

Synchronization of Uncertain Fractional Order Chaotic Systems via Adaptive Interval Type-2 Fuzzy Sliding Mode Control

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Abstract—In this paper, a novel adaptive interval type-2 fuzzy sliding mode control (AITFSMC) is proposed to handle high level uncertainties facing the fuzzy logic controller (FLC) in dynamic fractional order chaotic systems such as uncertainties in inputs to the FLC, uncertainties in control outputs, linguistic uncertainties and uncertainties associated with the noisy training data. Based on the learning algorithm combining Lyapunov approach and sliding mode control, free parameters of the AITSMC can be tuned on line by output feedback control law and adaptive law to synchronize two different uncertain fractional order chaotic systems. Meanwhile, the chattering phenomena in the control efforts can be reduced. During the design procedure, not only the stability and robustness can be guaranteed but also the external disturbance on the synchronization error can be attenuated. The numerical simulation is performed to illustrate the effectiveness of the proposed control strategy.

Keywords- Adaptive fuzzy, sliding mode control, fractional order, chaotic synchronization, Lyapunov synthesis

I. INTRODUCTION

In recent years, dynamics described by fractional differential equations are becoming more and more popular as the underlying facts about the differentiation and integration is significantly different from the integer counterparts and beyond this, many real life systems are described better by fractional order differential equations. For instance, electrochemical processes and flexible structures are modeled by fractional order models [1]-[4], the behavior of some biological systems is explored using fractional calculus [5] and dielectric polarization, electromagnetic waves and viscoelastic systems are involved with fractional order operators. It is observed that the description of some systems is more accurate when the fractional derivative is used [6]-[7]. Nowadays, many fractional-order differential systems behave chaotically, such as the fractional-order Chua's system [8-9], the fractional-order Duffing system [10-11], the fractional-order

Lu system [12], the fractional-order Chen's system [13-14], the fractional-order cellular neural network [15-16], and the fractional-order neural network [17].

Recently, synchronization of chaotic system is attractive in many applications such as power converters, biological systems, chemical reactions, information processing and secure communication. Over the past few years, fractional operators have been applied with satisfactory results in modeling and control of complex systems and synchronization of chaos has been investigated in many papers [18-20,40-42]. The algorithms based on active sliding mode controller employed to synchronize two different chaotic systems are proposed in [21]-[23]. However, there exists a chattering phenomenon implementing a sliding mode control (SMC), and this may excite high-frequency dynamics [24]-[26]. In order to eliminate chattering, Palm [27] noted the similarity between fuzzy controller and sliding mode controller with a boundary layer, and provided a fuzzy sliding mode design approach.

Although, type-1 fuzzy logic controllers (FLCs) have been successfully applied in many different applications, for dynamic unstructured environments and many real world applications, there is a need to cope with large amount of uncertainties such as uncertainties in inputs to the FLC, uncertainties in control outputs, linguistic uncertainties and uncertainties associated with the noisy training data. Type-2 FLCs have potential to overcome the limitations of type-1 FLCs and provide improved performance for many applications [32]-[36]. In this paper, a novel adaptive interval type-2 fuzzy sliding mode controller is developed to synchronize two different uncertain fractional order chaotic systems with high level uncertainties.

This paper is organized as follows: In section II, an introduction to fractional derivative and its relation to the approximation solution will be addressed. A brief of the interval type-2 fuzzy neural network (FNN) is presented in section III. Section IV generally proposes the employment of

the AITFSMC for synchronizing the fractional order chaotic system in presence of uncertainty and its stability analysis. In Section V, application of the proposed method on fractional order expression of Duffing–Holmes chaotic system is investigated. Finally, the simulation results and conclusion will be presented in Section VI.

II. BASIC DEFINITION AND PRELIMINARIES FOR FRACTIONAL ORDER SYSTEMS

Fractional calculus is a mathematical topic more than 300 years. It is a generalization of integration and differentiation to non-integer order fundamental operator, denoted by ${}_a D_t^q$, where a and t are the limits of the operator. This operator is a notation for taking both the fractional integral and functional derivative in a single expression defined as

$${}_a D_t^q = \begin{cases} \frac{d^q}{dt^q}, & q > 0 \\ 1 & q = 0 \\ \int_a^t (\tau)^{-q}, & q < 0 \end{cases} \quad (1)$$

There are some basic definitions of the general fractional integration and differentiation. The commonly used definitions are Grunwald-Letnikov and Riemann-Liouville. The Grunwald-Letnikov definition is expressed as

$${}_a D_t^q f(t) = \lim_{h \rightarrow 0} \sum_{j=0}^{\left[\frac{t-a}{h} \right]} (-1)^j \binom{q}{j} f(t-jh) \quad (2)$$

where $[\cdot]$ is the integer part. The simplest and easiest definition is Riemann-Liouville definition given as

$${}_a D_t^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau \quad (3)$$

where n is the first integer which is not less q , i.e., $n-1 < q < n$, and Γ is the Gamma function.

The numerical simulation of a fractional differential equation is not simple as that of an ordinary differential equation. In this paper, the algorithm which is an improved version of Adams–Bashforth–Moulton algorithm [28-30] to find an approximation for fractional order systems based on predictor-correctors [30-31] is given. Consider the following differential equation

$${}_0 D_t^q y(t) = r(y(t), t), \quad 0 \leq t \leq T \quad (4)$$

$$\text{and } y^{(k)}(0) = y_0^{(k)}, k = 0, 1, 2, \dots, m-1$$

where

$${}_0 D_t^q y(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q-m+1}} d\tau, & m-1 < q < m \\ \frac{d^m}{dt^m} y(t) & q = m \end{cases} \quad (5)$$

and m is the first integer larger the q . The solution of the equation (4) is equivalent to Volterra integral equation [32] described as

$$y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-\lambda)^{q-1} r(y(\lambda), \lambda) d\lambda \quad (6)$$

Let $h = T/N$, $t_n = nh$, $n = 0, 1, 2, \dots, N$. Then (6) can be discretized as follows.

$$y_h(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} r(y_h^p(t_{n+1}), t_{n+1}) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^n a_{j,n+1} r(y_h(t_j), t_j) \quad (7)$$

where predicted value $y_h^p(t_{n+1})$ is determined by

$$y_h^p(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q)} \sum_{j=0}^n b_{j,n+1} r(y_h(t_j), t_j) \quad (8)$$

and

$$a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1} & 1 \leq j \leq n \\ 1 & j = n+1 \end{cases} \quad (9)$$

$$b_{j,n+1} = \frac{h^q}{q} ((n+1-j)^q - (n-j)^q) \quad (10)$$

The approximation error is given as

$$\max_{j=0,1,2,\dots,N} |y(t_j) - y_h(t_j)| = O(h^p) \quad (11)$$

where $p = \min(2, 1+q)$. Therefore, the numerical solution of a fraction order system can be obtained by applying the above mentioned algorithm.

III. BRIEF DESCRIPTION OF INTERVAL TYPE-2 FNN [32]-[36]

In this section, we will describe the interval type-2 fuzzy set and the inference of type-2 fuzzy logic system which leads to interval type-2 fuzzy neural network (T2FNN). Due to the complexity of the type reduction, the general type-2 FLS becomes computationally intensive. In order to make things simpler and easier to compute meet and join operations, the secondary MFs of an interval type-2 FLS are all unity which leads finally to simplify type reduction. The 2-D interval type-2 Gaussian membership function (MF) with uncertain mean $m \in [m_1, m_2]$ and a fixed deviation σ is shown in Fig. 1.

$$\mu_{\tilde{A}}(x) = \exp \left[-\frac{1}{2} \left(\frac{x-m}{\sigma} \right)^2 \right], \quad m \in [m_1, m_2] \quad (12)$$

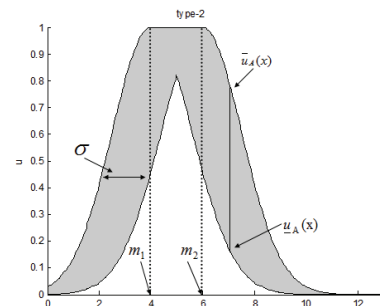


Figure 1. Interval type-2 fuzzy set with uncertain mean

It is obvious that the type-2 fuzzy set is in a region bounded by an upper MF and a lower MF denoted as $\bar{\mu}_i(x)$ and $\underline{\mu}_i(x)$, respectively, and is called a foot of uncertainty (FOU). In the meantime, the firing strength F^i for the i th rule can be an interval type-1 set [9] expressed as

$$F^i = \left[\frac{f^i}{\bar{f}^i} \right] \quad (13)$$

Where

$$\underline{f}^i = \underline{\mu}_{\tilde{F}_1^i}(x_1) * \dots * \underline{\mu}_{\tilde{F}_n^i}(x_n) = \prod_{j=1}^n \underline{\mu}_{\tilde{F}_j^i}(x_j) \quad (14)$$

$$\bar{f}^i = \bar{\mu}_{\tilde{F}_1^i}(x_1) * \dots * \bar{\mu}_{\tilde{F}_n^i}(x_n) = \prod_{j=1}^n \bar{\mu}_{\tilde{F}_j^i}(x_j) \quad (15)$$

Based on the center of sets type reduction, the defuzzified crisp output from an interval type-2 FLS is the average of y_l and y_r , i.e.,

$$\begin{aligned} y(x) &= \frac{y_l + y_r}{2} = \frac{1}{2}(\xi_r^T \Theta_r + \xi_l^T \Theta_l) \\ &= \frac{1}{2} \begin{bmatrix} \xi_r^T & \xi_l^T \end{bmatrix} \begin{bmatrix} \Theta_r \\ \Theta_l \end{bmatrix} = \xi^T \Theta \end{aligned} \quad (16)$$

where $(1/2)[\xi_r^T \ \xi_l^T] = \xi^T$ and $[\Theta_r^T \ \Theta_l^T] = \Theta^T$.

where y_l and y_r are the left most and right most points of the interval type-1 set which can be obtained as

$$\begin{aligned} y_l &= \frac{\sum_i^M f_l^i w_l^i}{\sum_i^M f_l^i} = \frac{\sum_{i=1}^L \bar{f}^i w_l^i + \sum_{i=L+1}^M \underline{f}^i w_l^i}{\sum_{i=1}^L \bar{f}^i + \sum_{i=L+1}^M \underline{f}^i} \\ &= \sum_{i=1}^L \underline{q}_l^i y_l^i + \sum_{i=L+1}^M \bar{q}_l^i y_l^i = [\underline{Q}_l \ \bar{Q}_l^T] \begin{bmatrix} y_l \\ \bar{y}_l \end{bmatrix} = \xi_l^T \Theta_l \end{aligned} \quad (17)$$

where $\underline{q}_l^i = \underline{f}^i / D_l$, $\bar{q}_l^i = \bar{f}^i / D_l$ and $D_l = (\sum_{i=1}^L \bar{f}^i + \sum_{i=L+1}^M \underline{f}^i)$. In

the meantime, we have $\underline{Q}_l = [\underline{q}_l^1, \underline{q}_l^2, \dots, \underline{q}_l^R]$, $\bar{Q}_l^T = [\bar{q}_l^1, \bar{q}_l^2, \dots, \bar{q}_l^R]$, $\xi_l^T = [\underline{Q}_l \ \bar{Q}_l^T]$ and $\Theta_l^T = [y_l \ \bar{y}_l^T]$.

and

$$\begin{aligned} y_r &= \frac{\sum_i^M f_r^i w_r^i}{\sum_i^M f_r^i} = \frac{\sum_{i=1}^R \underline{f}^i w_r^i + \sum_{i=R+1}^M \bar{f}^i w_r^i}{\sum_{i=1}^R \underline{f}^i + \sum_{i=R+1}^M \bar{f}^i} \\ &= \sum_{i=1}^R \underline{q}_r^i y_r^i + \sum_{i=R+1}^M \bar{q}_r^i y_r^i = [\underline{Q}_r \ \bar{Q}_r^T] \begin{bmatrix} y_r \\ \bar{y}_r \end{bmatrix} = \xi_r^T \Theta_r \end{aligned} \quad (18)$$

where $\underline{q}_r^i = \underline{f}^i / D_r$, $\bar{q}_r^i = \bar{f}^i / D_r$ and $D_r = (\sum_{i=1}^R \underline{f}^i + \sum_{i=R+1}^M \bar{f}^i)$.

In the meantime, we have $\underline{Q}_r = [\underline{q}_r^1, \underline{q}_r^2, \dots, \underline{q}_r^R]$, $\bar{Q}_r^T = [\bar{q}_r^1, \bar{q}_r^2, \dots, \bar{q}_r^R]$, $\xi_r^T = [\underline{Q}_r \ \bar{Q}_r^T]$ and $\Theta_r^T = [y_r \ \bar{y}_r^T]$, and M is the total number of rules in the rule base of the RIT2FNN. The weighting factors w_l^i and w_r^i of the consequent part represent

the centroid interval set of the consequent type-2 fuzzy set of the i th rule. In the meantime, R and L can be determined by using the iterative Karnik-Mendel procedure [32]-[36].

An interval T2FNN system which is an implementation of interval type-2 fuzzy logic system, and some of their parameters and components are presented by fuzzy logic terms, is shown in Fig. 2 [34]. The interval T2FNN is a four layers structure [34]. Layer I and layer II represent input nodes and type-2 fuzzification nodes, respectively, which form the antecedent part of this T2FNN. A classical 2-layer NN with fuzzy rule nodes and output nodes is used to construct layer III and layer IV, respectively, which compose the consequent part of this T2FNN. d_i , $i = 1 \sim z$ are reference signals.

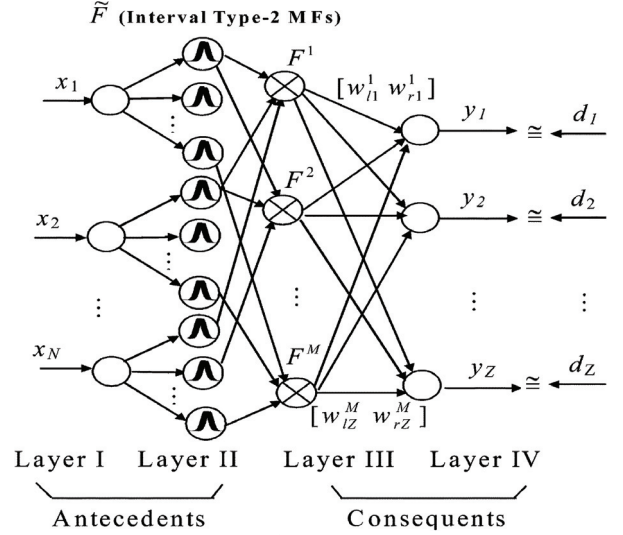


Figure 2. Interval T2FNN with antecedent part and consequent part.

IV. ADAPTIVE INTERVAL TYPE-2 FUZZY SLIDING MODEL SYNCHRONIZATION OF FRACTIONAL ORDER CHAOTIC SYSTEMS

Let's consider two chaotic systems with fractional order derivative, driven system and response system, as follows:

Driven system:

$$D^{(q)} x_1 = x_2$$

$$D^{(q)} x_2 = f(x, t) \quad (19)$$

Response system:

$$D^{(q)} y_1 = y_2$$

$$D^{(q)} y_2 = g(y, t) + u(t) + d(t) \quad (20)$$

where x_1 , x_2 , y_1 and y_2 are the state variables, $f(x, t)$ and $g(y, t)$ are unknown but bounded nonlinear functions which express system dynamics, $d(t)$ is the external bounded disturbance and $u(t)$ is the control input of the response system. The control objective to synchronize both driven and response systems by designing a nonlinear controller which obtains signals from driven system to tune behavior of the response system.

The synchronization error vector is defined as

$$e = [e_1, e_2] \quad (21)$$

where $e_i = y_i - x_i$. Then, in the space of the synchronization error, sliding surface is given by

$$S(t) = k_1 e_1 + k_2 e_2 \quad (22)$$

where k_1, k_2 are arbitrary constants which are chosen such that dynamic of the sliding surface vanished quickly. The process can be classified into two phases, approaching phase with $S(t) \neq 0$ and sliding phase with $S(t) = 0$. In order to guarantee the trajectory of the synchronization error vector e will move from approaching phase to sliding phase, a sufficient condition to is to design the control effort such that sliding condition

$$S(t)\dot{S}(t) \leq -\eta|S(t)|, \quad \eta > 0 \quad (23)$$

is satisfied. We have $S(t) = 0$ and $\dot{S}(t) = 0$ in the sliding phase, if $f(x, t)$ and $g(y, t)$ are known and free of external disturbance, $d(t) = 0$, the corresponding equivalent control effort u_{eq} to force the system dynamics to stay on the sliding surface can be derived from $\dot{S}(t) = 0$.

$$\dot{S}(t) = D^{(1-q)}(D^{(q)}(S(t))) = 0 \rightarrow D^{(q)}(S(t)) = 0 \quad (24)$$

Substituting (22) into (24), we have

$$\begin{aligned} D^{(q)}(S(t)) &= D^{(q)}(k_1 e_1 + k_2 e_2) = k_1 D^{(q)} e_1 + k_2 D^{(q)} e_2 \\ &= k_1 D^{(q)}(y_1 - x_1) + k_2 D^{(q)}(y_2 - x_2) \\ &= k_1(y_2 - x_2) + k_2[g(y, t) - f(x, t) + u_{eq}(t)] \\ &= k_1 e_2 + k_2[g(y, t) - f(x, t) + u_{eq}(t)] = 0 \end{aligned}$$

Then the equivalent control effort can be obtained as

$$u_{eq}(t) = -\frac{k_1}{k_2} e_2 + f(x, t) - g(y, t) \quad (25)$$

During the approaching phase, in order to satisfy the sliding condition (23), a switching control action $u_{sw} = \eta_{sw} \text{sgn}(S(t))$ must be included in the complete sliding control which can be expressed as

$$\begin{aligned} u(t) &= u_{eq}(t) - \eta_{sw} \text{sgn}(S(t)) \\ &= -\frac{k_1}{k_2} e_2 + f(x, t) - g(y, t) - \eta_{sw} \text{sgn}(S(t)) \end{aligned} \quad (26)$$

where η_{sw} is a positive constant such the approaching condition can be guaranteed for,

$$\eta_{sw} > |D^{1-q} d(t)| + |D^{1-q} \omega| \quad (27)$$

where ω is the minimum approximation error, $|d(t)|$ and $|\omega|$ are assumed to be bounded.

However, $f(x, t)$ and $g(y, t)$ are unknown and external disturbance, $d(t) \neq 0$, the ideal control effort (23) cannot be implemented. We replace $f(x, t)$ and $g(y, t)$ by the interval type-2 fuzzy logic system $f(x|\underline{\theta}_f)$ and $g(y|\underline{\theta}_g)$, respectively, in specified form as (16), i.e.,

$$\begin{aligned} f(x|\underline{\theta}_f) &= \frac{1}{2} \left(\underline{\theta}_{fr}^T \underline{\xi}_r(x) + \underline{\theta}_{fl}^T \underline{\xi}_l(x) \right) = \underline{\theta}_f^T \underline{\xi}(x), \\ g(y|\underline{\theta}_g) &= \frac{1}{2} \left(\underline{\theta}_{gr}^T \underline{\xi}_r(y) + \underline{\theta}_{gl}^T \underline{\xi}_l(y) \right) = \underline{\theta}_g^T \underline{\xi}(y) \end{aligned} \quad (28)$$

where $\underline{\theta}_f^T = [\underline{\theta}_{fr} \quad \underline{\theta}_{fl}]^T$ and $\underline{\theta}_g^T = [\underline{\theta}_{gr} \quad \underline{\theta}_{gl}]^T$. Here the parameters $\underline{\xi}_r(x)$, $\underline{\xi}_l(x)$, $\underline{\xi}_r(y)$ and $\underline{\xi}_l(y)$ depend on the fuzzy membership functions and are supposed to be fixed, while $\underline{\theta}_{fr}^T$, $\underline{\theta}_{fl}^T$, $\underline{\theta}_{gr}^T$ and $\underline{\theta}_{gl}^T$ are adjusted by adaptive laws based on Lyapunov stability criterion. Therefore, the resulting control effort can be obtained as

$$u(t) = -\frac{k_1}{k_2} e_2 + f(x|\underline{\theta}_f) - g(y|\underline{\theta}_g) - \eta_{sw} \text{sgn}(S(t)) \quad (29)$$

Following the proceeding consideration, the following theorem can be obtained.

Theorem: Consider the two fractional order chaotic systems, driven system (16) and response system (17), the control effort of the response system is given in (26) and the fuzzy-based adaptive laws are chosen as

$$D^{(q)} \underline{\theta}_{fr} = -\frac{1}{2} r_1 S(t) \underline{\xi}_r(x) \quad (30)$$

$$D^{(q)} \underline{\theta}_{fl} = -\frac{1}{2} r_2 S(t) \underline{\xi}_l(x) \quad (31)$$

$$D^{(q)} \underline{\theta}_{gr} = \frac{1}{2} r_3 S(t) \underline{\xi}_r(y) \quad (32)$$

$$D^{(q)} \underline{\theta}_{gl} = \frac{1}{2} r_4 S(t) \underline{\xi}_l(y) \quad (33)$$

Then, the overall adaptive scheme guarantees the global stability of the resulting closed-loop system in the sense that all signals involved are uniformly bounded and the synchronization error will converge to zero asymptotically.

Proof: The optimal parameter estimations $\underline{\theta}_f^*$ and $\underline{\theta}_g^*$ are defined as

$$\underline{\theta}_f^* = \arg \min_{\underline{\theta}_f \in \Omega_f} \left[\sup_{x \in \Omega_x} |f(x|\underline{\theta}_f) - f(x, t)| \right] \quad (34)$$

$$\underline{\theta}_g^* = \arg \min_{\underline{\theta}_g \in \Omega_g} \left[\sup_{y \in \Omega_y} |g(y|\underline{\theta}_g) - g(y, t)| \right] \quad (35)$$

where Ω_f , Ω_g , Ω_y and Ω_x are constraint sets of suitable bounds on $\underline{\theta}_f$, $\underline{\theta}_g$, y and x respectively and they are defined as $\Omega_f = \{\underline{\theta}_f \mid |\underline{\theta}_f| \leq M_f\}$, $\Omega_g = \{\underline{\theta}_g \mid |\underline{\theta}_g| \leq M_g\}$, $\Omega_y = \{y \mid |y| \leq M_y\}$ and $\Omega_x = \{x \mid |x| \leq M_x\}$, where M_f , M_g , M_y and M_x are positive constants. Then, we have

$$\begin{aligned} D^{(q)}(S(t)) &= D^{(q)}(k_1 e_1 + k_2 e_2) = k_1 D^{(q)} e_1 + k_2 D^{(q)} e_2 \\ &= k_1 e_2 + k_2 [g(y, t) - f(x, t) + u(t) + d(t)] \\ &= k_1 e_2 + k_2 \{ [g(y, t) - f(x, t)] + [-\frac{k_1}{k_2} e_2 + f(x|\underline{\theta}_f) - g(y|\underline{\theta}_g) - \eta_{sw} \text{sgn}(S(t))] + d(t) \} \\ &= k_2 \{ \frac{1}{2} [(\underline{\theta}_{gr}^* - \underline{\theta}_{gr})^T \underline{\xi}_r(y) + (\underline{\theta}_{gl}^* - \underline{\theta}_{gl})^T \underline{\xi}_l(y) - (\underline{\theta}_{fr}^* - \underline{\theta}_{fr})^T \underline{\xi}_r(x) - (\underline{\theta}_{fl}^* - \underline{\theta}_{fl})^T \underline{\xi}_l(x)] - k_2 [\eta_{sw} (\text{sgn}(S(t)) - d(t) - \omega)] \} \end{aligned}$$

$$= k_2 \frac{1}{2} [\tilde{\theta}_{gr}^T \xi_r(y) + \tilde{\theta}_{gl}^T \xi_l(y) - \tilde{\theta}_{fr}^T \xi_r(x) - \tilde{\theta}_{fl}^T \xi_l(x)] - k_2 [\eta_{sw} \operatorname{sgn}(S(t)) - d(t) - \omega] \quad (36)$$

where $\tilde{\theta}_{fr} = \theta_{fr}^* - \theta_{fr}$, $\tilde{\theta}_{fl} = \theta_{fl}^* - \theta_{fl}$, $\tilde{\theta}_{gr} = \theta_{gr}^* - \theta_{gr}$, $\tilde{\theta}_{gl} = \theta_{gl}^* - \theta_{gl}$ and the minimum approximation error is

$$\omega = [g(y, t) - g(y|\theta_g^*)] - [f(x, t) - f(x|\theta_f^*)] \quad (37)$$

Now consider the Lyapunov function candidate

$$V = \frac{1}{2} S^2(t) + \frac{k_2}{2r_1} \tilde{\theta}_{fr}^T \tilde{\theta}_{fr} + \frac{k_2}{2r_2} \tilde{\theta}_{fl}^T \tilde{\theta}_{fl} + \frac{k_2}{2r_3} \tilde{\theta}_{gr}^T \tilde{\theta}_{gr} + \frac{k_2}{2r_4} \tilde{\theta}_{gl}^T \tilde{\theta}_{gl} \quad (38)$$

where r_1, r_2, r_3 and r_4 are positive constants. Taking the derivative of the (38) with respect to time, we get

$$\begin{aligned} V^{(q)} &= S(t) S^{(q)}(t) + \frac{k_2}{r_1} (D^{(q)} \tilde{\theta}_{fr}^T)(\tilde{\theta}_{fr}) + \frac{k_2}{r_2} (D^{(q)} \tilde{\theta}_{fl}^T)(\tilde{\theta}_{fl}) \\ &\quad + \frac{k_2}{r_3} (D^{(q)} \tilde{\theta}_{gr}^T)(\tilde{\theta}_{gr}) + \frac{k_2}{r_4} (D^{(q)} \tilde{\theta}_{gl}^T)(\tilde{\theta}_{gl}) \\ &= S(t) k_2 \frac{1}{2} [\tilde{\theta}_{gr}^T \xi_r(y) + \tilde{\theta}_{gl}^T \xi_l(y) - \tilde{\theta}_{fr}^T \xi_r(x) - \tilde{\theta}_{fl}^T \xi_l(x)] \\ &\quad - k_2 [\eta_{sw} \operatorname{sgn}(S(t)) - d(t) - \omega] + \frac{k_2}{r_1} (D^{(q)} \tilde{\theta}_{fr}^T)(\tilde{\theta}_{fr}) \\ &\quad + \frac{k_2}{r_2} (D^{(q)} \tilde{\theta}_{fl}^T)(\tilde{\theta}_{fl}) + \frac{k_2}{r_3} (D^{(q)} \tilde{\theta}_{gr}^T)(\tilde{\theta}_{gr}) + \frac{k_2}{r_4} (D^{(q)} \tilde{\theta}_{gl}^T)(\tilde{\theta}_{gl}) \\ &= k_2 \tilde{\theta}_{gr}^T [\frac{1}{2} S(t) \xi_r(y) + \frac{1}{r_3} (D^{(q)} \tilde{\theta}_{gr})] + k_2 (\tilde{\theta}_{gl}^T) [\frac{1}{2} S(t) \xi_l(y) \\ &\quad + \frac{1}{r_4} (D^{(q)} \tilde{\theta}_{gl})] - k_2 \tilde{\theta}_{fr}^T [\frac{1}{2} S(t) \xi_r(x) - \frac{1}{r_1} (D^{(q)} \tilde{\theta}_{fr})] \\ &\quad - k_2 (\tilde{\theta}_{fl}^T) [\frac{1}{2} S(t) \xi_l(x) - \frac{1}{r_2} (D^{(q)} \tilde{\theta}_{fl})] \\ &\quad - k_2 \eta_{sw} |S(t)| + k_2 S(t) d(t) + k_2 S(t) \omega \end{aligned} \quad (39)$$

Substituting (30)-(33) into (39) and using (27), we have

$$\begin{aligned} V^{(q)} &\leq -k_2 \{ \eta_{sw} |S(t)| - |S(t)| |d(t)| - |S(t)| |\omega| \} \\ &= -k_2 |S(t)| \{ \eta_{sw} - |d(t)| - |\omega| \} < 0 \end{aligned} \quad (40)$$

The existence of adaptive fuzzy sliding mode dynamics is confirmed by (40) and the closed-loop system is globally asymptotically stable. The proof is completed.

V. SIMULATION EXAMPLE

In this section, we will apply our adaptive interval type-2 fuzzy sliding model controller to synchronize two different uncertain fractional order Duffing-Holmes chaotic systems, the drive system and the response system.

Example: Let us consider two different uncertain fractional order Duffing-Holmes chaotic systems, the drive system and the response system, as follows [37]:

Drive system:

$$\begin{cases} D^{(q)} x_1 = 2.2 x_2 \\ D^{(q)} x_2 = -0.40 x_1^3 - 0.40 x_1 - 0.10 x_2 + 22 \cos(1.29t) \end{cases} \quad (41)$$

Response system

$$\begin{cases} D^{(q)} y_1 = 2.2 y_2 \\ D^{(q)} y_2 = -0.55 y_1^3 - 0.30 y_1 - 0.20 y_2 + 18 \cos(1.29t) + u(t) + d(t) \end{cases} \quad (42)$$

The external disturbance is $d(t) = 0.5 \sin(t)$. The initial conditions of drive and response systems are chosen as:

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad \text{respectively.}$$

All design constants are selected as, $k_1=0.2$, $k_2=1$, $r_1=60$, $r_2=2$, $r_3=60$, $r_4=4$, $\eta_{sw}=1$ and step size $h=0.002$. The values of $q=0.98$ is considered.

For free of control input, the 3-D phase portrait of the drive and response systems is given in Figure. 3. It is obvious that both systems are not synchronized without control effort added in response system.

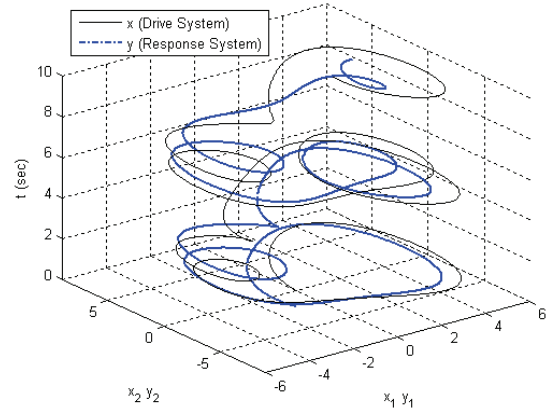


Figure 3. 3-D Phase portrait of chaotic drive and response systems.

From the adaptive laws (30)-(33), the control effort of the response system is given by

$$u(t) = -\frac{k_1}{k_2} e_2 + f(x|\theta_f) - g(y|\theta_g) - \eta_{sw} \operatorname{sgn}(S(t)) \quad (43)$$

The simulation results of both adaptive type-1 FNN SMC and adaptive interval type-2 FNN SMC are listed as follows: Figure 4 and Figure 5 show the trajectories of the states x_1 , y_1 and x_2 , y_2 , respectively. Control effort trajectory is given in Figure 6 and 3-D phase portrait, synchronization performance, of the drive and response systems is shown in Figure 7. Sliding surface trajectory, $S(t)$, is given in Figure 8. Figure 9 shows the graph of $V^{(q)}(t)$ which is always negative defined and consequently is stable.

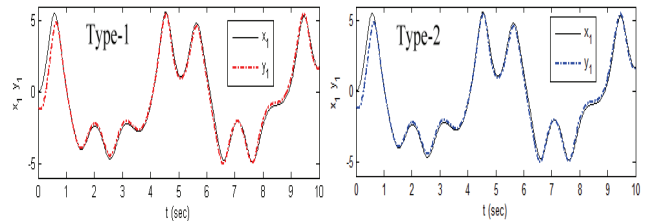


Figure 4. The trajectories of the states x_1 and y_1 .

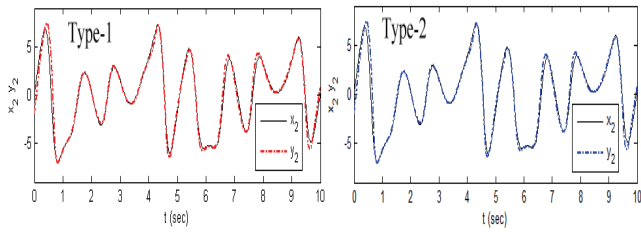


Figure 5. The trajectories of the states x_2 and y_2 .

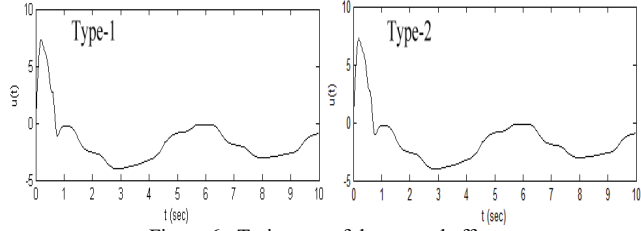


Figure 6. Trajectory of the control effort.

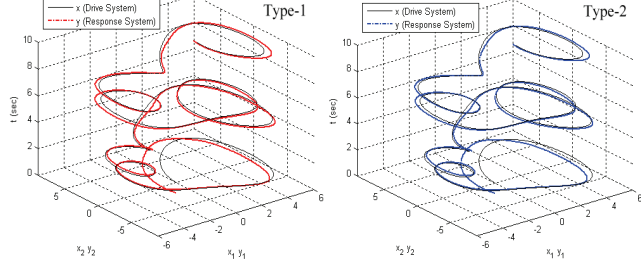


Figure 7. 3-D Phase portrait, synchronization performance, of the drive and response systems.

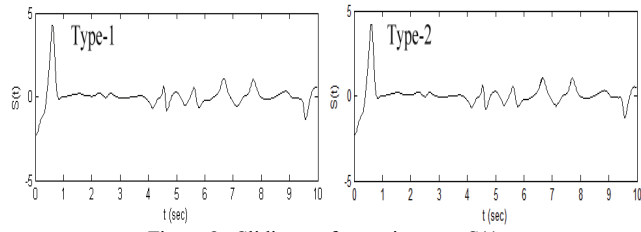


Figure 8. Sliding surface trajectory, $S(t)$.

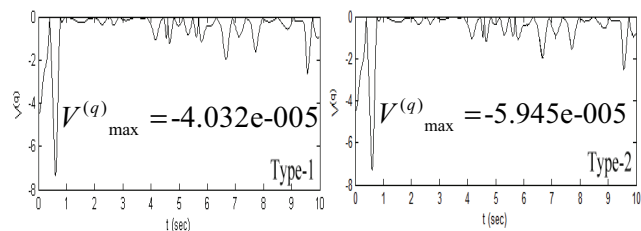


Figure 9. The graph of $V^{(q)}(t)$.

In order to show that the interval type-2 FLS can handle the measurement uncertainties, training data are corrupted by white Gaussian noise with signal-to-noise ratio (SNR) 20 dB. The trajectories of the states x_1 , y_1 and x_2 , y_2 , respectively, are given in Figure 10 and Figure 11. Figure 12 show the control effort trajectory and 3-D phase portrait, synchronization performance, of the drive and response systems is shown in Figure 13. Sliding surface trajectory, $S(t)$, is given in Figure 14. Figure 15 shows the graph of $V^{(q)}(t)$ which is always negative defined and consequently is stable.

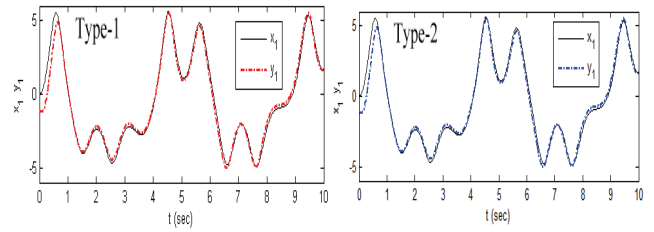


Figure 10. The trajectories of the states x_1 and y_1 .

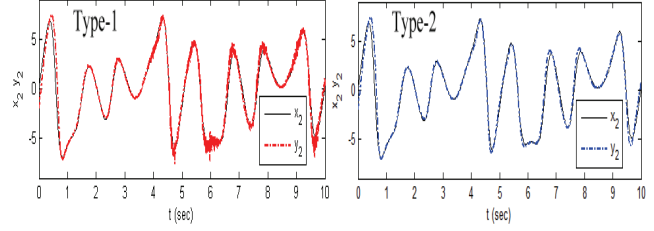


Figure 11. The trajectories of the states x_2 and y_2 .

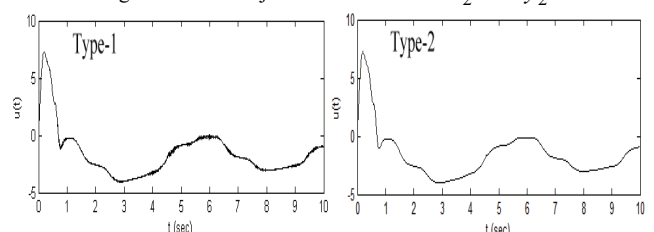


Figure 12. Trajectory of the control effort.

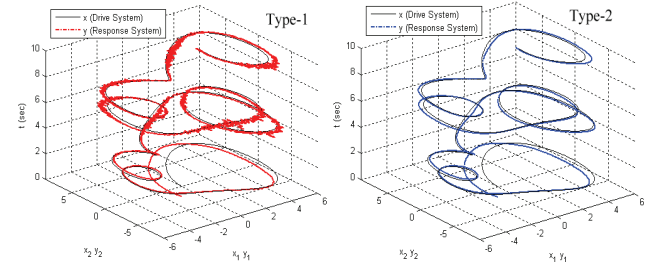


Figure 13. 3-D Phase portrait, synchronization performance, of the drive and response systems.

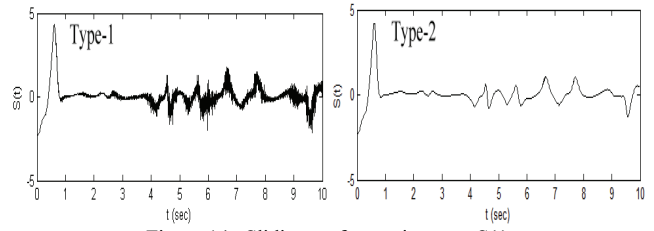


Figure 14. Sliding surface trajectory, $S(t)$.

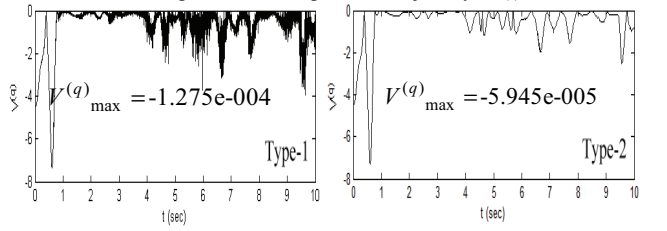


Figure 15. The graph of $V^{(q)}(t)$.

The synchronization performance, mean square errors of $\text{MESE1} = \sum_{k=0}^{5000} (y_1(k) - x_1(k))^2$ and $\text{MSE2} = \sum_{k=0}^{5000} (y_2(k) - x_2(k))^2$, for different values of q are shown in Figure 16 and Figure 17,

respectively. We can see that a fast synchronization of drive and response systems can be achieved and q is reduced the chaos is seen reduced, i.e., the synchronization error is reduced, accordingly. MSE1 and MSE2 of different q for type-1 and interval type-2 FNN controllers are given in Table I and Table II. MSU of different q for type-1 and interval type-2 FNN controllers are shown in Table III, where

$$MSU = \sum_{k=0}^{5000} u(k)^2.$$

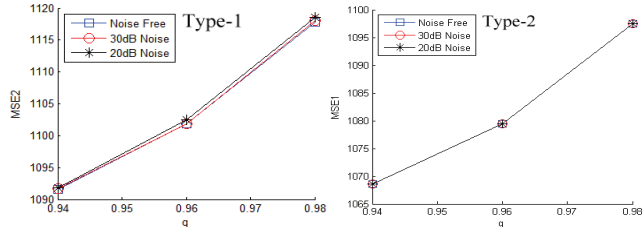


Figure 16. Mean square errors of $MSE1 = y_1 - x_1$ for different value of q .

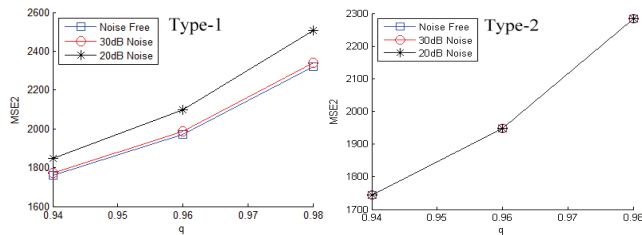


Figure 17. Mean square errors of $MSE2 = y_2 - x_2$ for different value of q .

Table I. MSE1 of the type-1 and the interval type-2 FNN controllers

<i>MSE1</i>	<i>q=0.98</i>		<i>q=0.96</i>		<i>q=0.94</i>	
	<i>Type-1</i>	<i>Type-2</i>	<i>Type-1</i>	<i>Type-2</i>	<i>Type-1</i>	<i>Type-2</i>
<i>Noise Free</i>	1117.8	1097.5	1101.8	1079.4	1091.6	1068.5
<i>30dB Noise</i>	1118.0	1097.5	1101.9	1079.4	1091.7	1068.5
<i>20dB Noise</i>	1118.6	1097.5	1102.5	1079.4	1091.8	1068.5

Table II. MSE2 of the type-1 and the interval type-2 FNN controllers

<i>MSE2</i>	<i>q=0.98</i>		<i>q=0.96</i>		<i>q=0.94</i>	
	<i>Type-1</i>	<i>Type-2</i>	<i>Type-1</i>	<i>Type-2</i>	<i>Type-1</i>	<i>Type-2</i>
<i>Noise Free</i>	2318.9	2283.5	1970.4	1947.8	1760.9	1743.5
<i>30dB Noise</i>	2341.5	2283.5	1987.8	1947.8	1772.0	1743.5
<i>20dB Noise</i>	2508.3	2283.5	2098.5	1947.8	1848.9	1743.5

Table III. MSU of the type-1 and the interval type-2 FNN controllers

<i>MSU</i>	<i>q=0.98</i>		<i>q=0.96</i>		<i>q=0.94</i>	
	<i>Type-1</i>	<i>Type-2</i>	<i>Type-1</i>	<i>Type-2</i>	<i>Type-1</i>	<i>Type-2</i>
<i>Noise Free</i>	34601	34452	38415	38298	40143	40004
<i>30dB Noise</i>	34606	34452	38427	38298	40145	40004
<i>20dB Noise</i>	34609	34452	38430	38298	40147	40004

VI. CONCLUSIONS

In this paper adaptive interval type-2 FNN SMC is proposed to deal with chaos synchronization between two different uncertain fractional order chaotic systems with linguistic uncertainties and internal noise. During the process of design, the free parameters of the adaptive fuzzy controller can be tuned on line by output feedback control law and adaptive law by using Lyapunov synthesis approach, and the chattering phenomena in the control efforts can be reduced. The simulation example, chaos synchronization of two different fractional order Duffing-Holmes chaotic systems, is given to demonstrate that interval type-2 FLC can overcome the limitations of type-1 FLC in high uncertain environments. The significance of the adaptive fuzzy sliding model control in the simulations for different values of q is manifest. Simulation results show that a fast synchronization of drive and response systems can be achieved and q is reduced the chaos is seen reduced, i.e., the synchronization error is reduced, accordingly.

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