

H_2 Guaranteed Cost of Uncertain Continuous T-S Fuzzy Systems by Multiple Lyapunov Function Approach

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Abstract—In this paper, the H_2 guaranteed cost control problem of uncertain continuous T-S fuzzy systems is investigated. We approach this problem by considering the multiple Lyapunov function and utilizing the non-PDC controller. Sufficient conditions for ensuring certain H_2 cost as well as asymptotical stability are derived and expressed in terms of LMIs. Unlike existing approaches, the assumption of known upper bounds on the time derivative of membership functions, which is quite unrealistic, is not required anymore in the present one. Numerical examples are also given to demonstrate our approach has lower H_2 guaranteed cost as well as control force in comparison with the existing ones.

Index Terms—linear matrix inequality, T-S fuzzy model, H_2 guaranteed cost control.

I. INTRODUCTION

In general, control of nonlinear systems can be a challenge task. The mathematic structure of non-linear systems makes the analysis and controller design very difficult. For the past decades, T-S fuzzy model [10] of non-linear systems has been demonstrated to be the powerful approach in non-linear system analysis and controller design. The T-S fuzzy is described fuzzy IF-THEN rules, which represent the linear input-output relations of nonlinear control systems. The main advantage of T-S fuzzy model is to express of the local dynamics of each fuzzy rule by linear system modeling to reduce the complexity of non-linear system analysis and controller design.

Stability and performance are most important issues in control systems. Once the the non-linear systems are expressed by T-S fuzzy model, stability analysis and performance design can be expressed by *linear Matrix Inequalities* (LMIs) [11], [1] and then can solved by available software. For example, the authors in [7], [5] studied the stabilization of T-S fuzzy systems. there also has been numerous results reported for design controller to ensure system performance such as mixed H_2/H_∞ [12], H_∞ [7], [8], [13], [22] and H_2 [19], [21]. On the other hand, the precise modeling of nonlinear system is sometimes impossible or difficult to achieve. The parametric uncertainties are always accompanied the system. The parametric uncertainties may degrade the system performance or even destabilize the system. Therefore, there are many researchers focusing on the issues such as robust stabilization, robust H_∞ and guaranteed cost

of H_2 [19] to ensure the system stability with adequate performance level subjecting uncertainties. Comparing to the rich results in robust stabilization and robust H_∞ control for T-S fuzzy system, there are few results [19], [25] regarding to H_2 guaranteed cost control for continuous-time T-S fuzzy systems. Furthermore, although [19], [25] proposed solutions to deal with H_2 guaranteed cost control of T-S fuzzy systems, there are still some open issues to be addressed. The most results are derived based on a quadratic Lyapunov function or so-called "common" Lyapunov function (CLF) or single Lyapunov function (SLF); that is to search single P matrix in Lyapunov function to satisfy all LMI conditions. This is too conservative for some cases, especially higher number of fuzzy rules, due to less degree of freedom on P matrix. To overcome this, a nonquadratic Lyapunov function [15] or multiple Lyapunov function (MLF) [18] (or fuzzy Lyapunov function), in which the membership functions are contained, is proposed to deal with discrete-time T-S fuzzy systems. In addition, in [18], a multiple Lyapunov function is also applied to study stability of T-S fuzzy systems. From the numerical examples shown in the report [18], this type of Lyapunov function further relaxes the conservatism of stability and improve the system performance. The drawback of the approach in [18] is the upper bounds of time derivative on membership functions $|\dot{h}_i(f(t))|$ must be known as prior to derive the stability conditions. In [24], [27], [26], the information of time derivative on membership functions must be evaluated in advance as well. Finding information of time derivative on membership functions is difficult or unrealistic in some cases. Thus in [21], a line integral of multiple Lyapunov function is employed which leads to the derivative of membership functions vanishing. However, the P_i matrices in Lyapunov function must have a special structure and the conditions for controller design are in terms of BMIs. Therefore, no efficiency software exists to solve them.

Unlike the previous approaches, we introduce a slack matrix and employ the property $\sum_{i=1}^r (h_i(t) \pm \dot{h}_i(t)) = 1$ to deal with time derivative of membership functions. The multiple Lyapunov function and nonparallel distributed compensation

(non-PDC) controller [18], [15] are proposed to derive H_2 guaranteed cost of uncertain T-S fuzzy system and to ensure closed-loop system asymptotically stable for all admissible parametric uncertainty. The conditions to ensure upper bound of H_2 cost and globally asymptotically stable are all in terms of LMI form. Thus, the guaranteed minimum H_2 cost can be reformulated as minimizing the upper bound of cost subjecting to LMI constraints and becomes a generalized eigenvalue problems (GEVP), which can be solved by efficiency tools such as MATLAB LMI toolbox.

The rest of paper is organized as follows. In section 2, the problem statement is made. Section 3 is the main results of this paper. The main theorem of guaranteed H_2 cost is presented in this section. Then numerical examples to demonstrate our result are given in section 4. Finally, conclusion are made in section 5.

The notation used in this paper is quite standard. $(X + \star) = X + X^T$. $X > 0$ (or $X \geq 0$) means X is positive definite (or positive semidefinite). Likewise, $X < 0$ (or $X \leq 0$) means X is negative definite (or negative semidefinite). For shorthand, $h(x(t))$ is denoted by h ; " $*$ " is denoted to be the symmetric term in the symmetric matrix.

II. PROBLEM FORMULATION AND PRELIMINARY

We begin with considering a class of uncertain continuous non-linear systems which can be governed by following IF-THEN rule based T-S fuzzy system,
Plant Rule i: IF $f_1(x(t))$ is M_1^i and ... and $f_s(x(t))$ is M_s^i THEN

$$\begin{aligned}\dot{x}(t) &= (A_i + \Delta A_i)x(t) + (B_{ui} + \Delta B_{ui})u(t), \\ z(t) &= (C_{zi} + \Delta C_{zi})x(t) + (D_{zui} + \Delta D_{zui})u(t),\end{aligned}\quad (1)$$

where $x(t) \in \mathbb{R}^{n \times n}$ is state vector with $x(0) = x_0$, $u(t) \in \mathbb{R}^m$ is the input vector, $z(t) \in \mathbb{R}^p$ is the controlled output vector, $A_i \in \mathbb{R}^{n \times n}$, $B_{ui} \in \mathbb{R}^{n \times m}$, $C_{zi} \in \mathbb{R}^{p \times n}$, and $D_{zui} \in \mathbb{R}^{p \times m}$ are known system matrices, M_j^i denotes an j -based fuzzy set used for the i th fuzzy rule, θ_i is i th the premise variable, $i = 1, \dots, r$ and r is the total number of fuzzy rules. Assume ΔA_i , ΔB_{ui} , ΔC_{zi} and ΔD_{zui} represent unknown time-varying parametric uncertainty with following structure

$$\begin{bmatrix} \Delta A_i & \Delta B_{ui} \\ \Delta C_{zui} & \Delta D_{zui} \end{bmatrix} = \begin{bmatrix} H_{xi} F_{\Delta} (E_{xi} & E_{ui}) \\ H_{zi} F_{\Delta} (E_{zi} & E_{zui}) \end{bmatrix}, \quad (2)$$

where H_{xi} , E_{xi} , E_{ui} , E_{zi} and E_{zui} are known matrices with appropriate dimension. F_{Δ} is unknown matrices with satisfying property

$$F_{\Delta} \in \Omega_F := \{F^T(t)F(t) < I\}$$

and the elements of $F(t)$ are Lebesgue measurable.

Given a pair of $x(t)$, $u(t)$, the final outputs of fuzzy system

can be inferred as:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^r h_i((A_i + \Delta A_i)x(t) + (B_{ui} + \Delta B_{ui})u(t)) \\ z(t) &= \sum_{i=1}^r h_i((C_{zi} + \Delta C_{zi})x(t) + (D_{zui} + \Delta D_{zui})u(t)),\end{aligned}\quad (3)$$

with $x(0) = x_0$ and $i = 1, \dots, r$ in which $f(x(t)) = [f_1(x(t)), f_2(x(t)), \dots, f_s(x(t))]$
 $\mu_i(f(x(t))) = \prod_{j=1}^s M_j^i(f_j(x(t))), h_i(x(t)) = \mu_i(f(x(t))) / \sum_{i=1}^r \mu_i(f(x(t)))$. In this paper, we will assume $\mu_i(f(x(t))) \geq 0$ for all t . Then we have following properties

$$h_i(x(t)) \geq 0, i = 1, \dots, r, \text{ and } \sum_{i=1}^r h_i(x(t)) = 1$$

for all t .

Next we define H_2 performance index function

$$J(H_2) = \int_0^\infty z^T(t)z(t)dt. \quad (5)$$

The goal is to design a feedback controller to the system described by (3) such that the closed-loop system is asymptotically stable and the upper bound of $J(H_2)$ is as small as possible for all admissible uncertainty described by (2).

Before we start to state our main results, two lemmas, which are repeatedly used through the derivation of main results, are given.

Lemma 1. *Schur Complement: Given matrices $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, and $S(x)$ with appropriate dimensions*

$$\begin{bmatrix} Q(x) & S^T(x) \\ S(x) & R(x) \end{bmatrix} < 0$$

$$\iff R(x) < 0 \text{ and } Q(x) - S^T(x)R^{-1}(x)S(x) < 0.$$

Lemma 2. *Given matrices S , H and E with proper dimensions and symmetric matrix S . Following inequality holds*

$$S + HFE + E^T F^T H^T < 0$$

for F satisfying $F^T F \leq R$ if and only if there exists $\varepsilon > 0$ such that

$$S + [\varepsilon^{-1}E^T \quad \varepsilon H] \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \varepsilon^{-1}E \\ \varepsilon H^T \end{bmatrix} < 0.$$

III. NON-PDC NONLINEAR CONTROLLER SYNTHESIS

The nonlinear controller for the nonlinear uncertain systems is type of Non-PDC controller [17] described by (3) is proposed as

$$u(t) = \sum_{i=1}^r h_i F_i P^{-1}(x(t))x(t), \quad (6)$$

and consider a multiple Lyapunov function

$$V(x(t)) = x^T(t)P^{-1}(x(t))x(t) \quad (7)$$

where $F_i \in \mathbb{R}^{m \times n}$ is state feedback gain and $P(x(t)) = \sum_{j=1}^r h_j(x(t))P_j$, $P_j = P_j^T \in \mathbb{R}^{n \times n} > 0$, $j = 1, \dots, r$. Note that here F_i and P_i are some matrices to be determined.

Remark 3. According to $P_j > 0, j = 1, \dots, r$ and the property of h_i , the invertibility and the positive definiteness of P are ensured, i.e. P^{-1} exists. In addition, by the property of $h_i(x(t)) \geq 0$ and $\sum_{i=1}^r h_i(x(t)) = 1$, we have $V(x) > 0 \forall x(t) \neq 0$ as well.

Remark 4. If let $P_1 = P_2, \dots, P_r = \bar{P}$, one has $P = \bar{P}$. With this assumption, multiple Lyapunov function $V(x(t))$ is reduced to common Lyapunov function and Non-PDC control law $u(t)$ is reduced to PDC one. Thus multiple Lyapunov function is more general than common Lyapunov function.

The following theorem gives sufficient conditions to ensure T-S fuzzy system represented by (3)) is globally asymptotically stable and H_2 cost is bounded by a prescribed level β .

Theorem 5. Assume the membership functions, h_i , $i = 1, \dots, r$, are differentiable. The system described by (3) is stable and $J(H_2) < \beta$ if there exist $\varepsilon_{xii} > 0$, $\varepsilon_{xij} > 0$, $\varepsilon_{zij} > 0$, $P_i > 0, F_i, i = 1, \dots, r; Q > 0; S > 0, X_{ii}, i = 1, \dots, r; X_{ij} = X_{ji}^T, i < j, i, j = 1, \dots, r; X_{(r+i)j} = X_{j(r+i)}^T, i = 1, \dots, r; X_{(r+i)(r+i)}, i = 1, \dots, r; X_{(r+i)(r+j)} = X_{j(r+i)(r+j)}^T, i < j, i, j = 1, \dots, r$ and satisfy

$$\begin{bmatrix} \Psi_{ii} + \varepsilon_{xii} H_{xi} H_{xi}^T & * \\ E_{xi} P_i + E_{ui} F_i & -\varepsilon_{xii} \end{bmatrix} < 0, \quad i = 1, \dots, r, \quad (8)$$

$$\begin{bmatrix} \Psi_{ij} + \varepsilon_{xij} H_{xi} H_{xi}^T + \varepsilon_{xji} H_{xj} H_{xj}^T & * & * \\ E_{xi} P_j + E_{ui} F_j & -\varepsilon_{xij} I & 0 \\ E_{xj} P_i + E_{uj} F_i & 0 & -\varepsilon_{xji} I \end{bmatrix} < 0, \quad (9)$$

$$-P_i < X_{(r+i)j} + X_{(r+i)j}^T, \quad i, j = 1, \dots, r, i < j, \quad (10)$$

$$-S < X_{(r+i)(r+i)}, \quad i = 1, \dots, r, \quad (11)$$

$$-2S < X_{(r+i)(r+j)} + X_{(r+i)(r+j)}^T, \quad i, j = 1, \dots, r, i < j, \quad (12)$$

$$\begin{bmatrix} -Q & * & * \\ C_{zi} P_j + D_{zui} F_j & -I + \varepsilon_{zij} H_{zi} H_{zi}^T & 0 \\ E_{zi} P_j + E_{zui} F_j & 0 & -\varepsilon_{zij} I \end{bmatrix} < 0, \quad (14)$$

$$\begin{bmatrix} \beta & x(0)^T \\ x(0) & \sum_i h_i(x(0)) P_i \end{bmatrix} > 0. \quad (15)$$

where

$$\begin{aligned} \Psi_{ii} &= ((A_i P_i + B_{ui} F_i) + \star) + Q + S - X_{ii} \\ \Psi_{ij} &= ((A_i P_j + B_{ui} F_j + A_j P_i + B_{uj} F_i) + \star) \\ &\quad + 2Q + 2S - X_{ij} - X_{ij}^T \end{aligned}$$

Proof: For simplicity, define

$$\begin{bmatrix} A_i + \Delta A_i & B_{ui} + \Delta B_{ui} \\ C_{zi} + \Delta C_{zi} & D_{zui} + \Delta D_{zui} \end{bmatrix} = \begin{bmatrix} \bar{A}_i & \bar{B}_{ui} \\ \bar{C}_{zi} & \bar{D}_{zui} \end{bmatrix}$$

With the feedback control law (6), the closed loop becomes

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^r \sum_{j=1}^r h_i h_j (\bar{A}_i + \bar{B}_{ui} F_j P^{-1}) x(t) \\ z(t) &= \sum_{i=1}^r \sum_{j=1}^r h_i h_j (\bar{C}_{zi} + \bar{D}_{zui} F_j P^{-1}) x(t). \end{aligned}$$

Next define a function $H(x)$

$$H(x) \triangleq \dot{V}(x) + z^T(t) z(t). \quad (16)$$

By taking derivative on $V(x)$, we obtain

$$\begin{aligned} &\dot{V}(x) + z^T(t) z(t) \\ &= \sum_i^r \sum_j^r h_i h_j (x^T(t) (\bar{A}_i + \bar{B}_{ui} F_j P^{-1})^T P^{-1} x(t) \\ &\quad + x^T(t) P^{-1} (A_i + B_{ui} F_j P^{-1}) x(t) - x^T(t) P^{-1} \dot{P} P^{-1} x(t) \\ &\quad + x^T(t) (\bar{C}_{zi} + \bar{D}_{zui} F_j P^{-1})^T (\bar{C}_{zi} + \bar{D}_{zui} F_j P^{-1}) x(t)) \end{aligned} \quad (17)$$

Let $y(t) = P^{-1} x(t)$. Thus, (17) can be rewritten as

$$\begin{aligned} H(x) &= \sum_{i=1}^r \sum_{j=1}^r h_i h_j y^T(t) ((\bar{A}_i P_j + \bar{B}_{ui} F_j) + (\star)) y(t) \\ &\quad - \sum_{i=1}^r \sum_{j=1}^r \dot{h}_i h_j y^T(t) P_i y(t) \\ &\quad + y^T(t) \left(\sum_{i=1}^r \sum_{j=1}^r h_i h_j (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j)^T \right. \\ &\quad \left. \cdot \sum_{i=1}^r \sum_{j=1}^r h_i h_j (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j) \right) y(t). \end{aligned} \quad (18)$$

Since $\sum_{i=1}^r \dot{h}_i = 0$, we have the properties $\sum_{i=1}^r (h_i + \dot{h}_i) = \sum_{j=1}^r (h_j - \dot{h}_j) = 1$. By introducing slack variable $S > 0$, and

$$\begin{bmatrix} X_{11} & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{r1} & \cdots & X_{rr} & * & \cdots & * \\ X_{(r+1)1} & \cdots & X_{(r+1)r} & X_{(r+1)(r+1)} & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{(r+1)r} & \cdots & X_{(r+r)r} & X_{(r+r)(r+1)} & \cdots & X_{(r+r)(r+r)} \end{bmatrix} < 0, \quad (13)$$

adding $\sum_{i=1}^r \sum_{j=1}^r (h_i + \dot{h}_i)(h_j - \dot{h}_j)S > 0$ to (18), one has

$$\begin{aligned} H(x) &< \sum_{i=1}^r \sum_{j=1}^r h_i h_j y^T(t) ((\bar{A}_i P_j + \bar{B}_{ui} F_j) + (\star)) \\ &+ Q + S y(t) - \sum_{i=1}^r \sum_{j=1}^r \dot{h}_i h_j y^T(t) (P_i + S - S) y(t) \\ &- \sum_{i=1}^r \sum_{j=1}^r \dot{h}_i \dot{h}_j y^T(t) S y(t) - y^T(t) Q y(t) \\ &+ \sum_{i=1}^r \sum_{j=1}^r y^T(t) h_i h_j (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j)^T \\ &\cdot \sum_{i=1}^r \sum_{j=1}^r h_i h_j (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j) y(t) \\ &= \sum_{i=1}^r h_i^2 y^T(t) ((\bar{A}_i P_i + \bar{B}_{ui} F_i) + (\star) + Q + S) y(t) \\ &+ \sum_{i=1}^r \sum_{j=1}^r h_i h_j y^T(t) ((\bar{A}_i P_j + \bar{B}_{ui} F_j) + (\bar{A}_j P_i + \bar{B}_{uj} F_i)) \\ &+ (\star) + 2Q + 2S y(t) - \sum_{i=1}^r \sum_{j=1}^r \dot{h}_i h_j y^T(t) (P_i) y(t) \\ &- \sum_{i=1}^r \dot{h}_i^2 y^T(t) S y - \sum_{i=1}^r \sum_{i < j}^r \dot{h}_{i=1}^r \dot{h}_j y^T(t) 2S y(t) \\ &- y^T(t) Q y + \sum_{i=1}^r \sum_{j=1}^r y^T(t) h_i h_j (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j)^T \\ &\cdot \sum_{i=1}^r \sum_{j=1}^r h_i h_j (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j) y(t). \end{aligned}$$

On the other hand, if (8) is satisfied, by Schur Complement, we have

$$\Psi_{ii} + \varepsilon_{xii} H_{xi} H_{xi}^T + (1/\varepsilon_{xii})(E_{xi} P_i + E_{ui} F_i)^T \cdot (E_{xi} P_i + E_{ui} F_i) < 0, \quad (19)$$

which is further equivalent to

$$(A_i P_i + B_{ui} F_i + H_{xi} F_{\Delta} (E_{xi} P_i + E_{ui} F_i) + (\star)) + Q + S < X_{ii} \quad (20)$$

for all $F_{\Delta}(t) \in \Omega$ by Lemma 2.

Furthermore, following the same manner, (9) is equivalent

to

$$\begin{aligned} &\Psi_{ij} + \varepsilon_{xij} H_{xi} H_{xi}^T + (1/\varepsilon_{xij})(E_{xi} P_j + E_{ui} F_j)^T \\ &\cdot (E_{xi} P_j + E_{ui} F_j) + \varepsilon_{xji} H_{xj} H_{xj}^T \\ &+ (1/\varepsilon_{xji}) E_{xj} P_i + E_{uj} F_i)^T (E_{xj} P_i + E_{uj} F_i) < 0, \\ &i, j = 1, \dots, r, i < j. \end{aligned} \quad (21)$$

By Lemma 2, (21) is equivalent to

$$\begin{aligned} &((A_i + H_{xi} F_{\Delta} E_{xi}) P_i + (B_{ui} + H_{xi} F_{\Delta} E_{ui}) F_j + \\ &(A_j + H_{xj} F_{\Delta} E_{xj}) P_i + (B_{uj} + H_{xj} F_{\Delta} E_{uj}) F_i) \\ &+ (\star) + 2Q + 2S < X_{ij} + X_{ij}^T, \quad i, j = \dots, r, i < j \end{aligned} \quad (22)$$

for all allowable F_{Δ} .

By (22) together with (10)-(13), one can readily obtain (23) as shown in the top of next page. Thus if (13) is held, we have $V(x) < 0 \quad \forall x(t) \neq 0$. This leads to imply the closed-loop system is asymptotically stable for all allowable F_{Δ} .

Next we consider the H_2 performance. By Lemma 1 Schur Complement, (14) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} -Q & * \\ C_{zi} P_j + D_{zui} F_j & -I \end{bmatrix} + \varepsilon_{zij} \begin{bmatrix} 0 \\ H_{zi} \end{bmatrix} \begin{bmatrix} 0 & H_{zi}^T \end{bmatrix} \\ &+ (1/\varepsilon_{zij}) \begin{bmatrix} (E_{zi} P_j + E_{zui} F_j)^T \\ 0 \end{bmatrix} \\ &\cdot \begin{bmatrix} E_{zi} P_j + E_{zui} F_j & 0 \end{bmatrix} < 0. \end{aligned} \quad (24)$$

Then, by Lemma 2, (24) is equivalent to

$$\begin{aligned} &\begin{bmatrix} -Q & * \\ C_{zi} P_j + D_{zui} F_j & -I \end{bmatrix} + \begin{bmatrix} 0 \\ H_{zi} \end{bmatrix} F_{\Delta} \\ &\cdot \begin{bmatrix} E_{zi} P_j + E_{zui} F_j & 0 \end{bmatrix} + (\star) < 0. \end{aligned} \quad (25)$$

for all $F_{\Delta} \in \Omega$. Again (25) is equivalent to

$$-Q + (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j)^T (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j) < 0$$

by Schur Complement. Hence we have

$$\begin{aligned} &-y^T(t) Q y + \sum_{i=1}^r \sum_{j=1}^r y^T(t) h_i h_j (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j)^T \\ &\cdot \sum_{i=1}^r \sum_{j=1}^r h_i h_j (\bar{C}_{zi} P_j + \bar{D}_{zui} F_j) y(t) < 0. \end{aligned}$$

Combining all together, we can obtain

$$\dot{V}(x(t)) + z^T(t) z(t) < 0 \quad (26)$$

$$V(x) < \begin{bmatrix} h_1 y \\ \vdots \\ h_r y \\ h_1 y \\ \vdots \\ h_r y \end{bmatrix}^T \begin{bmatrix} X_{11} & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{r1} & \cdots & X_{rr} & * & \cdots & * \\ X_{(r+1)1} & \cdots & X_{(r+1)r} & X_{(r+1)(r+1)} & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ X_{(r+1)r} & \cdots & X_{(r+r)r} & X_{(r+r)(r+1)} & \cdots & X_{(r+r)(r+r)} \end{bmatrix} \begin{bmatrix} h_1 y \\ \vdots \\ h_r y \\ h_1 y \\ \vdots \\ h_r y \end{bmatrix}. \quad (23)$$

Integrating both sides of (26) from 0 to ∞ , we readily obtain

$$\begin{aligned} \int_0^\infty z^T(t)z(t)dt &< V(x(0)) - V(\infty) < V(x(0)) \\ &= x(0)^T \left(\sum_{i=1}^r h_i(x(0))P_i \right)^{-1} x(0). \end{aligned}$$

Therefore, if (15) is satisfied, we have $\int_0^\infty z^T(t)z(t)dt < \beta$. $\square$
From theorem 5, to minimize H_2 cost (5) of system (3), we can reformulate the problem as

$$\min_{P_i, X_{ij}, Q, S, \varepsilon_{xij}, \varepsilon_{xji}, \varepsilon_{zij}} \beta \quad \text{subject to} \quad (8) - (15). \quad (27)$$

Remark 6. [19] If the system is stochastic system, the cost function can be written as

$$J(H_2) = \mathbb{E}\{z^T(t)z(t)\}, \quad (28)$$

where $\mathbb{E}$ is denoted as expectation operator. Furthermore if $x(0)$ is zero mean random variables and satisfies $\mathbb{E}\{x(0)x^T(0)\} = I$, the upper bound of cost can be expressed by

$$J(H_2) < \mathbb{E} \left\{ x^T(0) \left(\sum_{i=1}^r h_i(x(0))P_i \right)^{-1} x(0) \right\} = \text{Trace}\{\bar{P}^{-1}\}. \quad (29)$$

Moreover, by introducing an auxiliary variable $U > 0$, $\min \text{Trace}\{\bar{P}^{-1}\}$ subject to (8)-(15) can be equivalent to

$$\begin{aligned} \min_{P_i, X_{ij}, Q, S, U, \varepsilon_{xij}, \varepsilon_{xji}, \varepsilon_{zij}} \text{Trace}(U) \quad \text{subject to} \quad (8) - (15) \\ \text{and} \quad \begin{bmatrix} U & * \\ I & \bar{P} \end{bmatrix} > 0, \end{aligned} \quad (30)$$

where $\bar{P} = \sum_{i=1}^r h_i(x(0))P_i$.

Theorem 5 can be specialized to conditions of systems without uncertainties by simply setting all uncertain matrices to 0. Thus following corollary is made.

Corollary 7. For the systems without uncertainties, the (8), (9) and (14) are reduced to

$$((A_i P_i + B_{ui} F_i) + \star) + Q + S - X_{ii} < 0, \quad i = 1, \dots, r, \quad (31)$$

$$\begin{aligned} ((A_i P_j + B_{ui} F_j + A_j P_i + B_{uj} F_i) + \star) + 2Q + 2S \\ - X_{ij} - X_{ij}^T < 0, \quad i, j = 1, \dots, r, i < j, \end{aligned} \quad (32)$$

$$\begin{bmatrix} -Q & * \\ C_{zi} P_j + D_{zui} F_j & -I \end{bmatrix} < 0, \quad i, j = 1, \dots, r. \quad (33)$$

IV. ILLUSTRATIVE EXAMPLES

To show the performance of our proposed T-S fuzzy controllers, two examples are given. One is for nominal systems. the other one is for the system with uncertainties.

A. Example 1

The example one is cart-pole-typed inverted pendulum problem [9]. In this example, we consider uncertain-free model. Its dynamic equation is given by

$$\ddot{\theta} = \frac{g \sin(\theta(t)) - a m l \dot{\theta}^2(t) \sin(2\theta(t))/2 - a \cos(\theta(t))u(t)}{4l/3 - a m l \cos^2(\theta(t))}$$

where θ is angle of displacement in radians, M is the mass of cart, m is the mass of pendulum, g is the acceleration due to gravity, l is half length of pendulum, and $a = 1/(M+m)$. Here we chose $M = 8Kg$, $m = 2Kg$, $g = 9.8m/sec^2$, $l = 0.5m$. Its nonlinear model can be exactly represented by following T-S fuzzy rules,

Plant Rule1:

If $f_1(x(t))$ is M_1^1 and $f_2(x(t))$ is M_2^1 THEN

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + B_{u1} u(t) \\ z(t) &= C_{z1} x(t) + D_{u1} u(t), \end{aligned}$$

Plant Rule2:

If $f_1(x(t))$ is M_1^1 and $f_2(x(t))$ is M_2^2 THEN

$$\begin{aligned} \dot{x}(t) &= A_2 x(t) + B_{u2} u(t) \\ z(t) &= C_{z2} x(t) + D_{u2} u(t), \end{aligned}$$

Plant Rule3:

If $f_1(x(t))$ is M_1^2 and $f_2(x(t))$ is M_2^1 THEN

$$\begin{aligned} \dot{x}(t) &= A_3 x(t) + B_{u3} u(t) \\ z(t) &= C_{z3} x(t) + D_{u3} u(t), \end{aligned}$$

Plant Rule4:

If $f_1(x(t))$ is M_1^2 and $f_2(x(t))$ is M_2^2 THEN

$$\begin{aligned} \dot{x}(t) &= A_4 x(t) + B_{u4} u(t) \\ z(t) &= C_{z4} x(t) + D_{u4} u(t), \end{aligned}$$

in which $x(t) = [x_1(t) \ x_2(t)]^T = [\theta \ \dot{\theta}]^T$,

$$f_1(x(t)) = \frac{g - amlx_2^2(t) \cos(x_1(t))}{4l/3 - aml \cos^2(\theta(t))} \left(\frac{\sin(x_1(t))}{x_1(t)} \right),$$

$$f_2(x(t)) = \frac{-aml \cos(x_1(t))}{4l/3 - aml \cos^2(x_1(t))},$$

$$M_1^1(f_1(x(t))) = \frac{-f_1(x(t)) + 18}{9},$$

$$M_1^2(f_1(x(t))) = \frac{-f_1(x(t)) - 9}{9},$$

$$M_2^1(f_2(x(t))) = \frac{-f_2(x(t)) - 0.0052}{9},$$

$$M_2^2(f_2(x(t))) = \frac{-f_2(x(t)) + 0.1765}{0.1713},$$

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \quad A_3 = A_4 = \begin{bmatrix} 0 & 1 \\ 18 & 0 \end{bmatrix},$$

$$B_{u1} = B_{u3} = [0 \ -0.1765]^T, B_{u2} = B_{u4} = [0 \ -0.0052]^T, C_{z1} = C_{z2} = C_{z3} = C_{z4} = [1 \ 0], \text{ and } D_{zu1} = D_{zu2} = D_{zu3} = D_{zu4} = 0.001.$$

After using proposed approach, LMI toolbox gives the minimized upper bound $\beta = 0.4938$, P_i matrices, and feedback gain F_i matrices,

$$P_1 = \begin{bmatrix} 11.0330 & -93.0483 \\ -93.0483 & 863.8462 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 10.9363 & -44.8631 \\ -44.8631 & 184.0445 \end{bmatrix}$$

$$P_3 = 10^3 \begin{bmatrix} 0.0133 & -0.1136 \\ -0.1136 & 1.0096 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 13.7524 & -66.7003 \\ -66.7003 & 323.5029 \end{bmatrix}$$

$$F_1 = 10^4 [-0.4276 \ 6.9868], \quad F_2 = 10^4 [-0.4175 \ 2.1680]$$

$$F_3 = 10^4 [-0.6545 \ 9.0572], \quad F_4 = 10^4 [-0.6992 \ 4.3518]$$

with initial condition $x(0) = [\frac{22\pi}{50} \ -3.3]$. The time response is shown in Figure 1. With the proposed controller, we obtain $J(H_2) \approx 0.3811$. Thus $J(H_2) < \beta = 0.4938$.

Furthermore, the Table 1 shown below is the comparison of performance by our proposed approach and the approach in [19]. The comparison results show our approach obtains smaller upper bound and save more control energy.

Comparison	Theorem 5	SLF [19]
upper bond β	0.4938	4.0689
$\int_0^\infty u^T(t)u(t)dt$	$\approx 1.2454 \times 10^4$	$\approx 1.3951 \times 10^6$

TABLE I

PERFORMANCE COMPARISON OF CONTROL OF INVERTED PENDULUM

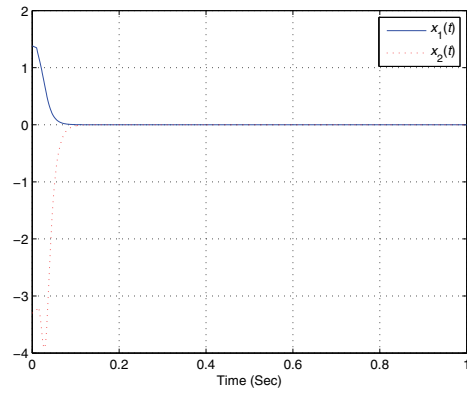
B. Example 2

In this example, an uncertain mass-spring-damped mechanic system is to be controlled to verify our proposed fuzzy controller. Its dynamic equation is depicted by

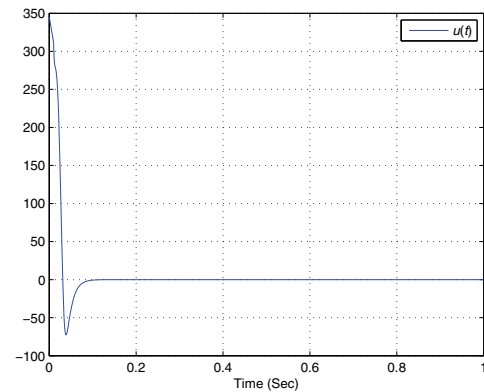
$$\ddot{x}(t) = c(t)\dot{x}(t) - 0.02x(t) - 0.67x^3(t) + u(t)$$

The system can be modeled by T-S fuzzy system,

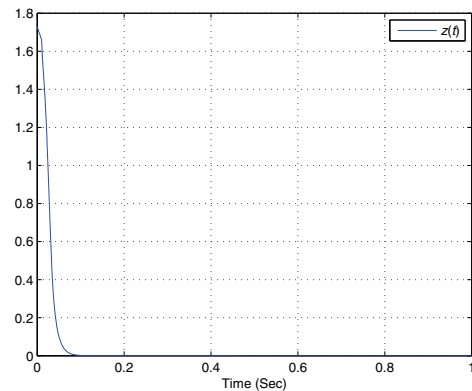
Plant Rule1:



(a) Time response of $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$



(b) Control action $u(t)$



(c) Controlled output $z(t)$

Fig. 1. Time response of $\theta(t)$, $\dot{\theta}(t)$, $u(t)$, and $z(t)$

If $x(t)$ is M_1 THEN

$$\begin{aligned} \dot{\tilde{x}}(t) &= (A_1 + \Delta A_1)\tilde{x}(t) + (B_{u1} + \Delta B_{u1})u(t) \\ z(t) &= C_{z1}x(t) + D_{u1}u(t) \end{aligned}$$

Plant Rule2:

If $x(t)$ is M_2 THEN

$$\dot{\tilde{x}}(t) = (A_2 + \Delta A_2)\tilde{x}(t) + (B_{u2} + \Delta B_{u2})u(t)$$

$$z(t) = C_{z2}\tilde{x}(t) + D_{u2}u(t)$$

where $\dot{\tilde{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{x}(t) \end{bmatrix}$, $A_1 = \begin{bmatrix} -0.1125 & -0.02 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.1125 & -1.527 \\ 1 & 0 \end{bmatrix}$, $B_{u1} = B_{u2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C_{z1} = C_{z2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D_{u1} = D_{u2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $H_{x1} = H_{x2} = \begin{bmatrix} -0.1125 \\ 0 \end{bmatrix}$, $E_{x1} = E_{x2} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $E_{u1} = E_{u2} = 0$.

By our derived theorem, the P_i matrices and feedback gains F_i are obtained as

$$P_1 = \begin{bmatrix} 0.5191 & -0.3431 \\ -0.3431 & 0.7620 \end{bmatrix}, P_2 = \begin{bmatrix} 1.3949 & -0.2965 \\ -0.2965 & 0.7639 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} -0.9690 & -0.0325 \end{bmatrix}, F_2 = \begin{bmatrix} -1.0026 & 0.0097 \end{bmatrix}.$$

Given the initial condition $\tilde{x}(0) = [-1 \quad -1.3]^T$, we obtain the optimal minimum bound $\beta = 4.4223$ and $J(H_2) \approx 3.4316$. Hence we see $J(H_2) < \beta$. The response of $\tilde{x}(t)$ and $u(t)$ is shown in figure 2.

The comparison of minimized upper bound obtained by our approach and common Lyapunov function approach is shown in Table II. Therefore, as our expectation, the guaranteed

Comparison	Theorem 5	SLF [19]
upper bond β	4.4223	9.4417
Trace(P^{-1})	2.4147	4.7176

TABLE II

PERFORMANCE COMPARISON OF MASS-SPRING DAMPED MECHANIC SYSTEMS

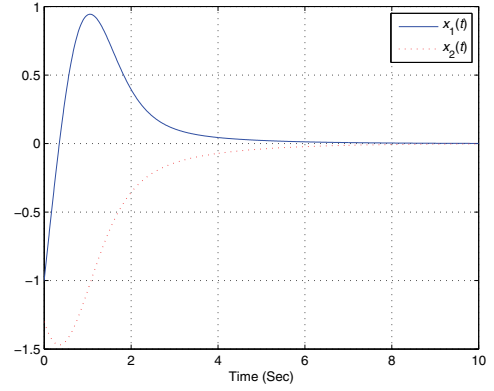
H_2 cost obtained by our proposed approach is lower than the previous common-Lyapunov function approach.

V. CONCLUSION

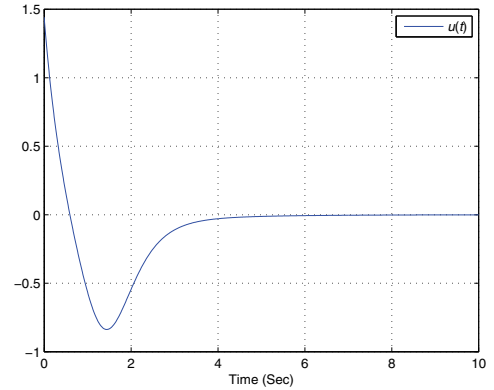
In this paper, we solve H_2 guaranteed cost of uncertain T-S fuzzy systems by multiple Lyapunov function. By our proposed approach, the priori information of derivative of membership functions can be completely removed. Thus, the controller is easy to implement and system is easy to analyze. Moreover, the guaranteed cost can be reformulated as minimizing an upper bound of cost function, which is a linear function, with some LMI constraints. Hence it can be efficiently solved by available software. From the numerical examples, it is also shown that the performance of designed system by our approach is better than the previous single Lyapunov function approach.

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(a) Time response of $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$



(b) Time response of $u(t)$

Fig. 2. Time response of state $\tilde{x}(t)$ and $u(t)$

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