

Natural Topology via Fuzzy Metric

Fagner Santana

Department of Mathematics

UFRN

RN, Brazil, postal code 59072-970

Email: fagner@cet.ufrn.br

Regivan Santiago

Department of Informatics and Applied Mathematics

UFRN

RN, Brazil, postal code 59072-970

Email: regivan@dimap.ufrn.br

Abstract—In this paper, we propose a natural way to define topologies by using fuzzy metrics; here the value of the distance between two points is a non-negative fuzzy number. With such topology, we also define notions like continuity, convergent sequence, Cauchy sequences and complete spaces. As an application, we provide a fixed point theorem quite similar to the classical Banach's theorem.

Keywords: Fuzzy metric, fixed point, fuzzy numbers, α -cuts, intervals, topology, continuity.

I. INTRODUCTION

The first concept of *fuzzy metrics* was introduced by Kramosil and Michalek [1], which adapted the notion of statistical metric spaces — or Menger spaces (see [2]) — to the fuzzy viewpoint. There, the distance between two points, a and b , is seen as a probabilistic distribution, and the expression $d(a, b)(t)$ denotes the assertion: “the probability of the distance from a to b is less than t ”. In [3], Kaleva and Seikkala introduced another concept of fuzzy metric; in this case, the distance between two points is a non-negative fuzzy number and the same expression $d(a, b)(t)$ now means: “the possibility (see [4]) of the distance between a and b is t ”.

There are several investigations concerning such approaches of fuzzy metric spaces, with results such as: fixed-point theorems, completions, separability, etc. (see [5], [6], [7]). In both cases, such generalizations are based on changing the usual valuation space $\langle \mathbb{R}^+, \leq \rangle$ to a suitable one. With the usual valuation, a metric on a set M is a function, satisfying some properties, which returns a quantitative information about the distance between two points. With such formalization it is possible to define a topology on M by using the concept of open ball; which is an entity with a strong geometric appeal. With this kind of topology, it is possible to abstract important concepts which reside in the world of real numbers; namely: continuity, completeness, compactness, etc.

The basic metric properties are stated in terms of the natural order, “ $\leq$ ”, and the sum on $\mathbb{R}$. This establishes a connection between the axiomatics and the structure of the valuation. Thus, in principle, to modify the valuation space in order to preserve basic properties, the new valuation space should retain both: a notion of sum and a notion of order. However, in [8], we introduced a new kind of generalization by modifying the usual valuation; $\mathbb{R}$ was replaced by an ordered (preordered) set $(G, \leq)$ with smallest element, also endowed with a new kind of relation; called: *semi-auxiliary relation* for $\leq$. The resulting valuation space was named *Informative Metric Valuation* or

just **IMV** and a function $d : M \times M \rightarrow G$ satisfying suitable axioms was designated by *Informative Metrics*. Such generalization produced topologies very similar to the usual case; since the basic building blocks (open balls) were easily recovered.

Kaleva and Seikkala showed in [3] that, in a particular case, their proposed fuzzy metrics is very similar to the usual metric when an addition and an order relation are defined on the set of non-negative fuzzy numbers. In that case, the resulting fuzzy metric space becomes a *generalized metric* in the sense of [9], which is a bridge to the IMV approach. Following such bridge, we apply the ideas of [8] to define a natural topology on a fuzzy metric space. With such topology, concepts like: continuity, completeness, convergent and Cauchy sequences arise naturally. They are investigated here and an application of them: a fixed-point theorem is provided. A very important aspect of such application is that its proof follows steps similar to those in the proof of classical Banach theorem (see [10]) which depend only on the topological concepts listed above. This does not happen with the most of the proofs of fixed-points theorems for fuzzy metric spaces, revealing that our approach does not only have an advantage, but that it also gives a way to obtain similar classical results and notions that are built on usual topological concepts.

mds

January 13, 2011

II. PRELIMINARIES

In this section, we introduce the basic concepts that will be used in the whole paper, such as: fuzzy numbers, fuzzy arithmetic, partial order on the fuzzy numbers and some required properties.

A. Fuzzy Numbers: arithmetic and order

A fuzzy number is a fuzzy set $A : \mathbb{R} \rightarrow [0, 1]$ satisfying:

- 1) A is normal (exists $t \in \mathbb{R}$ such that $A(t) = 1$);
- 2) The support of A — i.e. the set $\text{supp}A = \{t \in \mathbb{R}; A(t) > 0\}$ — is a bounded set;
- 3) For every $\alpha \in (0, 1]$, the α -cut $A_\alpha = \{t \in \mathbb{R}; A(t) \geq \alpha\}$ is a closed interval.

Every real number r can be seen as a fuzzy number, by the following membership function:

$$\mu_r(t) = \begin{cases} 1, & \text{if } t = r \\ 0, & \text{if } t \neq r \end{cases}.$$

This fuzzy numbers will be denoted by $\tilde{r}$. For other examples of fuzzy numbers, like triangular and trapezoidal fuzzy numbers, see [11].

The following theorem gives the characterization of fuzzy numbers.

Theorem II.1. *Let $A : \mathbb{R} \rightarrow [0, 1]$ be a fuzzy set. Then, A is a fuzzy number iff there exists a closed interval $[a, b]$ and functions $l : (-\infty, a) \rightarrow [0, 1]$ and $r : (b, \infty) \rightarrow [0, 1]$, where l is non-decreasing, continuous from the right and there exists ω_1 such that $l(t) = 0$, for all $t < \omega_1$ and r is non-increasing, continuous from the left and there exists $\omega_2 \in \mathbb{R}$ such that $r(t) = 0$ for all $t > \omega_2$ such that:*

$$A(t) = \begin{cases} 1, & \text{if } t \in [a, b] \\ l(t), & \text{if } t < a \\ r(t), & \text{if } t > b \end{cases}.$$

Proof: See [11]. ■

A non-negative fuzzy number A is a fuzzy number such that $A(t) = 0$ for all $t < 0$. If $r > 0$, then $\tilde{r}$ is a non-negative fuzzy number. The set of all non-negative fuzzy numbers will be denoted by G .

If $A \in G$ and $\alpha \in (0, 1]$, then we use the notation $A_\alpha = [a_\alpha, \bar{a}_\alpha]$.

Since the fuzzy sets are uniquely defined by your α -cuts, we can define an arithmetic and a partial order in G from an arithmetic and a partial order on the set of closed real intervals $\mathbb{IR}$.

Let $\mathbb{IR}^+$ be the set of all intervals $[a, b]$ such that $0 \leq a \leq b$. We consider the following operations on $\mathbb{IR}^+$:

- i) $[a, b] + [c, d] = [a + b, c + d]$;
- ii) $[a, b] \cdot [c, d] = [ac, bd]$.

This operations are the same operations defined by Moore (see [12]) for the set $\mathbb{IR}$ of all closed intervals. We define the partial order $\leq_{km}$ (Kulisch-Miranker order, see [13]) by: $[a, b] \leq_{km} [c, d] \Leftrightarrow a \leq c$ and $b \leq d$. The next proposition gives some properties of $\mathbb{IR}^+$.

Proposition II.1. *Let $A, B, C, D \in \mathbb{IR}^+$, where $A = [\underline{a}, \bar{a}]$, $B = [\underline{b}, \bar{b}]$, $C = [\underline{c}, \bar{c}]$ and $D = [\underline{d}, \bar{d}]$. The following properties are hold:*

- 1) $[0, 0] \leq_{km} A$, for all $A \in \mathbb{IR}^+$;
- 2) $A + (B + C) = (A + B) + C$ and $A \cdot (B \cdot C) = (A \cdot B) \cdot C$;
- 3) $A^n = [\underline{a}^n, \bar{a}^n]$;
- 4) If $A \leq_{km} B$ and $C \leq_{km} D$, then $A + C \leq_{km} B + D$ and $A \cdot C \leq_{km} B \cdot D$;
- 5) If $\bar{b} < 1$, then $A \cdot B <_{km} A$, ie, $A \cdot B \leq_{km} A$ and $A \cdot B \neq A$.

Proof: For items 2) and 3) see [12]. The proof of items 1), 4) and 5) are immediate. ■

From the above interval operations, we can define similar operations on G .

Definition II.1. *Let $A, B \in G$. We define:*

- 1) *The sum between A and B is the fuzzy number $A \oplus B \in G$ such that $(A \oplus B)_\alpha = A_\alpha + B_\alpha$, for all $\alpha \in (0, 1]$;*

- 2) *The product between A and B is the fuzzy number $A \odot B \in G$ such that $(A \odot B)_\alpha = A_\alpha \cdot B_\alpha$, for all $\alpha \in (0, 1]$;*
- 3) *The partial order $\preceq$ is defined by $A \preceq B \Leftrightarrow A_\alpha \leq_{km} B_\alpha$, for all $\alpha \in (0, 1]$.*

Proposition II.2. *Let $A, B, C, D \in G$. The following properties are hold:*

- 1) $\tilde{0} \preceq A$, for all $A \in G$;
- 2) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$ and $A \odot (B \odot C) = (A \odot B) \odot C$;
- 3) $(A^n)_\alpha = (A \odot \dots \odot A)_\alpha = [\underline{a}_\alpha^n, \bar{a}_\alpha^n]$, for all $\alpha \in (0, 1]$;
- 4) If $A \preceq B$ and $C \preceq D$, then $A \oplus C \preceq B \oplus D$ and $A \odot C \preceq B \odot D$;
- 5) If $\bar{b}_\alpha < 1$, for all $\alpha \in (0, 1]$, then $A \odot B \prec A$, ie, $A \odot B \leq_{km} A$ and $A \odot B \neq A$.

Proof: It's straightforward from the proposition II.1. ■

In an usual metric space, we define the open balls $B(a, r)$ using the strict relation $<$. The strict relation $\prec$ on G — namely $A \preceq B$ iff $A \preceq B$ and $A \neq B$ — has proved inadequate for this purpose, so we need to define a relation that will make the role of the strict relation.

Definition II.2. *Let $A, B \in G$. The relation $\prec\prec$ is defined by: $A \prec\prec B$ iff there exists $r_1, r_2 > 0$ such that $\underline{b}_\alpha - \underline{a}_\alpha > r_1$ and $\bar{b}_\alpha - \bar{a}_\alpha > r_2$, for all $\alpha \in (0, 1]$.*

Example II.1. *If $a, b \in \mathbb{R}^+$, with $a < b$, then $\tilde{a} \prec\prec \tilde{b}$.*

Example II.2. *A triangular fuzzy numbers is a fuzzy set with a “triangle shape”. Its membership function A is defined by: $A(x) = 0$, if $x \leq a_1$, $A(x) = l(x)$, if $a_1 \leq x \leq a_2$, where $l(x)$ is the straight line connecting the points $(a_1, 0)$ and $(a_2, 1)$, $A(x) = r(x)$, if $a_2 \leq x \leq a_3$, where $r(x)$ is the straight line connecting the points $(a_2, 1)$ and $(a_3, 0)$ and $A(x) = 0$, if $x \geq a_3$. The numbers a_1, a_2 and a_3 completely determine A , so we use the notation $A = (a_1, a_2, a_3)$. If $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ are two triangular fuzzy numbers, then $A \prec\prec B$ if, and only if, $a_1 < b_1$, $a_2 < b_2$ and $a_3 < b_3$.*

Proposition II.3. *Let $A, B, C, D \in G$. The following properties are valid:*

- 1) *If $A \prec\prec B$, then $A \prec B$;*
- 2) *If $A \preceq B \prec\prec C \preceq D$, then $A \prec\prec D$;*
- 3) *If $A \preceq C$ and $B \prec\prec D$, then $A \oplus B \prec\prec C \oplus D$.*

Proof:

- 1) It's straightforward.
- 2) There exists $r_1, r_2 > 0$, such that $\underline{c}_\alpha - \underline{b}_\alpha > r_1$ and $\bar{c}_\alpha - \bar{b}_\alpha > r_2$, for all $\alpha \in (0, 1]$. Since $\underline{b}_\alpha \geq \underline{a}_\alpha$, $\bar{b}_\alpha \geq \bar{a}_\alpha$, $\underline{d}_\alpha \geq \underline{c}_\alpha$ and $\bar{d}_\alpha \geq \bar{c}_\alpha$, for all $\alpha \in (0, 1]$, we can conclude that $\underline{d}_\alpha - \underline{a}_\alpha > r_1$ and $\bar{d}_\alpha - \bar{a}_\alpha > r_2$, for all $\alpha \in (0, 1]$, so $A \oplus B \prec\prec C \oplus D$.
- 3) There exists $r_1, r_2 > 0$ such that $\underline{d}_\alpha - \underline{b}_\alpha > r_1$ and $\bar{d}_\alpha - \bar{b}_\alpha > r_2$. We have $\underline{c}_\alpha - \underline{a}_\alpha \geq 0$ and $\bar{c}_\alpha - \bar{a}_\alpha \geq 0$, so $\underline{d}_\alpha - \underline{b}_\alpha + \underline{c}_\alpha - \underline{a}_\alpha > r_1$ and $\bar{d}_\alpha - \bar{b}_\alpha + \bar{c}_\alpha - \bar{a}_\alpha > r_2 \Rightarrow A \oplus B \prec\prec C \oplus D$. ■

Corollary II.2. Let $A, B, C \in G$. Then:

- 1) If $A \preceq B \prec\prec C$, then $A \prec\prec C$;
- 2) If $A \prec\prec B \preceq C$, then $A \prec\prec C$.

Proof: It's straightforward. ■

It also follows from the above results that the relation $\prec\prec$ is transitive.

The next results are very important to ensure that the class of open sets in the following is Hausdorff.

Theorem II.3. If $A, B \in G$, with $\tilde{0} \prec\prec A$ and $\tilde{0} \prec\prec B$ then exists $C \in G$, with $\tilde{0} \prec\prec C$, such that $C \preceq A$ and $C \preceq B$.

Proof: Since $\tilde{0} \prec\prec A$ and $\tilde{0} \prec\prec B$, then exists $r_1, r_2 > 0$ such that $\overline{a_\alpha} \geq \underline{a_\alpha} > r_1$ and $\overline{b_\alpha} \geq \underline{b_\alpha} > r_2$. So take $C \in G$ defined by $C = \tilde{r}$, where $r = \min\{r_1, r_2\} > 0$. So, trivially we have $\tilde{0} \prec\prec C$, $C \preceq A$ and $C \preceq B$. ■

Theorem II.4. If $A, B \in G$ with $A \prec\prec B$, then there exists $C \in G$, with $\tilde{0} \prec\prec C$ such that $A \oplus C \prec\prec B$.

Proof: There exists $r_1, r_2 > 0$ such that $\underline{b_\alpha} - \underline{a_\alpha} > r_1$ and $\overline{b_\alpha} - \overline{a_\alpha} > r_2$. It sufficient to take $C \in G$ defined by $C(t) = \begin{cases} 1, & \text{if } t = r/2 \\ 0, & \text{if } t \neq r/2 \end{cases}$, where $r = \min\{r_1, r_2\}$. ■

Theorem II.5. If $A \in G$, with $\tilde{0} \prec\prec A$, then exists $B, C \in G$, with $\tilde{0} \prec\prec B$ and $\tilde{0} \prec\prec C$ such that $B \oplus C \prec\prec A$.

Proof: Since $\tilde{0} \prec\prec A$, there exists $r > 0$ such that $\overline{a_\alpha} \geq \underline{a_\alpha} > r$. It sufficient to take $B, C \in G$ defined by $B(t) = C(t) = \begin{cases} 1, & \text{if } t \in [r/8, r/4] \\ 0, & \text{if } t \notin [r/8, r/4] \end{cases}$. ■

III. FUZZY METRIC

In their paper, Kaleva and Seikkala gave the following definition of fuzzy metric:

Definition III.1. Let be M a non-empty set, a mapping $d : M \times M \rightarrow G$ and $L, R : [0, 1] \times [0, 1] \rightarrow [0, 1]$ two mappings symmetric, non-decreasing on both arguments and satisfying $L(0, 0) = 0$ and $R(1, 1) = 1$. The quadruple (X, d, L, R) is called a fuzzy metric space and d a fuzzy metric, when:

- i) $d(x, y) = \tilde{0}$ if and only if $x = y$;
- ii) $d(x, y) = d(y, x)$, for all $x, y \in M$;
- iii) Given $x, y \in M$, use notation $[d(x, y)]_\alpha = [\lambda_\alpha(x, y), \rho_\alpha(x, y)]$. Thus, for all $x, y, z \in M$,
 - 1) $d(x, y)(s + t) \geq L(d(x, z)(s), d(y, z)(t))$, whenever $s \leq \lambda_1(x, z)$, $t \leq \lambda_1(z, y)$ and $s + t \leq \lambda_1(x, y)$;
 - 2) $d(x, y)(s + t) \leq R(d(x, z)(s), d(y, z)(t))$, whenever $s \geq \lambda_1(x, z)$, $t \geq \lambda_1(z, y)$ and $s + t \geq \lambda_1(x, y)$;

Example III.1 (Kaleva and Seikkala). Let E be the set of all fuzzy numbers. The function $d : E \times E \rightarrow G$ defined by

$$d(x, y) = \begin{cases} |x - y| & , \text{ if } x \neq y \\ \tilde{0} & , \text{ if } x = y \end{cases} \quad (1)$$

is a fuzzy metric where $L \equiv 0$ and $R = \max$

There are several functions L and R satisfying the conditions in Definition III.1. For example, for $L = \min$ and $R = \max$ the following result is hold:

Theorem III.1. In a fuzzy metric space $(X, d, \min, \max)$, the condition iii) of the definition III.1 is equivalent to:

$$d(x, z) \preceq d(x, y) \oplus d(y, z).$$

Proof: See [3]. ■

In this case, a function $d : M \times M \rightarrow G$ is a fuzzy metric if and only if it satisfies:

- i) $d(x, y) = \tilde{0}$ if and only if $x = y$;
- ii) $d(x, y) = d(y, x)$, for all $x, y \in M$;
- iii') $d(x, z) \preceq d(x, y) \oplus d(y, z)$.

Thus, a fuzzy metric is a generalized metric in the sense given in [9]. The condition iii'), or triangle inequality, takes a closer way than usual. In this paper, we consider only fuzzy metric in this sense and we will call them gf-metrics.

Example III.2. If $d : M \times M \rightarrow \mathbb{R}$ is an usual metric, then $d_f : M \times M \rightarrow G$ defined by $d_f(x, y) = d(x, y)$ is a gf-metric.

Example III.3. Let $d_1, d_2, d_3, d_4 : M \times M \rightarrow \mathbb{R}$ be usual metrics such that $d_1 \leq d_2 \leq d_3 \leq d_4$. The function $d_f : M \times M \rightarrow G$ defined by:

$$d_f(x, y)(t) = \begin{cases} 1/2, & \text{if } d_1(x, y) \leq t < d_2(x, y) \\ 1, & \text{if } d_2(x, y) \leq t \leq d_3(x, y) \\ 1/2, & \text{if } d_3(x, y) < t \leq d_4(x, y) \\ 0, & \text{in other cases} \end{cases}$$

is a gf-metric.

Theorem III.2. Let $d : M \times M \rightarrow G$ be a gf-metric. If $d(a, b) \prec\prec R$, where $\tilde{0} \prec\prec R$, then exists $\tilde{0} \prec\prec D$ such that $d(b, c) \prec\prec D \Rightarrow d(a, c) \prec\prec R$.

Proof: Since $d(a, b) \prec\prec R$, it follows from theorem II.4 that there exists $\tilde{0} \prec\prec D$ such that $d(a, b) \oplus D \prec\prec R$. Suppose that $d(b, c) \prec\prec D$. So, follows from the proposition II.3 that $d(a, b) \oplus d(b, c) \prec\prec d(a, b) \oplus D \prec\prec R \Rightarrow d(a, b) \oplus d(b, c) \prec\prec R$. From the triangle inequality, it follows that $d(a, c) \preceq d(a, b) \oplus d(b, c) \prec\prec R \Rightarrow d(a, c) \prec\prec R$. ■

The above theorem guarantees that all fuzzy metric is an informative metric, where the IMV is $(G, \preceq, \prec\prec, \tilde{0})$ (see [8]).

A. Topology via Fuzzy Metric

We begin this section with a definition of open ball that has the same geometric sense of open balls in usual metric spaces.

Definition III.2. Let (M, d) be an gf-metric space. Given $a \in M$ and $R \in G$, with $\tilde{0} \prec\prec R$, the open ball of center a and radius R is the set $B(a, R) = \{b \in M; d(a, b) \prec\prec R\}$. A subset $O \subseteq M$ is an **open set**, if for every $a \in O$ exists an open ball $B(a, R)$, such that $B(a, R) \subseteq O$.

Observation III.1. The empty set is an open set by convention.

Theorem III.3. Let (M, d) be an gf-metric space. The class $\mathfrak{S}(M)$ of the open sets of M is a topology on M .

Proof: It must be proved that: $\emptyset, M \in \mathfrak{S}(M)$, if $\{A_\lambda\}_{\lambda \in L} \subseteq \mathfrak{S}(M)$, then $\bigcup_{\lambda \in L} A_\lambda \in \mathfrak{S}(M)$, and if $A, B \in \mathfrak{S}(M)$, then $A \cap B \in \mathfrak{S}(M)$. The first two conditions are immediate. Let $A, B \in \mathfrak{S}(M)$. Take $a \in A \cap B$. Since A and B are open sets, there are two open balls $B(a, R_1)$ and $B(a, R_2)$ such that $B(a, R_1) \subseteq A$ and $B(a, R_2) \subseteq B$. By the definition of open ball, we have $\tilde{0} \prec R_1$ and $\tilde{0} \prec R_2$, so, from the theorem II.3, there is $D \in G$, with $\tilde{0} \prec D$ such that $D \preceq R_1$ and $D \preceq R_2$. Thus, consider the open ball $B(a, D)$. Take $b \in B(a, D)$, which implies that $d(a, b) \prec D$. Thus, follows from the proposition II.3 that $d(a, b) \prec R_1 \Rightarrow b \in B(a, R_1)$ which implies that $B(a, D) \subseteq B(a, R_1)$. Similarly it is proved that $B(a, D) \subseteq B(a, R_2)$. Thus, $B(a, D) \subseteq A \cap B$, and so $A \cap B \in \mathfrak{S}(M)$. ■

Proposition III.1. *Let (M, d) be an gf-metric space. Every open ball are open sets.*

Proof: Follows directly from theorem III.2. ■

The class of all open balls in a gf-metric space (M, d) is a basis for the topology $\mathfrak{S}(M)$.

In [3], the authors defined a topology generated by a fuzzy metric space (M, d, L, R) , it is based on the following definition of open ball:

$$B(a, \varepsilon, \alpha) = \{y \in M; \rho_\alpha(x, y) < \varepsilon\} \quad (2)$$

Note that, the definition in 2 depends only on the upperbound of α -cuts $d(x, y)_\alpha$. Thus, the topology generated does not fully incorporate the vagueness expressed by the fuzzy number $d(x, y)$. However, the definition proposed here incorporates all the information contained in $d(x, y)$, since the relation used to define the open balls depends both: on upper and lowerbounds of each α -cut.

Remark III.1. *In usual metric spaces the strong order “ $<$ ” is used to define open balls. Those spaces has the property: “ $0 < d(x, y)$, whenever $x \neq y$ ”. In the last section we defined the notion of open balls in terms of “ $\prec$ ” relation and in what follows we define gf-spaces which share the corresponding property.*

Definition III.3 (Hausdorff gf-metric spaces). *A gf-metric space which satisfies: “ $\tilde{0} \prec d(x, y)$, whenever $x \neq y$ ” is called a **Hausdorff gf-metric space**.*

Theorem III.4. *Let (M, d) be a Hausdorff gf-metric space. The resulting topology, $\mathfrak{S}(M)$, is a Hausdorff topology.*

Proof: Let (M, d) be a gf-metric space. Take $a, b \in M$, where $a \neq b$. Thus, we have $\tilde{0} \prec d(a, b)$. From the theorem II.5 follows that there exists $B, C \in G$, with $\tilde{0} \prec B$ and $\tilde{0} \prec C$ such that $B \oplus C \prec d(a, b)$. Define the open balls $B(a, B)$ and $B(b, C)$. Suppose there exists $c \in B(a, B) \cap B(b, C)$, which implies $d(a, c) \prec B$ and $d(c, b) \prec C$, so $d(a, b) \preceq d(a, c) \oplus d(b, c) \prec B \oplus C \prec d(a, b) \Rightarrow d(a, b) \prec d(a, b)$, so $d(a, b) \prec d(a, b)$, which is a contradiction. ■

From now on we consider just Hausdorff gf-metric spaces.

Definition III.4. *Let x_n be a sequence in a gf-metric space (M, d) . We shall say that x_n **converges to** $L \in M$ if for every $\tilde{0} \prec R$, there exists $n_0 \in \mathbb{N}$, such that if $n \geq n_0$, then $d(x_n, L) \prec R$. In this case, we say x_n is **gf-convergent** and L is the **gf-limit** of x_n ; we use the notation: “ $x_n \longrightarrow L$ ”. A sequence x_n is said to be a **gf-cauchy sequence**, if for every $\tilde{0} \prec R$, there exists $n_0 \in \mathbb{N}$, such that if $m, n \geq n_0$, then $d(x_m, x_n) \prec R$.*

Observation III.2. *The theorem III.4 guarantees the unicity of the limit of a convergent sequence.*

Definition III.5. *A gf-metric space is said to be complete if every gf-cauchy sequence is convergent.*

Example III.4. *In a gf-metric space, every convergent sequence is a gf-cauchy sequence.*

In fact, if $x_n \longrightarrow L$, then for every $\tilde{0} \prec R$, there exists $n_1 \in \mathbb{N}$ such that $n > n_1 \Rightarrow d(x_n, L) \prec R_1$, and there exists $n_2 \in \mathbb{N}$, such that $m > n_2 \Rightarrow d(x_m, L) \prec R_2$, where $\tilde{0} \prec R_1$, $\tilde{0} \prec R_2$ and $R_1 \oplus R_2 \prec R$. Define $n_0 = \max\{n_1, n_2\}$. Thus, if $n > n_0$, then $d(x_n, x_m) \preceq d(x_n, L) \oplus d(L, x_m) \prec R_1 \oplus R_2 \preceq R \Rightarrow d(x_n, x_m) \prec R$.

B. gf-Continuity

In this section, we present the concept of gf-continuity, which represents the idea of continuity in the fuzzy context.

Definition III.6. *A function $f : M \longrightarrow N$ between the two gf-metric spaces (M, d_1) and (N, d_2) is called **gf-continuous at** $a \in M$, if for every $\tilde{0} \prec R$ there exist $\tilde{0} \prec D$ such that, if $d_1(a, b) \prec D$, then $d_2(f(a), f(b)) \prec R$. If f is gf-continuous for all $a \in M$, then we will simply say that f is **gf-continuous**.*

The next theorem shows how the gf-continuity is linked with the usual topological concept of continuity.

Theorem III.5. *A function $f : M \longrightarrow N$ is continuous with respect to the topologies $\mathfrak{S}(M)$ and $\mathfrak{S}(N)$ iff f is gf-continuous.*

Proof: The proof follows the same steps of the usual case (see [10]). ■

If $(\mathbb{R}^n, d)$ is a gf-metric space then constant functions, the identity function, sums and products (for $n = 1$) of gf-continuous functions are all gf-continuous functions.

Theorem III.6. *Let (M, d_1) and (N, d_2) be two gf-metric spaces and consider the mapping $f : M \longrightarrow N$. If f is gf-continuous at $a \in M$ and x_n is a sequence in M such that $x_n \longrightarrow a$, then $f(x_n) \longrightarrow f(a)$.*

Proof: Given $\tilde{0} \prec R$, there exists $\tilde{0} \prec D$ such that $d(a, x) \prec D \Rightarrow d(f(a), f(x)) \prec R$. Since $x_n \longrightarrow a$, there exists $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow d(x_n, a) \prec D$, so $d(f(x_n), f(a)) \prec R$, ie, $f(x_n) \longrightarrow f(a)$. ■

IV. APPLICATION: FIXED POINT THEOREM

The fixed point theorems play an important role in applications. All these theorems involves convenient definitions of

contractions. Below, we present the definition of contractions in gf-metric spaces.

Definition IV.1. Let (M, d) be a gf-metric space and $T : M \rightarrow M$ a mapping. We say that T is a **gf-contraction** if exists $K \in G$, with $K \prec \tilde{1}$, such that $d(T(x), T(y)) \preceq K \odot d(x, y)$.

Observation IV.1. Note that if $K \prec \tilde{1}$, then there exists $0 < r < 1$ such that $\overline{k_\alpha} < r$, for all $\alpha \in (0, 1]$.

Proposition IV.1. Let (M, d) be a gf-metric space and $T : M \rightarrow M$ a mapping. If T is a gf-contraction, then T is gf-continuous

Proof: Let $a \in M$. Since T is a gf-contraction, we have $d(T(x), T(y)) \preceq d(x, y)$, for all $x, y \in M$. Thus, given $\tilde{0} \prec R$, take $D = R$ and suppose that $d(a, x) \prec \prec D = R$. So, $d(T(a), T(x)) \preceq d(a, x) \prec \prec D = R \Rightarrow d(T(a), T(x)) \prec \prec R$. Thus, T is gf-continuous at any $a \in M$ and so, T is gf-continuous. ■

Below, we proof a fixed point theorem in gf-metric spaces which is very similar to the classical Banach's theorem.

Theorem IV.1 (Fixed Point Theorem). Let (M, d) be a complete gf-metric space and $T : M \rightarrow M$ a contraction. Thus, T has a unique fixed point.

Proof: Take $x_0 \in M$ and define the sequence x_n by $x_1 = T(x_0)$ and $x_{n+1} = T(x_n)$. Thus, $d(x_2, x_1) = d(T(x_1), T(x_0)) \preceq K \odot d(x_1, x_0)$, $d(x_3, x_2) = d(T(x_2), T(x_1)) \preceq K \odot d(x_2, x_1) \preceq K \odot (K \odot d(x_1, x_0)) = K^2 \odot d(x_1, x_0)$, etc.. In general, we have

$$d(x_{n+1}, x_n) \preceq K^n \odot d(x_0, x_1). \quad (3)$$

Given $m, n \in \mathbb{N}$, with $m > n$, using the triangle inequality and (3) we have:

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_n, x_{n+1}) \oplus d(x_{n+1}, x_{n+2}) \\ &\oplus \dots \oplus d(x_{m-1}, x_m) \\ &\preceq (K^n \odot d(x_1, x_0)) \oplus (K^{n+1} \odot d(x_1, x_0)) \\ &\oplus \dots \oplus (K^{m-1} \odot d(x_1, x_0)) \\ &= (K^n \oplus \dots \oplus K^{m-1}) \odot d(x_1, x_0), \text{ so} \\ d(x_m, x_n) &\preceq (K^n \oplus \dots \oplus K^{m-1}) \odot d(x_1, x_0) \end{aligned} \quad (4)$$

Consider $r > 0$ as in the observation IV.1.

If $K_\alpha = [\underline{k}_\alpha, \overline{k}_\alpha]$, for all $\alpha \in (0, 1]$, then $K_\alpha^n = [\underline{k}_\alpha^n, \overline{k}_\alpha^n]$, so

$$\begin{aligned} (K^n \oplus \dots \oplus K^{m-1})_\alpha &= \left[\sum_{i=n}^{m-1} \underline{k}_\alpha^i, \sum_{i=n}^{m-1} \overline{k}_\alpha^i \right] \\ &\leq_{km} \left[\sum_{i=n}^{m-1} r^i, \sum_{i=n}^{m-1} r^i \right]. \end{aligned} \quad (5)$$

Since the serie $\sum r^n$ is convergent, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $m > n > n_0 \Rightarrow \sum_{i=n}^{m-1} \underline{k}_\alpha^i \leq \sum_{i=n}^{m-1} \overline{k}_\alpha^i \leq$

$\sum_{i=n}^{m-1} r^i < \varepsilon$, for all $\alpha \in (0, 1]$, so

$$\left[\sum_{i=n}^{m-1} \underline{k}_\alpha^i, \sum_{i=n}^{m-1} \overline{k}_\alpha^i \right] \leq_{km} [\varepsilon, \varepsilon], \text{ for all } \alpha \in (0, 1]. \quad (6)$$

Since $\text{supp } d(x_1, x_0)$ is bounded, there exists $l > 0$ such that $0 < \lambda_\alpha(x_1, x_0) \leq \rho_\alpha(x_1, x_0) < l$, for all $\alpha \in (0, 1]$, where $(d(x_1, x_0))_\alpha = [\lambda_\alpha(x_1, x_0), \rho_\alpha(x_1, x_0)]$.

Given $\tilde{0} \prec \prec R$, we know there is $d > 0$ such that $d < \frac{r_\alpha}{\alpha} \leq \frac{\overline{r}_\alpha}{\alpha}$, for all $\alpha \in (0, 1]$. Take $0 < \varepsilon < \frac{d}{2l}$. Thus, there exists $n_0 \in \mathbb{N}$ such that:

$$\begin{aligned} m > n > n_0 &\Rightarrow (K^n \oplus K^{n+1} \oplus \dots \oplus K^{m-1})_\alpha \\ &\leq_{km} [\varepsilon, \varepsilon] \\ &\Rightarrow (K^n \oplus K^{n+1} \oplus \dots \oplus K^{m-1})_\alpha \\ &\quad \cdot [\lambda_\alpha(x_1, x_0), \rho_\alpha(x_1, x_0)] \\ &\leq_{km} [\varepsilon, \varepsilon] \cdot [\lambda_\alpha(x_1, x_0), \rho_\alpha(x_1, x_0)] \end{aligned} \quad (7)$$

Thus, we have $[(K^n \oplus \dots \oplus K^{m-1}) \odot d(x_1, x_0)]_\alpha \leq_{km} [\varepsilon \lambda_\alpha(x_1, x_0), \varepsilon \rho_\alpha(x_1, x_0)]$, for all $\alpha \in (0, 1]$, ie,

$$(K^n \oplus \dots \oplus K^{m-1}) \odot d(x_1, x_0) \preceq [\varepsilon, \varepsilon] \odot d(x_1, x_0). \quad (8)$$

Note that $\lambda_\alpha(x_1, x_0) < l \Rightarrow \varepsilon \lambda_\alpha(x_1, x_0) < \varepsilon l < \frac{d}{2l} l = \frac{d}{2}$, so, from $\frac{r_\alpha}{\alpha} > d$, follows that $\frac{r_\alpha}{\alpha} - \frac{d}{2} > \frac{d}{2} \Rightarrow \frac{r_\alpha}{\alpha} - \varepsilon \lambda_\alpha(x_1, x_0) > \frac{d}{2}$. Analogously, we proof that $\frac{\overline{r}_\alpha}{\alpha} - \varepsilon \rho_\alpha(x_1, x_0) > \frac{d}{2}$. Thus, we can conclude that $[\varepsilon \lambda_\alpha(x_1, x_0), \varepsilon \rho_\alpha(x_1, x_0)] \prec \prec R$, so, for all $m > n > n_0$, we have

$$\begin{aligned} d(x_m, x_n) &\preceq (K^n \oplus K^{n+1} \oplus \dots \oplus K^{m-1}) \odot d(x_1, x_0) \\ &\preceq [\varepsilon, \varepsilon] \odot d(x_1, x_0) \\ &\prec \prec R. \end{aligned}$$

Thus, we conclude that x_n is a gf-Cauchy sequence. Since the gf-metric space (M, d) is complete, we have that x_n is convergent. Let $a \in M$ such that $x_n \rightarrow a$. Since T is continuous, we have $T(x_n) \rightarrow T(a)$, but, $T(x_n) = x_{n+1}$, so $T(x_n) \rightarrow a$. Thus, as the limit of x_n is unique, we have $T(a) = a$, ie, a is a fixed point of T .

Now, suppose that $x, y \in M$ and $T(x) = x$, $T(y) = y$, with $x \neq y$. Follows that:

$$\begin{aligned} d(x, y) &= d(T(x), T(y)) \preceq K \odot d(x, y) \prec d(x, y) \\ &\Rightarrow d(x, y) \prec d(x, y), \text{ which is a contradiction.} \end{aligned}$$

So, the fixed point must to be unique. ■

V. CONCLUSION

The fuzzy metric proposed by Kaleva and Seikkala with $L = \min$ and $R = \max$ satisfies a very natural triangle inequality. In this case we showed how to define a Hausdorff topology, which carries more information than the original one defined in [3], since it depends on $d(x, y)$, and not just on a piece of $d(x, y)$, as in the original definition. With this definition it was possible to create natural notions of fuzzy continuity and fuzzy convergent sequence, from which we proved a fixed point theorem very similar to the classical Banach's theorem. In future works, we attempted to verify other classical results

on metric spaces in the fuzzy context and, mainly, to investigate ways to reduce the triangle inequality iii) for a more natural form, where L and R are arbitrary functions satisfying the conditions of the Kaleva and Seikkala's definition. We also intend to propose a concept of fuzzy quasi-metric, following the same ideas developed in this paper. Achieving that, we have in hands a topology that can not be Hausdorff, but will have other properties of separability, like the T_1 spaces (see [10]).

REFERENCES

- [1] I. Kramosil and J. Michalek, *Fuzzy Metric and Statistical Metric Spaces*, Kybernetika, vol. 11, pp. 336-344, 1975.
- [2] K. Menger, *Statistical Metric*, Proc. Nat. Acad. Sci., USA, vol. 28, pp. 535-537, 1942.
- [3] O. Kaleva and S. Seikkala, *On Fuzzy Metric Spaces*, Fuzzy sets and Systems, 12, pp. 215-229, 1984.
- [4] L. Zadeh, *Fuzzy Sets as a Basis for a Theory of Possibility*, Fuzzy Sets and Systems, vol. 100 (suplement), pp. 9-34, 1999.
- [5] L. Ciri, *Solving the Banach Fixed Point Principle for Nonlinear Contractions in Probabilistic Metric Spaces*, Nonlinear Analysis: Theory, Methods & Applications, vol. 72, pp 2009-2018, 2010.
- [6] B. Singh and S. Jain, *Semicompactibility and Fixed Point Theorems in Fuzzy Metric Space Using Implicit Relation*, International Journal of Mathematics and Mathematical Sciences, vol. 16, pp 2617-2629, 2005.
- [7] B. Lee, S. Lee and K. Park, *The Completions of Fuzzy Metric Spaces and Fuzzy Normed Linear Spaces*, Fuzzy Sets and Systems, vol. 106, pp 469-473, 1999.
- [8] F. Santana and R. Santiago, *Informative Metrics. Introducing Interval Metrics*, Interval Mathematics and Connections in Teaching and Scientific Development, edited by R. Reiser and M. Pilla, Federal University of Pelotas, University Publisher, pp 131-144, 2010.
- [9] M. Khamsi, V. Kreinovich and D. Misane, *A New Method of Proving the Existence of Answer Sets for Disjunctive Logic Programs: a Metric Fixed-point Theorem for Multivalued Mappings*, Vancouver: Proc. Workshop on logic programming with Incomplete Information, pp. 58-73, 1993.
- [10] J. Dugundji, *Topology*, Allyn and Bacon Inc, Boston, 1966.
- [11] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, New Jersey, Prentice Hall, 1995.
- [12] R. Moore, *Interval Analysis*, Englewood Cliffs, Prentice Hall, 1966.
- [13] U. Kulisch and W. Miranker, *Computer Arithmetic in Theory and Practice*, New York: Academic press, 1981.