly true. We can also determine the necessity of the event

$Z \in \tilde{A}$ in the following way:

$$N(Z \in \tilde{A}) = 1 - \Pi(Z \in \tilde{A}'), \quad (3)$$

where $\tilde{A}'$ is the complement of $\tilde{A}$ with the membership function $1 - \mu_{\tilde{A}}(z)$. Hence, $N(Z \in \tilde{A})$ evaluates the extent to which “ $Z \in \tilde{A}$ ” is certainly true. If A is a crisp subset of $\mathbb{R}$, then it is easy to verify that $\Pi(Z \in A) = \sup_{z \in A} \pi_Z(z)$ and $N(Z \in A) = \inf_{z \notin A} (1 - \pi_Z(z))$.

IV. POSSIBILISTIC MIN-MAX PROBLEM

Consider a situation in which the costs of the elements $e_1, \dots, e_n$ are uncertain. This uncertainty is modeled by specifying a scenario set Γ which is defined by explicitly listing all the possible element cost vectors. Thus $\Gamma = \{S_1, \dots, S_K\}$ is finite and contains exactly K scenarios, where a scenario S_j , $j = 1, \dots, K$, is a vector of the element costs $S_j = (c_{1j}, c_{2j}, \dots, c_{nj})$. Additionally, a possibility distribution on the scenario set is specified. This possibility distribution expresses the fact that some scenarios are more plausible than others. Hence, for each scenario S_j a number $\pi(S_j) \in [0, 1]$ is given, which denotes the possibility of the event that S_j will occur. The value of $\pi(S_j)$ can also be seen as an upper bound on the probability of occurring S_j [15]. It is convenient to represent the uncertain costs in the form of Table I whose columns contain all the cost scenarios in Γ .

TABLE I
THE SCENARIO REPRESENTATION OF THE UNCERTAIN COSTS

	$\pi(S_1)$ S_1	$\pi(S_2)$ S_2	...	$\pi(S_K)$ S_K
e_1	c_{11}	c_{12}	...	c_{1K}
e_2	c_{21}	c_{22}	...	c_{2K}
e_3	c_{31}	c_{32}	...	c_{3K}
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
e_n	c_{n1}	c_{n2}	...	c_{nK}

A. Linear Sum Criterion

Let $f(X, S_j) = \sum_{e_i \in X} c_{ij}$ be the cost of solution X under scenario S_j . We will denote by $f(X)$ the uncertain cost of X , i.e. a real variable whose value depends on scenario $S_j \in \Gamma$. The uncertain cost $f(X)$ has the following possibility distribution $\pi_{f(X)}$:

$$\pi_{f(X)}(z) = \max_{\{j: f(X, S_j) = z\}} \pi(S_j) \quad (4)$$

and $\pi_{f(X)}(z) = 0$ if $f(X, S_j) \neq z$ for all $j = 1, \dots, K$. Hence, $\pi_{f(X)}(z) = \Pi(f(X) = z)$ is the possibility of the event that the cost of X will take value of z . To illustrate this idea, let us consider the example of the shortest path problem presented in Fig. 1.

In this problem $E = \{e_1, \dots, e_5\}$ is the set of arcs of a given graph G and $\Phi = \{\{e_1, e_4\}, \{e_1, e_3, e_5\}, \{e_2, e_5\}\}$ contains all directed paths between s and t in G . There are six cost scenarios listed in the table and each scenario has some possibility of occurrence. The possibility distribution of

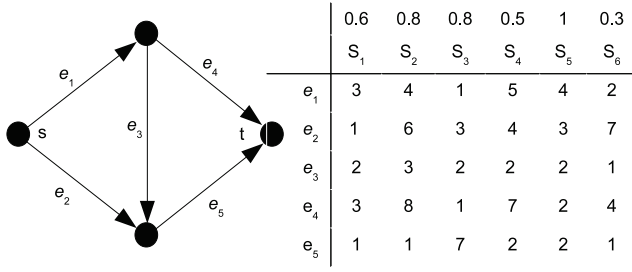


Fig. 1. An example of the shortest path problem with uncertain costs

the cost $f(X)$ for the path $X = \{e_1, e_3, e_5\}$ is shown in Fig. 2. Observe that the path X has a cost of 8 under two scenarios S_2 and S_5 and, according to (4), $\pi_{f(X)}(8) = \Pi(f(X) = 8) = \max\{\pi(S_2), \pi(S_5)\} = 1$.

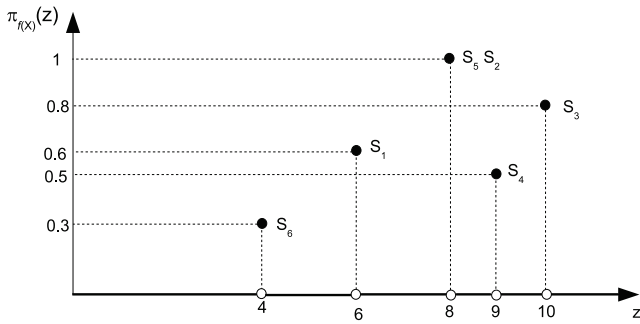


Fig. 2. The possibility distribution for $f(X)$, where $X = \{e_1, e_3, e_5\}$

Now assume that a decision maker expresses her/his preferences about solution costs by means of a fuzzy goal $\tilde{G}$. The membership function of $\tilde{G}$ has the following form:

$$\mu_{\tilde{G}}(z) = \begin{cases} 1 & \text{if } z \leq g \\ 1 - \frac{g-z}{\beta} & \text{for } z \in [g, g + \beta] \\ 0 & \text{if } z \geq g + \beta \end{cases} \quad (5)$$

The interpretation of $\tilde{G}$ is as follows. The solution costs less than g are totally accepted and the solution costs greater than $g + \beta$ are not at all accepted. The acceptance of the solution costs decreases linearly between g and $g + \beta$. In general, a fuzzy goal $\tilde{G}$ can be any fuzzy set in $\mathbb{R}$ with a bounded support and the membership function $\mu_{\tilde{G}}$ being a nonincreasing mapping from $[0, \infty)$ into $[0, 1]$ such that $\mu_{\tilde{G}}(z) = 1$ for $z \in [0, g]$. In this paper we use the membership function (5) to simplify the presentation. We will denote by $\tilde{G}'$ the complement of $\tilde{G}$ with the membership function $1 - \mu_{\tilde{G}}(z)$.

Now, we are ready to give a criterion of choosing a robust solution in problem $\mathcal{P}$ with the uncertain costs. Having a fuzzy goal $\tilde{G}$ we would like to find a solution $X \in \Phi$, whose cost belongs to $\tilde{G}$ with the highest necessity (certainty). Thus, we would like to solve the following problem:

$$\text{POSS MIN-MAX } \mathcal{P} : \quad \max_{X \in \Phi} N(f(X) \in \tilde{G}). \quad (6)$$

There are no doubts that every optimal solution to (6) is robust, since the cost of such a solution falls within $\tilde{G}$ with the highest

degree of certainty. Since $N(f(X) \in \tilde{G}) = 1 - \Pi(f(X) \in \tilde{G}')$, problem (6) is equivalent to

$$\min_{X \in \Phi} \Pi(f(X) \in \tilde{G}'). \quad (7)$$

By (2) and (4) we obtain

$$\begin{aligned} \Pi(f(X) \in \tilde{G}') &= \sup_{z \in \mathbb{R}} \min\{\pi_{f(X)}(z), \mu_{\tilde{G}'}(z)\} \\ &= \max_{j=1, \dots, K} \min\{\pi(S_j), \mu_{\tilde{G}'}(f(X, S_j))\}, \end{aligned} \quad (8)$$

which is illustrated in Fig. 3.

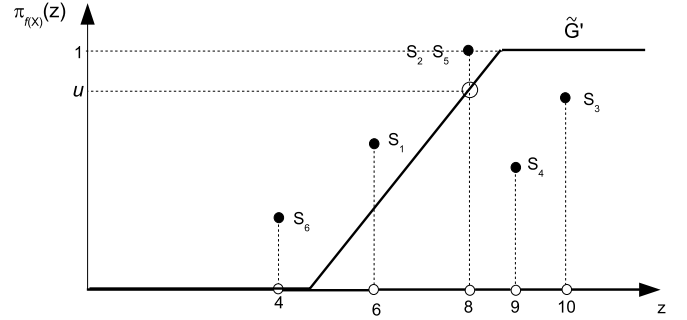


Fig. 3. $\Pi(f(X) \in \tilde{G}') = u$ for the path $X = \{e_1, e_3, e_5\}$

Observe that $N(f(X) \in \tilde{G}) = 1$ if $\Pi(f(X) \in \tilde{G}') = 0$, i.e. when the cost of X under every scenario $S_j \in \Gamma$ such that $\pi(S_j) > 0$ is totally accepted. On the other hand, $N(f(X) \in \tilde{G}) = 0$ if $\Pi(f(X) \in \tilde{G}') = 1$, i.e. when the cost of X under some scenario S_j such that $\pi(S_j) = 1$ is not at all accepted. In the example presented in Fig. 3 an intermediate situation is shown, where $N(f(X) \in \tilde{G}) \in (0, 1)$. Note that the solution costs under scenarios S_3 and S_4 , which are not at all accepted (i.e. belong to $\tilde{G}'$ with degree 1), occur with possibility less than 1. Using (7) and (8), we can represent POSS MIN-MAX $\mathcal{P}$ as the following mathematical programming problem:

$$\begin{aligned} \min u \\ u &\geq \min\{\pi(S_j), \mu_{\tilde{G}'}(f(X, S_j))\}, \quad 1 \leq j \leq K, \\ X &\in \Phi, \\ u &\geq 0 \end{aligned} \quad (9)$$

Let us introduce additional binary variables $\delta_j \in \{0, 1\}$, $j = 1, \dots, K$, and a sufficiently large constant M . We then get the following equivalent formulation:

$$\begin{aligned} \min u \\ u &\geq \pi(S_j) - M\delta_j, & 1 \leq j \leq K, \\ u &\geq \mu_{\tilde{G}'}(f(X, S_j)) - M(1 - \delta_j), & 1 \leq j \leq K, \\ X &\in \Phi, \\ \delta_j &\in \{0, 1\}, & 1 \leq j \leq K, \\ u &\geq 0. \end{aligned} \quad (10)$$

From (5) and the equality $f(X, S_j) = \sum_{i=1}^n c_{ij}x_i$, we obtain:

$$\begin{aligned} \min u \\ u &\geq \pi(S_j) - M\delta_j, & 1 \leq j \leq K, \\ u &\geq \frac{1}{\beta} \sum_{i=1}^n c_{ij}x_i - \frac{g}{\beta} - M(1 - \delta_j), & 1 \leq j \leq K, \\ (x_1, \dots, x_n) &\in \Phi', \\ \delta_j &\in \{0, 1\} & 1 \leq j \leq K, \\ x_i &\in \{0, 1\}, & 1 \leq i \leq n, \\ u &\geq 0, \end{aligned} \quad (11)$$

where x_i is a binary variable equal to 1 if and only if element e_i belongs to the solution computed. For a particular problem, $(x_1, \dots, x_n) \in \Phi'$ can be replaced with a set of linear constraints. As the result we obtain a mixed integer linear programming (MIP) model which can be solved by the available and powerful software such as CPLEX. If X^* is an optimal solution to (11) with an objective value of u^* , then $N(f(X^*) \in \tilde{G}) = 1 - u^*$. We now illustrate the approach using two examples. In the first example the deterministic problem is polynomially solvable while in the second one it is NP-hard.

Example 1: Consider the following *shortest path problem* with imprecise costs (POSS MIN-MAX SHORTEST PATH). We are given a digraph $G = (V, A)$ with uncertain arc costs. Namely, for each arc $(k, l) \in A$ a cost $c_{(k,l)j}$ of this arc under scenario S_j , $j = 1, \dots, K$, is specified. We distinguish two nodes s and t in G and seek a directed path X from s to t which maximizes $N(f(X) \in \tilde{G})$. Let us introduce a binary variable $x_{(k,l)}$ for each arc $(k, l) \in A$, $x_{(k,l)}$ is equal to 1 if and only if the arc (k, l) belongs to the path computed. The set Φ' is described by the standard mass balance constraints (see e.g., [14]). Hence, we obtain the following MIP model for the problem:

$$\begin{aligned} \min u \\ u &\geq \pi(S_j) - M\delta_j, & 1 \leq j \leq K, \\ u &\geq \frac{1}{\beta} \sum_{(k,l) \in A} c_{(k,l)j} x_{(k,l)} - \frac{g}{\beta} - M(1 - \delta_j), & 1 \leq j \leq K, \\ \sum_{\{l: (k,l) \in A\}} x_{(k,l)} - \sum_{\{l: (l,k) \in A\}} x_{(l,k)} &= b(k), & k \in V, \\ x_{(k,l)} &\in \{0, 1\}, & (k, l) \in A, \\ \delta_j &\in \{0, 1\}, & 1 \leq j \leq K, \\ u &\geq 0, \end{aligned}$$

where $b(s) = 1$, $b(t) = -1$ and $b(k) = 0$ for all $k \in V \setminus \{s, t\}$.

Example 2: Consider the following *minimum cost knapsack problem* (POSS MIN-MAX MINIMUM COST KNAPSACK). We are given a set of items $E = \{e_1, \dots, e_n\}$. Each item e_i has a precise weight w_i and imprecise cost, where c_{ij} is the cost of e_i under scenario S_j , $j = 1, \dots, K$. We would like to find a subset X of the items of the total weight at most W which maximizes $N(f(X) \in \tilde{G})$. Contrary to the shortest path problem discussed in the previous example, the deterministic knapsack problem with one scenario is NP-hard. Nevertheless, by introducing binary variable x_i for each item $e_i \in E$, where x_i is equal to 1 if and only if item e_i belongs to the subset computed, we can propose the following mixed integer programming model for the problem:

$$\begin{aligned} \min u \\ u &\geq \pi(S_j) - M\delta_j, & 1 \leq j \leq K, \\ u &\geq \frac{1}{\beta} \sum_{i=1}^n c_{ij} x_i - \frac{g}{\beta} - M(1 - \delta_j), & 1 \leq j \leq K, \\ \sum_{i=1}^n w_i x_i &\leq W, \\ x_i &\in \{0, 1\}, & 1 \leq i \leq n, \\ \delta_j &\in \{0, 1\}, & 1 \leq j \leq K, \\ u &\geq 0. \end{aligned}$$

Examples 1 and 2 show that our approach is general and can also be applied to the problems whose deterministic versions are NP-hard. The obtained MIP model can be solved by using some standard off-the-shelf MIP solvers.

Now we show that POSS MIN-MAX $\mathcal{P}$ is a generalization of the min-max robust problem $\mathcal{P}$, i.e. the problem $\mathcal{P}$ with the

min-max criterion, where no probability and possibility distributions on the scenario set are given. Suppose that $\pi(S_j) = 1$ for all scenarios $S_j \in \Gamma$, which means that a decision maker believes that all the scenarios in Γ are equally possible to occur. Let us choose a fuzzy goal $\tilde{G}$ such that $g = 0$ and β is sufficiently large, so that the inequality $\frac{1}{\beta} \sum_{i=1}^n c_{ij} x_i \leq 1$ always holds. We can set, for example, $\beta = \sum_{i=1}^n \sum_{j=1}^K c_{ij}$. Then the model (11) can be simplified as follows:

$$\begin{aligned} \min u \\ u &\geq \frac{1}{\beta} \sum_{i=1}^n c_{ij} x_i & 1 \leq j \leq K \\ (x_1, \dots, x_n) &\in \Phi' \\ u &\geq 0 \end{aligned} \quad (12)$$

Substituting $z = \beta u$ in (12) yields:

$$\begin{aligned} \min z \\ z &\geq \sum_{i=1}^n c_{ij} x_i & 1 \leq j \leq K \\ (x_1, \dots, x_n) &\in \Phi' \\ z &\geq 0 \end{aligned} \quad (13)$$

which is the well known min-max robust combinatorial optimization problem with scenario set Γ [2]. This implies that all the negative complexity results known for (13) hold true for POSS MIN-MAX $\mathcal{P}$. The robust min-max problems are typically NP-hard and also hard to approximate. Hence, POSS MIN-MAX $\mathcal{P}$ cannot be easier. We refer the reader to [6], [2], [16], [17] for some complexity results for the min-max versions of the classical combinatorial optimization problems.

B. Bottleneck Criterion

It turns out that the problem becomes easier to solve if we replace the linear sum criterion with the bottleneck one. Assume that the cost of solution X under scenario S_j is expressed as $f(X, S_j) = \max_{e_i \in X} c_{ij}$. The uncertain cost $f(X)$ of solution X is defined exactly the same as (4) and we seek a solution X maximizing $N(f(X) \in \tilde{G})$ (or, equivalently, minimizing $\Pi(f(X) \in \tilde{G}')$). From (8) we get:

$$\Pi(f(X) \in \tilde{G}') = \max_{j=1, \dots, K} \min\{\pi(S_j), \mu_{\tilde{G}'}(\max_{e_i \in X} c_{ij})\}.$$

Function $\mu_{\tilde{G}'}$ is nondecreasing ($\mu_{\tilde{G}}$ is nonincreasing), therefore

$$\begin{aligned} \Pi(f(X) \in \tilde{G}') &= \max_{j=1, \dots, K} \min\{\pi(S_j), \max_{e_i \in X} \mu_{\tilde{G}'}(c_{ij})\} \\ &= \max_{e_i \in X} \max_{j=1, \dots, K} \min\{\pi(S_j), \mu_{\tilde{G}'}(c_{ij})\} \end{aligned}$$

Set $c_i^* = \max_{j=1, \dots, K} \min\{\pi(S_j), \mu_{\tilde{G}'}(c_{ij})\}$, $e_i \in E$, and in consequence

$$\min_{X \in \Phi} \Pi(f(X) \in \tilde{G}') = \min_{X \in \Phi} \max_{e_i \in X} c_i^*,$$

which is a deterministic bottleneck combinatorial optimization problem. So, POSS MIN-MAX $\mathcal{BP}$ can be reduced to problem $\mathcal{BP}$ with deterministic costs c_i^* , $e_i \in E$. We have thus proved the following theorem:

Theorem 1: If problem $\mathcal{BP}$ is polynomially solvable, then problem POSS MIN-MAX $\mathcal{BP}$ is also polynomially solvable.

The above result holds true for a wide class of classical polynomially solvable bottleneck combinatorial optimization problems [18], [19], [20], [21].

V. POSSIBILISTIC TWO-STAGE PROBLEM

Suppose that we are given first stage element costs C_i for all $e_i \in E$ and these costs are precisely known. In the second stage the costs are imprecise and all the possible cost scenarios are listed in scenario set $\Gamma = \{S_1, \dots, S_K\}$. Additionally, a possibility distribution π is specified in Γ and its interpretation is the same as for the single-stage problem discussed in the previous section. An instance of the problem can be represented in the form of Table II.

TABLE II
THE REPRESENTATION OF COSTS IN TWO STAGES

	stage I	stage II				
		$\pi(S_1)$	$\pi(S_2)$	$\dots$	$\pi(S_K)$	
		S_1	S_2	$\dots$	S_K	
e_1	C_1	c_{11}	c_{12}	$\dots$	c_{1K}	
e_2	C_2	c_{21}	c_{22}	$\dots$	c_{2K}	
e_3	C_3	c_{31}	c_{32}	$\dots$	c_{3K}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	
e_n	C_n	c_{n1}	c_{n2}	$\dots$	c_{nK}	

Let $X \subseteq E$ be a part of a feasible solution determined in the first stage. The cost of X under scenario $S_j \in \Gamma$ is then

$$f(X, S_j) = \sum_{e_i \in X} C_i + \min_{\{Y_j: X \cup Y_j \in \Phi\}} \sum_{e_i \in Y_j} c_{ij}.$$

So, a decision consists of choosing a partial solution X in the first stage and augmenting it optimally to a complete feasible solution $X \cup Y_j$ after observing that scenario S_j has occurred in the second stage. The cost $f(X)$ of a partial solution X is uncertain and its possibility distribution is defined in the same way as (4).

Consider the following example. A set of eight elements $E = \{e_1, \dots, e_8\}$ is given and we must select exactly 5 out of them. We can select any $X \subset E$ such that $|X| \leq 5$ in the first stage and complete it, after observing which of the six scenarios in $\Gamma = \{S_1, \dots, S_6\}$ has occurred. The representation of the costs in both stages for this problem is shown in Table III.

TABLE III
THE REPRESENTATION OF COSTS FOR THE SAMPLE PROBLEM

	stage I	stage II					
		0.4	0.3	0.6	0.8	1	0.7
		S_1	S_2	S_3	S_4	S_5	S_6
e_1	5	5	1	2	1	3	2
e_2	4	5	3	2	9	3	5
e_3	2	8	1	6	1	1	9
e_4	7	4	4	7	8	3	5
e_5	4	9	1	1	8	7	7
e_6	7	6	3	1	9	8	8
e_7	5	1	1	4	6	1	5
e_8	3	1	1	1	2	1	2

Suppose that a decision maker decides to select two elements, say e_3 and e_5 , in the first stage. Therefore, $X =$

$\{e_3, e_5\}$ and the cost in the first stage is 6. The cost $f(X, S_j)$ depends now on a scenario S_j that will occur. For instance, if scenario S_1 is observed, then the subset X is completed to form a feasible solution (a five element subset of E) by choosing the elements in $Y_1 = \{e_5, e_8, e_7\}$, which gives the smallest cost $f(X, S_1)$ equal to 12. A similar reasoning can be performed for the remaining scenarios. The possibility distribution for $f(X)$ is presented in Fig. 4.

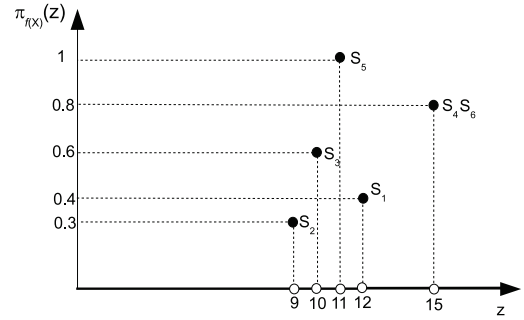


Fig. 4. The possibility distribution for $f(X)$ where $X = \{e_3, e_5\}$

In order to choose a solution, we use exactly the same solution concept as in Section IV. Namely, a decision maker specifies a fuzzy goal $\tilde{G}$ and we seek a solution which maximizes the necessity that its costs will fall within $\tilde{G}$. Thus POSS 2-STAGE is the optimization problem $\max_{X \in \Phi} N(f(X) \in \tilde{G})$ or equivalently $\min_{X \in \Phi} \Pi(f(X) \in \tilde{G}')$. The same reasoning as in Section IV leads to a model similar to (10). Let us introduce binary variables x_i and y_{ij} , $i = 1, \dots, n$, $j = 1, \dots, K$. Variable x_i is equal to 1 if and only if element e_i is selected in the first stage, y_{ij} is equal to 1 if and only if element e_i is selected in the second stage if scenario S_j occurs. Then a feasible solution $X \cup Y_j$ under scenario S_j is described by variables $x_1, \dots, x_n, y_{1j}, \dots, y_{nj}$, which form a vector $(x_1, \dots, x_n, y_{1j}, \dots, y_{nj}) \in \Phi'$. The cost $f(X, S_j)$ can be then expressed as $\sum_{i=1}^n C_i x_i + \sum_{i=1}^n c_{ij} y_{ij}$. The model takes the following form:

$$\begin{aligned}
 & \min u \\
 & u \geq \pi(S_j) - M\delta_j, & 1 \leq j \leq K, \\
 & u \geq \frac{1}{\beta} \left(\sum_{i=1}^n C_i x_i + \sum_{i=1}^n c_{ij} y_{ij} \right) - \frac{g}{\beta} - M(1 - \delta_j), & 1 \leq j \leq K, \\
 & (x_1, \dots, x_n, y_{1j}, \dots, y_{nj}) \in \Phi', & 1 \leq j \leq K, \\
 & \delta_j \in \{0, 1\}, & 1 \leq j \leq K, \\
 & u \geq 0.
 \end{aligned} \tag{14}$$

For a particular problem we should only replace $(x_1, \dots, x_n, y_{1j}, \dots, y_{nj}) \in \Phi'$ with some linear constraints involving the variables $x_1, \dots, x_n, y_{1j}, \dots, y_{nj}$. We now apply the above MIP approach to the following examples of combinatorial problems.

Example 3: Consider the 2-stage *minimum cost selecting items* with imprecise costs (POSS 2-STAGE MINIMUM COST SELECTING ITEMS). A set of items $E = \{e_1, \dots, e_n\}$ and integer p , $1 < p < n$ are given. The set of feasible solutions consists of all the subsets of E with cardinality p . We select

some elements in the first stage, where the item costs C_i are known for each $e_i \in E$ and complete it to a feasible solution after a scenario from Γ is observed. The cost of item e_i under scenario S_j is c_{ij} . This problem can be solved by using the following MIP formulation:

$$\begin{aligned}
& \min u \\
& u \geq \pi(S_j) - M\delta_j, & 1 \leq j \leq K, \\
& u \geq \frac{1}{\beta} \left(\sum_{i=1}^n C_i x_i + \sum_{i=1}^n c_{ij} y_{ij} \right) - \frac{g}{\beta} - M(1 - \delta_j), & 1 \leq j \leq K, \\
& \sum_{i=1}^n x_i + \sum_{i=1}^n y_{ij} = p, & 1 \leq j \leq K, \\
& \delta_j \in \{0, 1\}, & 1 \leq j \leq K, \\
& x_i \in \{0, 1\}, & 1 \leq i \leq n, \\
& y_{ij} \in \{0, 1\}, & 1 \leq i \leq n, \\
& & 1 \leq j \leq K, \\
& u \geq 0.
\end{aligned}$$

Example 4.: Consider the 2-stage *minimum cost matching* with imprecise costs (POSS 2-STAGE MINIMUM COST MATCHING). Let E be the set of the edges of a complete bipartite graph $G = (U \cup V, E)$, where $|U| = |V| = n$. The solution set Φ consists of all perfect matchings (assignments) in G . We can build a partial assignment in the first stage X , when the edge costs are equal to $C_{(k,l)}$ for $(k,l) \in E$ and complete it in the second stage when a cost scenario from Γ is observed. The cost of edge (k,l) under scenario S_j is equal to $c_{(k,l)j}$. This problem can be solved by using the following MIP formulation:

$$\begin{aligned}
& \min u \\
& u \geq \pi(S_j) - M\delta_j, & 1 \leq j \leq K, \\
& u \geq \frac{1}{\beta} \left(\sum_{(k,l) \in E} C_{(k,l)} x_{(k,l)} \right. \\
& \quad \left. + \sum_{(k,l) \in E} c_{(k,l)j} y_{(k,l)j} \right) - \frac{g}{\beta} - M(1 - \delta_j), & 1 \leq j \leq K, \\
& \sum_{k \in U} (x_{(k,l)} + y_{(k,l)j}) = 1, & l \in V, 1 \leq j \leq K, \\
& \sum_{l \in V} (x_{(k,l)} + y_{(k,l)j}) = 1, & k \in U, 1 \leq j \leq K, \\
& x_{(k,l)} \in \{0, 1\}, & (k,l) \in E, \\
& y_{(k,l)j} \in \{0, 1\}, & (k,l) \in E, \\
& & 1 \leq j \leq K, \\
& \delta_j \in \{0, 1\}, & 1 \leq j \leq K, \\
& u \geq 0.
\end{aligned}$$

By assuming that $\pi(S_j) = 1$ for each $S_j \in \Gamma$ and choosing a suitable fuzzy goal, we can show that POSS 2-STAGE $\mathcal{P}$ is a generalization of the robust two-stage problem $\mathcal{P}$ known from the literature (see, e.g., [7], [8], [17]). This implies that all the negative complexity results for robust two-stage problem $\mathcal{P}$ hold true for POSS 2-STAGE $\mathcal{P}$.

VI. CONCLUSIONS

In this paper, we have proposed a general framework for dealing with possibilistic combinatorial optimization problems with uncertain costs in the single and two-stage decision models. The uncertainty has been modeled by a specified scenario set with a possibility distribution given on it. We have provided the solution concepts that lead to robust solutions in the setting of possibility theory. The concepts are

consistent with popular and widely accepted robust approach to single and two stage optimization, where decision makers are interested in minimizing a solution cost in the worst case. Apart from showing the general framework, we have provided some methods based on mixed integer models for computing a solution with respect to the given solution concepts and we have illustrated them by examples.

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