

Fuzzy Delta Separation Axioms

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Abstract—We introduce a new type of separation axioms, which is called fuzzy δ -separation axioms by using the concept of fuzzy δ -open sets. Also we investigate the relation between the separation property and the subspaces. We show that fuzzy δ -separation axioms are hereditary in fuzzy regular open subspaces.

Index Terms—fuzzy logic, fuzzy topology, separation axiom

I. INTRODUCTION AND PRELIMINARIES

The usual notion of a set was generalized with the introduction of fuzzy sets by Zadeh in the classical paper [1]. The concept of fuzzy topology was first defined by Chang[2] and later redefined in a somewhat different way by Lowen[3] and Hutton and Reilly[4] and others. Several authors have expansively developed the theory of fuzzy sets and its applications. Closure and interior are alternative approaches to open sets in topology. So, many researchers have studied for these notions. The notions of fuzzy δ -closure and fuzzy θ -closure of a fuzzy set in a fuzzy topological space were introduced by Ganguly and Saha[5] and Mukherjee and Sinha[6], respectively. Furthermore, the notion of strong δ -continuity was investigated in [7] and the concept of δ -continuity on function spaces was studied in [8]. The notion of θ -closure and θ -interior also expanded in more generalized spaces, intuitionistic fuzzy topological spaces [9]. In [10] we also have introduced fuzzy δ -topology and fuzzy δ -compactness, and studied their properties.

In order to prove many important theorems of topology we need the Hausdorff condition, because Hausdorff condition implies the uniqueness of limit. Moreover most of the important spaces in mathematics are Hausdorff spaces. For this reason, many different definitions are introduced and studied as in [11], [12], [13], [14], [15], [16], [17], [18]. The aim of this paper is to introduce separation axioms in a fuzzy δ -topological space, which is both harmonious in itself and compatible with fuzzy δ -compactness. In this manner, we will define a new type of separation axioms, which is called fuzzy δ -separation axioms by using the concept of fuzzy δ -open sets introduced in [10]. We will investigate the relation between the separation property and the subspaces. We will show that fuzzy δ -separation axioms are hereditary in fuzzy regular open subspaces. It is interesting that there are some deviations in the behavior of these axioms as compared to those in the general topology.

Throughout this paper by $(X, \mathcal{T})$ or simply by X , we mean a fuzzy topological space (briefly fts) due to Chang [2]. A fuzzy point in X with support $x \in X$ and value α ($0 < \alpha \leq 1$) is denoted by x_α . When no confusion will arise, we omit the value α and write the fuzzy point x_α simply as x . For a fuzzy set A in X , a fuzzy point $x_\alpha \in A$ iff $\alpha \leq A(x)$. A fuzzy point x_α is said to be quasi-coincident (q -coincident, for short) with A , denoted by $x_\alpha q A$, iff $\alpha > A^c(x)$, or $\alpha + A(x) > 1$, where A^c denotes the complement of A , defined by $A^c = 1 - A$. A fuzzy set A in a fts X is said to be a q -neighbourhood (q -nbd, for short) of a fuzzy point x_α iff there exists a fuzzy open set B such that $x_\alpha q B$ and $B \leq A$. For two fuzzy sets A and B , $A \leq B$ iff $A \tilde{q} B^c$. A fuzzy point $x_\alpha \in \text{cl}(A)$ iff each q -nbd of x_α is q -coincident with A (See [19]). A fuzzy set A in a fts X is called a fuzzy regular open set iff $A = \text{int}(\text{cl}(A))$, and the complement of a fuzzy regular open set is called a fuzzy regular closed set (See [20]).

Definition 1.1: ([5]) A fuzzy point x_α is said to be a fuzzy δ -cluster point of a fuzzy set A iff every fuzzy regular open q -nbd U of x_α is q -coincident with A . The set of all fuzzy δ -cluster points of A is called the fuzzy δ -closure of A , and denoted by $\text{cl}_\delta(A)$.

Remark 1.2: ([10]) For any fuzzy set A in a fts $(X, \mathcal{T})$, the δ -closure of A is represented as follows;

$$\text{cl}_\delta(A) = \bigwedge \{ \text{cl}(U) \mid A \leq \text{cl}(U), U \in \mathcal{T} \}.$$

Definition 1.3: ([6]) A fuzzy set A is said to be a fuzzy δ -nbd of a fuzzy point x_α iff there exists a fuzzy regular open q -nbd V of x_α such that $V \leq A$.

Definition 1.4: ([5]) A fuzzy set A is said to be fuzzy δ -closed iff $A = \text{cl}_\delta(A)$, and the complement of a fuzzy δ -closed set is called a fuzzy δ -open set.

Since δ -open set is the complement of an δ -closed set, G is δ -open iff $G = \text{int}_\delta(G)$. And we know that $1 - \text{int}_\delta(1 - A) = \text{cl}_\delta(A)$. A fuzzy set A is fuzzy δ -open in a fts X if and only if for each fuzzy point x_α with $x_\alpha q A$, A is a fuzzy δ -nbd of x_α ([21]).

It is easy to show that $\text{cl}(A) \leq \text{cl}_\delta(A) \leq \text{cl}_\theta(A)$ for any fuzzy set A in a fts X . However, for a fuzzy open set A in a fts $(X, \mathcal{T})$, we have $\text{cl}(A) = \text{cl}_\delta(A)$ (See [6]). Moreover, it is clear that any regular open set is δ -open, and any δ -open set is open.

For any fuzzy set A in a fts $(X, \mathcal{T})$, $\text{cl}_\delta(A)$ is a fuzzy δ -closed set. That is, $\text{cl}_\delta(\text{cl}_\delta(A)) = \text{cl}_\delta(A)$ (See [10]).

Definition 1.5: ([5]) A function $f : X \rightarrow Y$ is said to be fuzzy δ -continuous (f.d.c., for short) if and only if for each fuzzy point x_α in X and for any regular open q -nbd V of $f(x_\alpha)$ in Y , there exists a regular open q -nbd U of x_α such that $f(U) \leq V$.

Remark 1.6: ([10]) $f : X \rightarrow Y$ is fuzzy δ -continuous if and only if for each fuzzy δ -open set U of Y , $f^{-1}(U)$ is fuzzy δ -open in X . Therefore the composition of two fuzzy δ -continuous mappings is also fuzzy δ -continuous.

Definition 1.7: Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{H})$ be a fuzzy mapping.

- (1) f is fuzzy δ -open if for each fuzzy δ -open set A in X , $f(A)$ is a fuzzy δ -open in Y .
- (2) f is fuzzy δ -closed if for each fuzzy δ -closed set B in X , $f(B)$ is fuzzy δ -closed in Y .

Theorem 1.8: ([10]) If $f : X \rightarrow Y$ is fuzzy δ -continuous, then the following are equivalent.

- (a) $f(\text{cl}_\delta(A)) \leq \text{cl}_\delta(f(A))$.
- (b) $\text{cl}_\delta(f^{-1}(A)) \leq f^{-1}(\text{cl}_\delta(A))$.
- (c) For each fuzzy δ -closed set A in Y , $f^{-1}(A)$ is fuzzy δ -closed in X .
- (d) For each fuzzy δ -open set A in Y , $f^{-1}(A)$ is fuzzy δ -open in X .

In 1993, M. Macho Stadler and M.Á. De Prada Vicente[22] have introduced a new concept of subspaces in fuzzy topological spaces. In Section 4 we will define the concept of fuzzy δ -separation axioms in this subspace.

Definition 1.9: ([22]) Let $(X, \mathcal{T})$ be a fts and $\mu \in I^X$. Let $X^\mu = \{V \wedge \mu \mid V \in I^X\}$. The family $\mathcal{T}^\mu = \{V \wedge \mu \mid V \in \mathcal{T}\}$ is called the fuzzy μ -topology induced by $\mathcal{T}$ over μ . The ordered pair $(\mu, \mathcal{T}^\mu)$ is called the subspace. The elements of $\mathcal{T}^\mu$ are called fuzzy open sets in the subspace μ . $V \in X^\mu$ is called fuzzy closed in μ if $\mu - V \in \mathcal{T}^\mu$.

Definition 1.10: Let x_r be a fuzzy point in μ , i.e. $x_r \in \mu$. We say that $N \in X^\mu$ is a fuzzy neighborhood of x_r in the subspace μ if there is a $U \in \mathcal{T}^\mu$ such that $x_r \in U \leq N$.

Remark 1.11: In [22] " $x_r \in U$ " means that $r < U(x)$. But in this paper it means that $r \leq U(x)$. And if $\mu = \chi_A$ for any crisp fuzzy subset $A \subseteq X$, then $\mathcal{T}^\mu$ is a subspace in the sense of general topology.

Theorem 1.12: ([22]) Let $U \in X^\mu$. Then $U \in \mathcal{T}^\mu$ iff for each $x_r \in U$, U is a fuzzy neighborhood of x_r in μ .

Definition 1.13: ([22]) Let $V \in X^\mu$. We define the interior of V in the subspace μ as the largest fuzzy open set in μ contained in V , that is

$$\text{int}^\mu(V) = \sup_{U \in \mathcal{T}^\mu} \{U \mid U \leq V\}.$$

Similarly we define the closure of V in μ as the smallest closed set in μ contains V , that is

$$\text{cl}^\mu(V) = \inf_{U \in \mathcal{T}^\mu} \{\mu - U \mid V \leq \mu - U\}.$$

Definition 1.14: ([23]) Let $A \in X^\mu$ and $x_r \in X$. x_r is said to be q -coincident with A in the subspace μ if $r + A(x) > \mu(x)$. And $x_r q A[\mu]$ represents it.

Definition 1.15: ([23]) Let $x_r \in \mu$. We say that $N \in X^\mu$ is a fuzzy q -nbd of x_r in μ if there is a $U \in \mathcal{T}^\mu$ such that $x_r q U[\mu]$ and $U \leq N$.

Theorem 1.16: ([23]) Let $U \in X^\mu$. Then $U \in \mathcal{T}^\mu$ iff for each $x_r q U[\mu]$, U is a q -neighborhood of x_r in μ .

Lemma 1.17: ([22]) Let $(X, \mathcal{T})$ be a fuzzy topological space and $(\mu, \mathcal{T}^\mu)$ a fuzzy subspace.

- (1) If $A \in X^\mu$, then $\text{int}(A) \leq \text{int}^\mu(A)$.
- (2) $\text{int}(A) = \text{int}^\mu(A) \wedge \text{int}(\mu)$ for any fuzzy subset $A \in X^\mu$.
- (3) If $\mu \in \mathcal{T}$ and $A \in X^\mu$, $\text{int}(A) = \text{int}^\mu(A)$.

II. FUZZY δ -SEPARATION AXIOMS

In this section, we construct new separation axioms, by using the concept of fuzzy δ -open sets. As for the fuzzy disjointness, we know that only the implication $\mu_1 \wedge \mu_2 = 0 \Rightarrow \mu_1 \leq \mu_2^c$ holds, but the reverse implication does not hold. With this principle, we now define a new fuzzy δ -separation axioms.

Definition 2.1: A fts X is called fuzzy δ - T_0 if for any pair of fuzzy points p and q with different supports in X , there is a fuzzy δ -open set U with $p \leq U \leq q^c$ or $q \leq U \leq p^c$.

This separation axiom is different from the fuzzy T_0 axiom as in the following example.

Example 2.2: Let $X = [0, 1]$ and for any $\alpha \in (0, 1)$

$$U_\alpha(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \alpha \\ 0 & \text{if } \alpha < x \leq 1. \end{cases}$$

Let $\mathcal{T} = \{\bar{0}, \bar{1}\} \cup \{U_\alpha \mid \alpha \in (0, 1)\}$. Then clearly $\mathcal{T}$ is a fuzzy topology on X and the set of all fuzzy δ -open sets in $(X, \mathcal{T})$ is $\{\bar{0}, \bar{1}\}$. Therefore for any two distinct fuzzy points of $(X, \mathcal{T})$ there is a fuzzy open subset of X which contains one but not the other. Hence $(X, \mathcal{T})$ is fuzzy T_0 but not fuzzy δ - T_0 .

Theorem 2.3: Let $f : X \rightarrow Y$ be injective and fuzzy δ -continuous. If Y is fuzzy δ - T_0 , then so is X .

Proof: Take any two points p and q with different supports in X . Since f is injective, $f(p)$ and $f(q)$ are two fuzzy points with different supports in Y . Since Y is fuzzy δ - T_0 , there is a fuzzy δ -open set U with $f(p) \leq U \leq f(q)^c$ or $f(q) \leq U \leq f(p)^c$. Therefore $p \leq f^{-1}(U) \leq q^c$ or $q \leq f^{-1}(U) \leq p^c$. And since f is fuzzy δ -continuous, $f^{-1}(U)$ is fuzzy δ -open. Hence X is fuzzy δ - T_0 . $\square$

Definition 2.4: A fts X is called fuzzy δ - T_1 if for any pair of fuzzy points p and q with different supports in X , there are two fuzzy δ -open sets U_1, U_2 with $p \leq U_1 \leq q^c$ and $q \leq U_2 \leq p^c$.

Clearly every fuzzy δ - T_1 space is fuzzy δ - T_0 .

Example 2.5: Let $X = \{a, b\}$ and $\mathcal{T} = \{b_s \mid s \in [0, 1]\} \cup \{\bar{1}, \bar{0}\}$, where b_s is the fuzzy point with value s at the support b . Then clearly $\mathcal{T}$ is a fuzzy topology and all elements in $\mathcal{T}$ are fuzzy regular open, so they are fuzzy δ -open. Take any two fuzzy points a_r and b_s where r and s are nonzero. Then there is a fuzzy δ -open set b_s such that $b_s \leq b_s \leq a_r^c$, and $\bar{1}$ is an only fuzzy δ -open set with $a_r \leq \bar{1}$. Clearly, for any s , $\bar{1} \not\leq b_s^c$. Hence $(X, \mathcal{T})$ is fuzzy δ - T_0 , but it is not fuzzy δ - T_1 .

Theorem 2.6: A fts X be fuzzy δ - T_1 iff any crisp fuzzy point in X is fuzzy δ -closed.

Proof: Let X be fuzzy δ - T_1 . Take any crisp fuzzy point p in X . We will show that p^c is fuzzy δ -open. We can take a fuzzy point $q \in p^c$ with different support from p . Since X is fuzzy δ - T_1 , there is a fuzzy δ -open set U with $q \leq U \leq p^c$. Thus $p^c = \bigcup_{q \in p^c} \{U \mid q \leq U \leq p^c, U \text{ is fuzzy } \delta\text{-open in } X\}$. Since $\bigcup_{q \in p^c} \{U \mid q \leq U \leq p^c, U \text{ is fuzzy } \delta\text{-open in } X\}$ is fuzzy δ -open, p^c is fuzzy δ -open. Hence p is fuzzy δ -closed. $\square$

Corollary 2.7: Let X be a fuzzy topological space. X is fuzzy δ - T_1 if and only if $x_\alpha = \bigcap \{\text{cl}_\delta(U) \mid x_\alpha \leq \text{cl}_\delta(U)\}$.

Theorem 2.8: Let $f : X \rightarrow Y$ be injective and fuzzy δ -continuous. If Y is fuzzy δ - T_1 , then so is X .

Proof: Take any two fuzzy points p and q with different supports in X . Since f is injective, $f(p)$ and $f(q)$ are two fuzzy points with different supports in Y . Since Y is fuzzy δ - T_1 , there are two fuzzy δ -open sets U_1, U_2 with $f(p) \leq U_1 \leq f(q)^c$ and $f(q) \leq U_2 \leq f(p)^c$. Therefore $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are fuzzy δ -open sets in X with $p \leq f^{-1}(U_1) \leq q^c$ and $q \leq f^{-1}(U_2) \leq p^c$. Hence X is fuzzy δ - T_1 . $\square$

Definition 2.9: A fts X is called fuzzy δ -Hausdorff, or fuzzy δ - T_2 , if for any pair of fuzzy points p and q with different supports in X , there are fuzzy δ -open sets U_1 and U_2 with $p \leq U_1 \leq q^c$, $q \leq U_2 \leq p^c$ and $U_1 \leq U_2^c$.

Obviously every fuzzy δ - T_2 space is fuzzy δ - T_1 .

Theorem 2.10: Let $(X, \mathcal{T})$ be a fuzzy δ - T_1 space. If the complement of each fuzzy δ -open set is also fuzzy δ -open, then $(X, \mathcal{T})$ is fuzzy δ - T_2 .

Proof: Take any two fuzzy points p and q in X with different supports. Since X is fuzzy δ - T_1 , there is a fuzzy δ -open set U such that $p \leq U \leq q^c$ or $q \leq U \leq p^c$. We may assume that $p \leq U \leq q^c$. Then we have that $q \leq U^c \leq p^c$. By the assumption, U^c is fuzzy δ -open. Hence $(X, \mathcal{T})$ is fuzzy δ - T_2 . $\square$

Theorem 2.11: If for any $x \in X$ the crisp fuzzy point x_1 is fuzzy δ -open in $(X, \mathcal{T})$, then $(X, \mathcal{T})$ is fuzzy δ - T_2 .

Proof: Take any two fuzzy points p and q with different supports. Then $p \leq p_1 \leq q^c$ and $q \leq q_1 \leq p^c$. Clearly $p_1 \leq q_1^c$, and by the assumption p_1 and q_1 are fuzzy δ -open. Hence $(X, \mathcal{T})$ is fuzzy δ - T_2 . $\square$

Theorem 2.12: Let $f : X \rightarrow Y$ be injective and fuzzy δ -continuous. If Y is fuzzy δ - T_2 , then so is X .

Proof: Take any two points p and q with different supports in X . Since f is injective, $f(p)$ and $f(q)$ are

two fuzzy points with different supports in Y . Since Y is fuzzy δ - T_2 , there are two fuzzy δ -open sets U_1, U_2 with $f(p) \leq U_1 \leq f(q)^c$, $f(q) \leq U_2 \leq f(p)^c$ and $U_1 \leq U_2^c$. Therefore $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are fuzzy δ -open sets in X with $p \leq f^{-1}(U_1) \leq q^c$ and $q \leq f^{-1}(U_2) \leq p^c$. Furthermore $f^{-1}(U_1) \leq (f^{-1}(U_2))^c$. Hence X is fuzzy δ - T_2 . $\square$

Definition 2.13: A fts X is called fuzzy δ -regular if for any fuzzy point p in X and any fuzzy δ -closed set K with $p \leq K^c$, there are fuzzy δ -open sets U_1 and U_2 with $p \leq U_1$, $K \leq U_2$ and $U_1 \leq U_2^c$. A fts X is called fuzzy δ - T_3 if it is fuzzy δ -regular and fuzzy δ - T_1 .

We can easily prove that every fuzzy δ - T_3 space is fuzzy δ - T_2 .

We have known that for any fuzzy closed set K , $\text{int}(K)$ is fuzzy regular open. Thus it is also fuzzy δ -open. Therefore the following theorem is hold.

Theorem 2.14: For a fts $(X, \mathcal{T})$, the following are equivalent:

- (1) X is fuzzy δ -regular.
- (2) For any fuzzy point p and any fuzzy δ -open set V containing p , there is a fuzzy δ -open set U such that $p \leq U \leq \text{cl}_\delta(U) \leq V$.
- (3) For any fuzzy δ -closed set K and any fuzzy point p such that $p \leq K^c$, there are fuzzy δ -open sets U_1 and U_2 such that $p \leq U_1$, $K \leq U_2$ and $\text{cl}(U_1) \leq (\text{cl}(U_2))^c$.
- (4) For any fuzzy δ -closed set K and any fuzzy point p such that $p \leq K^c$, there are fuzzy open sets U_1 and U_2 such that $p \leq U_1$, $K \leq U_2$ and $\text{cl}(U_1) \leq (\text{cl}(U_2))^c$.

Proof: (1) $\Rightarrow$ (2) Let p be a fuzzy point set and V a fuzzy δ -open set containing p . Then there exist fuzzy δ -open sets U_1 and U_2 such that $p \leq U_1$, $1 - V \leq U_2$ and $U_1 \leq 1 - U_2$. So $p \leq U_1 \leq 1 - U_2 \leq V$. Thus $p \leq U_1 \leq \text{cl}_\delta(U_1) \leq \text{cl}_\delta(1 - U_2) = 1 - U_2 \leq V$.

(2) $\Rightarrow$ (3) Let K be a fuzzy δ -closed subset of X and p a fuzzy point set such that $p \leq K^c$. Then $1 - K$ is a fuzzy δ -open set with $p \leq K^c$. By (2) there is a fuzzy δ -open set U such that $p \leq U \leq \text{cl}_\delta(U) \leq 1 - K$. Since U is a fuzzy δ -open set containing p , there is a fuzzy δ -open set V such that $p \leq V \leq \text{cl}_\delta(V) \leq U \leq \text{cl}_\delta(U) \leq 1 - K$. Put $U_1 = V$ and $U_2 = (\text{cl}_\delta(U))^c$. Then U_1 and U_2 are fuzzy δ -open sets with $p \leq U_1$, $K \leq U_2$. Furthermore $\text{cl}(U_2) \leq \text{cl}((\text{cl}_\delta(U))^c) \leq \text{cl}(U^c) = U^c$. Hence $\text{cl}(U_1) = \text{cl}(V) \leq U \leq (\text{cl}(U_2))^c$.

(3) $\Rightarrow$ (4) it is obvious.

(4) $\Rightarrow$ (1) Let K be a fuzzy δ -closed subset of X and p a fuzzy point set such that $p \leq K^c$. By (4), there are fuzzy open sets U and V such that $p \leq U$, $K \leq V$ and $\text{cl}(U) \leq (\text{cl}(V))^c$. Since $p \leq U \leq \text{cl}(U)$, $p \leq \text{int}(U) = U \leq \text{int}(\text{cl}(U))$. Put $U_1 = \text{int}(\text{cl}(U))$, then U_1 is fuzzy δ -open and $p \leq U_1$. Since $K \leq V \leq \text{cl}(V)$, $K \leq \text{int}(V) = V \leq \text{int}(\text{cl}(V))$. Put $U_2 = \text{int}(\text{cl}(V))$, then U_2 is fuzzy δ -open and $K \leq U_2$. Furthermore, since $\text{int}(\text{cl}(U)) \leq \text{cl}(U) \leq (\text{cl}(V))^c \leq (\text{int}(\text{cl}(V)))^c$, we have $U_1 \leq U_2^c$. $\square$

Definition 2.15: A fts X is called fuzzy δ -normal if for any pair of fuzzy δ -closed subsets K, L with $K \leq L^c$ in X ,

there are fuzzy δ -open sets U_1 and U_2 in X with $K \leq U_1$, $L \leq U_2$ and $U_1 \leq U_2^c$. A fts X is called fuzzy δ - T_4 if it is fuzzy δ - T_1 and fuzzy δ -normal.

Clearly every fuzzy δ - T_4 space is a fuzzy δ - T_3 space.

Theorem 2.16: For a fts $(X, \mathcal{T})$, the following are equivalent:

- (1) X is fuzzy δ -normal.
- (2) For any fuzzy δ -closed set K and any fuzzy δ -open set U containing K , there exists a fuzzy δ -open set V such that $K \leq V \leq \text{cl}(V) \leq U$.
- (3) For any fuzzy δ -closed set K and any fuzzy δ -open set U containing K , there exists a fuzzy open set V such that $K \leq V \leq \text{cl}(V) \leq U$.
- (4) For any pair of fuzzy δ -closed subsets K, L with $K \leq L^c$ in X , there are fuzzy open sets U_1 and U_2 with $K \leq U_1$, $L \leq U_2$ and $\text{cl}(U_1) \leq (\text{cl}(U_2))^c$.

Proof: (1) $\Rightarrow$ (2) Let K be a fuzzy δ -closed set and U a fuzzy δ -open set containing K . Then U^c is a fuzzy δ -closed set with $U^c \leq K^c$. Thus there are fuzzy δ -open sets V_1 and V_2 such that $K \leq V_1, U^c \leq V_2$ and $V_1 \leq V_2^c$. So $K \leq V_1 \leq V_2^c \leq U$ and $V_1 \leq \text{cl}_\delta(V_1) \leq \text{cl}_\delta(V_2^c) = V_2^c$. Hence $K \leq V_1 \leq \text{cl}(V_1) \leq \text{cl}_\delta(V_1) \leq U$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (4) Let K and L be fuzzy δ -closed subsets of X with $K \leq L^c$. Then L^c is a fuzzy δ -open set containing K . By (3) there is a fuzzy open set V such that $K \leq V \leq \text{cl}(V) \leq L^c$. Since $\text{cl}(V)$ is fuzzy δ -closed and L^c is a fuzzy δ -open set containing $\text{cl}(V)$, there is a fuzzy open set U such that $K \leq V \leq \text{cl}(V) \leq U \leq \text{cl}(U) \leq L^c$. Let $U_1 = V$ and $U_2 = (\text{cl}(U))^c$, then U_1 and U_2 are fuzzy open sets with $K \leq U_1, L \leq U_2$. And $\text{cl}(U_2) = \text{cl}((\text{cl}(U))^c) \leq \text{cl}(U^c) = U^c$. So $\text{cl}(U_1) = \text{cl}(V) \leq U \leq (\text{cl}(U_2))^c$.

(4) $\Rightarrow$ (1) Let K and L be fuzzy δ -closed subsets of X with $K \leq L^c$. Then, by (4), there are fuzzy open sets U and V such that $K \leq U, L \leq V$ and $\text{cl}(U) \leq (\text{cl}(V))^c$. Furthermore since $K \leq U \leq \text{cl}(U)$, $K \leq U = \text{int}(U) \leq \text{int}(\text{cl}(U))$. Put $U_1 = \text{int}(\text{cl}(U))$, then U_1 is fuzzy δ -open and $K \leq U_1$. Similarly put $U_2 = \text{int}(\text{cl}(V))$, then U_2 is fuzzy δ -open and $L \leq U_2$. And since $\text{int}(\text{cl}(U)) \leq \text{cl}(U) \leq (\text{cl}(V))^c \leq (\text{int}(\text{cl}(V)))^c$, $U_1 \leq U_2^c$. $\square$

Example 2.17: Let $X = [0, 1]$ and for each $\alpha \in X$, $U_\alpha(x) = \alpha$ for all $x \in X$. And let $\mathcal{T} = \{U_\alpha \mid \alpha \in X\}$. Then $\mathcal{T}$ is a fuzzy topology and each U_α is fuzzy δ -open. Therefore $\mathcal{T}$ is fuzzy δ -normal and fuzzy δ -regular. But it is not fuzzy δ - T_1 . So it is neither fuzzy δ - T_3 nor fuzzy δ - T_4 .

Example 2.18: Let $X = [0, 1]$ and for each $\alpha \in X$

$$U_\alpha(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \alpha \\ 0 & \text{if } \alpha < x \leq 1, \end{cases}$$

$$V_\alpha(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \alpha \\ 1 & \text{if } \alpha < x \leq 1. \end{cases}$$

Let $\mathcal{T}$ be a fuzzy topology on X generated by the subbase $\{U_\alpha \mid \alpha \in X\} \cup \{V_\alpha \mid \alpha \in X\}$. Then $\mathcal{T}$ is a fuzzy topology

and $\text{int}(\text{cl}(U_\alpha)) = U_\alpha$ for all $\alpha \in X$. So every U_α is fuzzy δ -open. Similarly every V_α is also fuzzy δ -open. Therefore $(X, \mathcal{T})$ is fuzzy δ - T_4 and also fuzzy δ - T_3 .

III. FUZZY δ -CLOSURE AND δ -INTERIOR IN THE FUZZY SUBSPACE μ

Let $(X, \mathcal{T})$ be a fuzzy topological space and μ a fuzzy subset of X . We denote by $(\mu, \mathcal{T}^\mu)$ the fuzzy subspace on μ . If μ is fuzzy regular open(regular closed) in X , then $(\mu, \mathcal{T}^\mu)$ is said to be a fuzzy regular open(resp. regular closed) subspace.

Definition 3.1: Let $A \in X^\mu$. We say that A is fuzzy regular open(regular closed) in the subspace μ if $A = \text{int}^\mu(\text{cl}^\mu(A))$ (resp. $\text{cl}^\mu(\text{int}^\mu(A))$).

Definition 3.2: Let $A \in X^\mu$. A fuzzy point $x_r \in \mu$ is said to be a fuzzy δ -cluster point of A in μ iff every fuzzy regular open q -nbd U of x_r in μ is q -coincident with A in μ . The set of all fuzzy δ -cluster point of A in μ is called the fuzzy δ -closure of A in μ and is denoted by $\text{cl}_\delta^\mu(A)$.

Theorem 3.3: Let $A \in X^\mu$ and $x_r \in \mu$. $x_r \in \bigwedge \{F \in X^\mu \mid A \leq F, F = \text{cl}^\mu(\text{int}^\mu(F))\}$ if and only if every fuzzy regular open q -nbd U of x_r in μ is q -coincident with A in μ .

Proof: Suppose that N is a fuzzy regular open q -nbd of x_r with $N \tilde{q} A$. Then N is also a fuzzy open set in μ such that $x_r q N \leq N$ and $A \tilde{q} N$. Since N^c is fuzzy regular closed and $A \leq N^c$, $\bigwedge \{F \in X^\mu \mid A \leq F, F = \text{cl}^\mu(\text{int}^\mu(F))\} \leq N^c$. And since $x_r \notin N^c$, we have $x_r \notin \bigwedge \{F \in X^\mu \mid A \leq F, F = \text{cl}^\mu(\text{int}^\mu(F))\}$. Conversely suppose that $x_r \notin \bigwedge \{F \in X^\mu \mid A \leq F, F = \text{cl}^\mu(\text{int}^\mu(F))\}$. Then there is a fuzzy regular closed set F such that $x_r \notin F$ and $A \leq F$. Thus F^c is a fuzzy regular open set with $x_r q F^c$ and $A \tilde{q} F^c$. Hence x_r is not a fuzzy δ -cluster point of A in μ . $\square$

By the above theorem, in a fuzzy subspace $(\mu, \mathcal{T}^\mu)$, we have

$$\text{cl}_\delta^\mu(A) = \bigwedge \{F \in X^\mu \mid A \leq F, F = \text{cl}^\mu(\text{int}^\mu(F))\}$$

for any set $A \in X^\mu$. Now we define the δ -interior in a subspace.

Definition 3.4: Let $A \in X^\mu$. The δ -interior of A in μ is defined as follows;

$$\text{int}_\delta^\mu(A) = \mu - \text{cl}_\delta^\mu(\mu - A).$$

We have known that $\text{cl}^\mu(A) = \mu - \text{int}^\mu(\mu - A)$ for any fuzzy subset A of X^μ . So we have the following remark.

Remark 3.5:

$$\begin{aligned} \text{int}_\delta^\mu(A) &= \mu - \text{cl}_\delta^\mu(\mu - A) \\ &= \mu - \bigwedge \{F \in X^\mu \mid \mu - A \leq F, \\ &\quad F = \text{cl}^\mu(\text{int}^\mu(F))\} \\ &= \bigvee \{\mu - F \in X^\mu \mid \mu - F \leq A, \\ &\quad \mu - F = \mu - \text{cl}^\mu(\text{int}^\mu(F))\} \\ &= \bigvee \{U \in X^\mu \mid U \leq A, U = \text{int}^\mu(\text{cl}^\mu(U))\}. \end{aligned}$$

We will show that for any fuzzy set A in a fuzzy subspace $(\mu, \mathcal{T}^\mu)$,

$$\text{cl}_\delta^\mu(A) = \bigwedge \{ \text{cl}^\mu(U) \mid A \leq \text{cl}^\mu(U), U \in \mathcal{T}^\mu \}.$$

For the sake, we will prove two lemmas.

Lemma 3.6: Let $(\mu, \mathcal{T}^\mu)$ be a fuzzy subspace. If $V \in \mathcal{T}^\mu$, then $\text{cl}^\mu(V)$ is fuzzy regular closed in μ .

Proof: Since $V \leq \text{cl}^\mu(V)$, $V = \text{int}^\mu(V) \leq \text{int}^\mu(\text{cl}^\mu(V))$ and hence $\text{cl}^\mu(V) \leq \text{cl}^\mu(\text{int}^\mu(\text{cl}^\mu(V)))$. Conversely since $\text{int}^\mu(\text{cl}^\mu(V)) \leq \text{cl}^\mu(V)$, $\text{cl}^\mu(\text{int}^\mu(\text{cl}^\mu(V))) \leq \text{cl}^\mu(\text{cl}^\mu(V)) = \text{cl}^\mu(V)$. Hence $\text{cl}^\mu(V) = \text{cl}^\mu(\text{int}^\mu(\text{cl}^\mu(V)))$. $\square$

Lemma 3.7: Let $(\mu, \mathcal{T}^\mu)$ is a fuzzy subspace. Then $\{ \text{cl}^\mu(U) \mid U \in \mathcal{T}^\mu \} = \{ F \in X^\mu \mid F \text{ is fuzzy regular closed in } \mu \}$.

Proof: We know that for any fuzzy open set U in μ , $\text{cl}^\mu(U)$ is fuzzy regular closed in μ . Conversely, take any fuzzy regular closed set F in μ . Then $F = \text{cl}^\mu(\text{int}^\mu(F)) = \text{cl}^\mu(\bigvee \{ U \mid U \leq F, U \in \mathcal{T}^\mu \}) \in \{ \text{cl}^\mu(U) \mid U \in \mathcal{T}^\mu \}$. $\square$

We may have a problem in finding the fuzzy δ -closure of any fuzzy set. But from the above lemmas we have the clue to find it.

Theorem 3.8: For any fuzzy set A in a fuzzy subspace $(\mu, \mathcal{T}^\mu)$,

$$\text{cl}_\delta^\mu(A) = \bigwedge \{ \text{cl}^\mu(U) \mid A \leq \text{cl}^\mu(U), U \in \mathcal{T}^\mu \}.$$

Proof: The proof is straightforward. $\square$

In general, $\text{cl}^\mu(A) \neq \text{cl}(A) \wedge \mu$ and $\text{int}(A) = \text{int}^\mu(A) \wedge \text{int}(\mu)$ for any fuzzy subset $A \in X^\mu$. But for any crisp fuzzy subset Y of X , if $\mu = \chi_Y$, then $\text{cl}^\mu(A) = \text{cl}(A) \wedge \mu$ for any fuzzy subset $A \in X^\mu$ (See [19]).

Furthermore, if μ is fuzzy open in X and $\mu = \chi_Y$, $\text{cl}_\delta^\mu(A) = \text{cl}_\delta(A) \wedge \mu$ for any fuzzy subset A of μ . Now we will prove it.

Lemma 3.9: Let X be a fuzzy topological space and $\mu = \chi_Y$ a fuzzy open subset of X . Suppose that $A \in X^\mu$. If A is fuzzy regular open in X , then A is fuzzy regular open in μ .

Proof: For any fuzzy subset $A \in X^\mu$, the following holds.

$$\begin{aligned} \text{int}^\mu(\text{cl}^\mu(A)) &= \text{int}^\mu(\text{cl}(A) \wedge \mu) \\ &= \text{int}(\text{cl}(A) \wedge \mu) \\ &= \text{int}(\text{cl}(A)) \wedge \text{int}(\mu) \\ &= \text{int}(\text{cl}(A)) \wedge \mu. \end{aligned}$$

Hence if $A = \text{int}(\text{cl}(A))$, then $\text{int}^\mu(\text{cl}^\mu(A)) = \text{int}(\text{cl}(A)) \wedge \mu = A \wedge \mu = A$. $\square$

Theorem 3.10: Let $(X, \mathcal{T})$ is a fuzzy topological space and Y a crisp fuzzy subset of X . If $\mu = \chi_Y$ and μ is fuzzy regular open in X , then

$$\text{cl}_\delta^\mu(A) = \text{cl}_\delta(A) \wedge \mu$$

for any fuzzy subset $A \in X^\mu$.

Proof: Suppose that $x_r \notin \text{cl}_\delta^\mu(A)$. Then there is a fuzzy regular open q -nbd N of x_r in μ with $N \tilde{q} A[\mu]$. i.e. $\text{int}^\mu(\text{cl}^\mu(N)) \leq A^c$. Since $x_r q N[\mu]$, $x_r q N$ and N is fuzzy open in X , $N = \text{int}(N) \leq \text{int}(\text{cl}(N))$. Note that $\text{int}(\text{cl}(N))$ is a fuzzy regular open q -nbd of x_r in X . Since $\text{int}(\text{cl}(N)) = \text{int}(\text{cl}(N \wedge \mu)) = \text{int}(\text{cl}^\mu(N)) \leq \text{int}^\mu(\text{cl}^\mu(N)) \tilde{q} A[\mu]$, we have $\text{int}(\text{cl}(N)) \tilde{q} A$. Thus $x_r \notin \text{cl}_\delta(A) \wedge \mu$. Conversely take $x_r \in \text{cl}_\delta^\mu(A)$ and a fuzzy regular open q -nbd N of x_r in X . Then $N(x) + r > 1$ and so $(N \wedge \mu)(x) + r > 1$. Thus $N \wedge \mu$ is a fuzzy regular open q -nbd of x_r in X . By the above lemma, $N \wedge \mu$ is also a fuzzy regular open q -nbd of x_r in μ . Since $x_r \in \text{cl}_\delta^\mu(A)$, $(N \wedge \mu) q A$. Hence $N q A$. Therefore x_r is a fuzzy δ -cluster point of A in X . $\square$

In general every fuzzy regular open subset in μ is not fuzzy regular open in X . On the other hand, if $\mu = \chi_Y$ is a fuzzy regular open set in X , then a fuzzy δ -open set in X which is contained in X^μ is also fuzzy δ -open in μ . The following theorem shows it.

Theorem 3.11: Let $(X, \mathcal{T})$ be a fuzzy topological space and Y a crisp fuzzy subset of X . Let $\mu = \chi_Y$ be fuzzy regular open in X and $A \in X^\mu$. Then

$$\text{int}_\delta^\mu(A) = \text{int}_\delta(A).$$

Proof:

$$\begin{aligned} \text{int}_\delta^\mu(A) &= \mu - \text{cl}_\delta^\mu(\mu - A) \\ &= \mu - (\text{cl}_\delta(\mu - A) \wedge \mu) \\ &= \mu \wedge (\text{cl}_\delta(\mu - A) \wedge \mu)^c \\ &= \mu \wedge ((\text{cl}_\delta(\mu - A))^c \vee \mu^c) \\ &= \mu \wedge (\text{cl}_\delta(\mu - A))^c \\ &= \mu \wedge (1 - \text{cl}_\delta(\mu - A)) \\ &= \mu \wedge \text{int}_\delta(1 - (\mu - A)) \\ &= \text{int}_\delta(\mu \wedge (1 - (\mu - A))) \\ &= \text{int}_\delta(A). \end{aligned}$$

$\square$

IV. FUZZY δ -SEPARATION AXIOMS IN THE FUZZY SUBSPACE μ

Now we define the fuzzy δ -separation axioms on the fuzzy subspaces. Note that $A^c = \mu - A$ in the fuzzy subspace μ , for any set $A \in X^\mu$.

Definition 4.1: A fuzzy subspace $(\mu, \mathcal{T}^\mu)$ is called fuzzy δ - T_0 if for any pair of fuzzy points p and q with different supports in μ , there is a fuzzy δ -open set U in μ with $p \leq U \leq q^c$ or $q \leq U \leq p^c$.

Theorem 4.2: Let $(X, \mathcal{T})$ be a fuzzy δ - T_0 space. Suppose that Y is a crisp fuzzy subset of X and a fuzzy regular open set of X . Then $(\mu = \chi_Y, \mathcal{T}^\mu)$ is fuzzy δ - T_0 .

Proof: Let X be a fuzzy δ - T_0 space and $\mu = \chi_Y$ a fuzzy regular open subset of X . Take any fuzzy points p and q with different supports in a subspace $(\mu, \mathcal{T}^\mu)$. Then p and q

are also fuzzy points with different supports in the space X . Since X is fuzzy δ - T_0 , there is a fuzzy δ -open set U in X with $p \leq U \leq q^c$ or $q \leq U \leq p^c$. And since $U \wedge \mu$ is fuzzy δ -open in X with $U \wedge \mu \leq \mu$, $U \wedge \mu$ is also fuzzy δ -open in μ . Furthermore $p \leq U \wedge \mu \leq (\mu - q)$ or $q \leq U \wedge \mu \leq (\mu - p)$. Hence $(\mu, \mathcal{T}^\mu)$ is fuzzy δ - T_0 . $\square$

Definition 4.3: A fuzzy subspace $(\mu, \mathcal{T}^\mu)$ is called fuzzy δ - T_1 if for any pair of fuzzy points p and q with different supports in μ , there are two fuzzy δ -open sets U_1, U_2 in μ with $p \leq U_1 \leq q^c$ and $q \leq U_2 \leq p^c$.

Theorem 4.4: A fuzzy subspace $(\mu, \mathcal{T}^\mu)$ is fuzzy δ - T_1 iff every crisp fuzzy point in μ is fuzzy δ -closed in μ .

Proof: Take any crisp fuzzy point p in μ . We will show that $\mu - p$ is fuzzy δ -open in μ . We can take a fuzzy point $q \in \mu - p$ with different support from p . Since $(\mu, \mathcal{T}^\mu)$ is fuzzy δ - T_1 , there is a fuzzy δ -open set U in μ with $q \leq U \leq \mu - p$. Thus $\mu - p = \bigcup_{q \in \mu - p} \{U \mid q \leq U \leq \mu - p, U \text{ is fuzzy } \delta\text{-open in } \mu\}$. Since $\bigcup_{q \in \mu - p} \{U \mid q \leq U \leq \mu - p, U \text{ is fuzzy } \delta\text{-open in } \mu\}$ is fuzzy δ -open in μ , $\mu - p$ is fuzzy δ -open in μ . Hence p is fuzzy δ -closed in μ . $\square$

Corollary 4.5: Let $(\mu, \mathcal{T}^\mu)$ be a fuzzy subspace. μ is fuzzy δ - T_1 if and only if $x_\alpha = \bigcap \{\text{cl}_\delta(U) \mid x_\alpha \leq \text{cl}_\delta(U)\}$ for any $x_\alpha \in \mu$.

Theorem 4.6: Let $(X, \mathcal{T})$ be a fuzzy δ - T_1 space. Suppose that Y is a crisp fuzzy subset of X and fuzzy regular open in X . Then $(\mu = \chi_Y, \mathcal{T}^\mu)$ is fuzzy δ - T_1 .

Proof: It is obvious. $\square$

Definition 4.7: A fuzzy subspace $(\mu, \mathcal{T}^\mu)$ is called fuzzy δ -Hausdorff, or fuzzy δ - T_2 , if for any pair of fuzzy point p and q with different supports in μ , there are fuzzy δ -open sets U_1 and U_2 in μ with $p \leq U_1 \leq q^c$, $q \leq U_2 \leq p^c$ and $U_1 \leq U_2^c$.

Obviously every fuzzy δ - T_2 subspace is fuzzy δ - T_1 .

Theorem 4.8: Let $(X, \mathcal{T})$ be a fuzzy δ - T_2 space. Suppose that Y is a crisp fuzzy subset of X and fuzzy regular open in X . Then $(\mu = \chi_Y, \mathcal{T}^\mu)$ is fuzzy δ - T_2 .

Proof: Take any fuzzy points p and q with different supports in a subspace $(\mu, \mathcal{T}^\mu)$. Then p and q are also fuzzy points with different supports in the space X . Since X is fuzzy δ - T_2 , there are fuzzy δ -open sets U_1 and U_2 in X such that $p \leq U_1 \leq 1 - q$, $q \leq U_2 \leq 1 - p$ and $U_1 \leq 1 - U_2$. Thus there are fuzzy δ -open sets $U_1 \wedge \mu$ and $U_2 \wedge \mu$ in μ such that $p \leq (U_1 \wedge \mu) \leq (1 - q) \wedge \mu = \mu - q$, $q \leq (U_2 \wedge \mu) \leq (1 - p) \wedge \mu = \mu - p$ and $(U_1 \wedge \mu) \leq (1 - U_2) \wedge \mu = \mu - (U_2 \wedge \mu)$. So μ is fuzzy δ - T_2 . $\square$

Definition 4.9: A fuzzy subspace $(\mu, \mathcal{T}^\mu)$ is called fuzzy δ -regular if for every pair consisting of a fuzzy point p in μ and a fuzzy δ -closed set K in μ with $p \leq K^c$, there are fuzzy δ -open sets U_1, U_2 in μ with $p \leq U_1$, $K \leq U_2$ and $U_1 \leq U_2^c$. A fuzzy subspace $(\mu, \mathcal{T}^\mu)$ is called fuzzy δ - T_3 if it is fuzzy δ -regular and fuzzy δ - T_1 .

We can easily prove that every fuzzy δ - T_3 subspace is fuzzy δ - T_2 .

We have known that for any fuzzy closed set K in μ , $\text{int}^\mu(K)$ is fuzzy regular open in μ . Thus it is also fuzzy δ -open. Therefore the following theorem holds.

Theorem 4.10: For a fuzzy subspace $(\mu, \mathcal{T}^\mu)$, the following are equivalent:

- (1) $(\mu, \mathcal{T}^\mu)$ is fuzzy δ -regular.
- (2) For any fuzzy point p and any fuzzy δ -open set V containing p in $(\mu, \mathcal{T}^\mu)$, there is a fuzzy δ -open set U in μ such that $p \leq U \leq \text{cl}_\delta^\mu(U) \leq V$.
- (3) For any fuzzy δ -closed set K and any fuzzy point p in $(\mu, \mathcal{T}^\mu)$ such that $p \leq (\mu - K)$, there are fuzzy δ -open sets U_1 and U_2 in μ such that $p \leq U_1, K \leq U_2$ and $\text{cl}^\mu(U_1) \leq (\mu - \text{cl}^\mu(U_2))$.
- (4) For any fuzzy δ -closed set K and each fuzzy point p in $(\mu, \mathcal{T}^\mu)$ such that $p \leq (\mu - K)$, there are fuzzy open sets U_1 and U_2 in $(\mu, \mathcal{T}^\mu)$ such that $p \leq U_1, K \leq U_2$ and $\text{cl}^\mu(U_1) \leq (\mu - \text{cl}^\mu(U_2))$.

Proof: (1) $\Rightarrow$ (2) Let p be a fuzzy point set in μ and V a fuzzy δ -open set in μ containing p . Then there exist fuzzy δ -open sets U_1 and U_2 in μ such that $p \leq U_1, \mu - V \leq U_2$ and $U_1 \leq \mu - U_2$. So $p \leq U_1 \leq \mu - U_2 \leq V$. Thus $p \leq U_1 \leq \text{cl}_\delta^\mu(U_1) \leq \text{cl}_\delta^\mu(\mu - U_2) = \mu - U_2 \leq V$.

(2) $\Rightarrow$ (3) Let K be a fuzzy δ -closed subset in μ and p a fuzzy point in μ such that $p \leq \mu - K$. Then $\mu - K$ is a fuzzy δ -open set in μ with $p \leq \mu - K$. By (2) there is a fuzzy δ -open set U in μ such that $p \leq U \leq \text{cl}_\delta^\mu(U) \leq \mu - K$. Since U is a fuzzy δ -open set in μ containing p , there is a fuzzy δ -open set V in μ such that $p \leq V \leq \text{cl}_\delta^\mu(V) \leq U \leq \text{cl}_\delta^\mu(U) \leq \mu - K$. Put $U_1 = V$ and $U_2 = \mu - \text{cl}_\delta^\mu(U)$. Then U_1 and U_2 are fuzzy δ -open sets in μ with $p \leq U_1, K \leq U_2$. Furthermore $\text{cl}^\mu(U_2) \leq \text{cl}^\mu(\mu - \text{cl}_\delta^\mu(U)) \leq \text{cl}^\mu(\mu - U) = \mu - U$. Since $\text{cl}^\mu(V) \leq \text{cl}_\delta^\mu(V)$, $\text{cl}^\mu(U_1) = \text{cl}^\mu(V) \leq U \leq \mu - \text{cl}^\mu(U_2)$.

(3) $\Rightarrow$ (4) it is obvious.

(4) $\Rightarrow$ (1) Let K be a fuzzy δ -closed subset in μ and p a fuzzy point in μ such that $p \leq \mu - K$. By (4), there are fuzzy open sets U and V in μ such that $p \leq U, K \leq V$ and $\text{cl}^\mu(U) \leq \mu - \text{cl}^\mu(V)$. Since $p \leq U \leq \text{cl}^\mu(U)$, $p \leq \text{int}^\mu(U) = U \leq \text{int}^\mu(\text{cl}^\mu(U))$. Put $U_1 = \text{int}^\mu(\text{cl}^\mu(U))$, then U_1 is fuzzy δ -open in μ and $p \leq U_1$. Since $K \leq V \leq \text{cl}^\mu(V)$, $K \leq \text{int}^\mu(V) = V \leq \text{int}^\mu(\text{cl}^\mu(V))$. Put $U_2 = \text{int}^\mu(\text{cl}^\mu(V))$, then U_2 is fuzzy δ -open in μ and $K \leq U_2$. Furthermore, since $\text{int}^\mu(\text{cl}^\mu(U)) \leq \text{cl}^\mu(U) \leq \mu - \text{cl}^\mu(V) \leq \mu - \text{int}^\mu(\text{cl}^\mu(V))$, $U_1 \leq \mu - U_2$. $\square$

Definition 4.11: A fuzzy subspace $(\mu, \mathcal{T}^\mu)$ is called fuzzy δ -normal if for any pair of fuzzy δ -closed subsets K, L in μ with $K \leq L^c$, there are fuzzy δ -open sets U_1, U_2 in μ with $K \leq U_1, L \leq U_2$ and $U_1 \leq U_2^c$. A fuzzy subspace $(\mu, \mathcal{T}^\mu)$ is called fuzzy δ - T_4 if it is fuzzy δ - T_1 and fuzzy δ -normal.

Clearly every fuzzy δ - T_4 subspace is fuzzy δ - T_3 .

Theorem 4.12: For a fuzzy subspace $(\mu, \mathcal{T}^\mu)$, the following are equivalent:

- (1) $(\mu, \mathcal{T}^\mu)$ is fuzzy δ -normal.
- (2) For any fuzzy δ -closed set K and any fuzzy δ -open set U containing K in $(\mu, \mathcal{T}^\mu)$, there exists a fuzzy δ -open set V in

μ such that $K \leq V \leq \text{cl}_\delta^\mu(V) \leq U$.

(3) For any fuzzy δ -closed set K and any fuzzy δ -open set U containing K in $(\mu, \mathcal{T}^\mu)$, there exists a fuzzy δ -open set V in μ such that $K \leq V \leq \text{cl}^\mu(V) \leq U$.

(4) For any fuzzy δ -closed set K and any fuzzy δ -open set U containing K in $(\mu, \mathcal{T}^\mu)$, there exists a fuzzy open set V in μ such that $K \leq V \leq \text{cl}^\mu(V) \leq U$.

(5) For any pair of fuzzy δ -closed subsets K and L with $K \leq \mu - L$ in $(\mu, \mathcal{T}^\mu)$, there are fuzzy open sets U_1 and U_2 in μ with $K \leq U_1$, $L \leq U_2$ and $\text{cl}^\mu(U_1) \leq (\mu - \text{cl}^\mu(U_2))$.

Proof: (1) $\Rightarrow$ (2) Let K be a fuzzy δ -closed set in μ and U a fuzzy δ -open set in μ containing K . Then $\mu - U$ is a fuzzy δ -closed set in μ with $\mu - U \leq \mu - K$. Thus there are fuzzy δ -open sets V_1 and V_2 in μ such that $K \leq V_1$, $\mu - U \leq V_2$ and $V_1 \leq \mu - V_2$. So $K \leq V_1 \leq \mu - V_2 \leq U$ and $V_1 \leq \text{cl}_\delta^\mu(V_1) \leq \text{cl}_\delta^\mu(\mu - V_2) = \mu - V_2$. Hence $K \leq V_1 \leq \text{cl}_\delta^\mu(V_1) \leq U$.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) It is obvious.

(4) $\Rightarrow$ (5) Let K and L be fuzzy δ -closed subsets in μ with $K \leq \mu - L$. Then $\mu - L$ is a fuzzy δ -open set in μ containing K . By (3) there is a fuzzy open set V in μ such that $K \leq V \leq \text{cl}^\mu(V) \leq \mu - L$. Since $\text{cl}^\mu(V)$ is fuzzy δ -closed in μ and $\mu - L$ is a fuzzy δ -open set in μ containing $\text{cl}^\mu(V)$, there is a fuzzy open set U in μ such that $K \leq V \leq \text{cl}^\mu(V) \leq U \leq \text{cl}^\mu(U) \leq \mu - L$. Let $U_1 = V$ and $U_2 = \mu - \text{cl}^\mu(U)$, then U_1 and U_2 are fuzzy open sets in μ with $K \leq U_1$ and $L \leq U_2$. Also $\text{cl}^\mu(U_2) = \text{cl}^\mu(\mu - \text{cl}^\mu(U)) \leq \text{cl}^\mu(\mu - U) = \mu - U$. So $\text{cl}^\mu(U_1) = \text{cl}^\mu(V) \leq U \leq \mu - \text{cl}^\mu(U_2)$.

(5) $\Rightarrow$ (1) Let K and L be fuzzy δ -closed subsets in μ with $K \leq \mu - L$. Then, by (4), there are fuzzy open sets U and V in μ such that $K \leq U$, $L \leq V$ and $\text{cl}^\mu(U) \leq \mu - \text{cl}^\mu(V)$. Since $K \leq U \leq \text{cl}^\mu(U)$, we have $K \leq U = \text{int}^\mu(U) \leq \text{int}^\mu(\text{cl}^\mu(U))$. Put $U_1 = \text{int}^\mu(\text{cl}^\mu(U))$, then U_1 is fuzzy δ -open in μ and $K \leq U_1$. Similarly put $U_2 = \text{int}^\mu(\text{cl}^\mu(V))$, then U_2 is fuzzy δ -open in μ and $L \leq U_2$. And since $\text{int}^\mu(\text{cl}^\mu(U)) \leq \text{cl}^\mu(U) \leq \mu - \text{cl}^\mu(V) \leq \mu - \text{int}^\mu(\text{cl}^\mu(V))$, we have $U_1 \leq \mu - U_2$. $\square$

Theorem 4.13: Let $(X, \mathcal{T})$ be a fuzzy δ -regular space. Suppose that Y is a crisp fuzzy subset of X and fuzzy regular open in X . Then $(\mu = \chi_Y, \mathcal{T}^\mu)$ is fuzzy δ -regular.

Proof: Let U be a fuzzy δ -open set in μ and p a fuzzy point in μ with $p \leq U$. Since μ is fuzzy regular open in X , U is also fuzzy δ -open in X . Since X is fuzzy δ -regular, there is a fuzzy δ -open set W of X such that $p \leq W \leq \text{cl}_\delta(W) \leq U$. Thus there is a fuzzy δ -open set $W_\mu = W \wedge \mu$ in μ such that $p \leq W_\mu \leq \text{cl}_\delta(W) \wedge \mu = \text{cl}_\delta^\mu(W_\mu) \leq U \wedge \mu = U$. Hence μ is fuzzy δ -regular. $\square$

Lemma 4.14: Let $(X, \mathcal{T})$ be a fuzzy δ -normal space. Suppose that μ is a crisp fuzzy subset of X and fuzzy regular closed in X . If $K \in X^\mu$ is fuzzy regular closed in μ , then K is also fuzzy regular closed in X .

Proof: $K = \text{cl}_\delta^\mu(K) = \text{cl}_\delta(K) \wedge \mu = \text{cl}_\delta(K) \wedge \text{cl}_\delta(\mu) \geq \text{cl}_\delta(K \wedge \mu) = \text{cl}_\delta(K) \geq K$. $\square$

Theorem 4.15: Let $(X, \mathcal{T})$ be a fuzzy δ -normal space. Suppose that Y is a crisp fuzzy subset of X and fuzzy regular

clopen in X . Then $(\mu = \chi_Y, \mathcal{T}^\mu)$ is fuzzy δ -normal.

Proof: Let X be a fuzzy δ -normal space and μ a fuzzy δ -clopen subspace of X . Let K, L be fuzzy δ -closed subsets in μ with $K \leq (\mu - L)$. Since μ is fuzzy δ -closed in X , K and L are also fuzzy δ -closed in X with $K \leq (1 - L)$. Since X is fuzzy δ -normal, there exist fuzzy δ -open sets U_1 and U_2 in X with $K \leq U_1$, $L \leq U_2$ and $U_1 \leq U_2^c$. So there exist fuzzy δ -open sets $(\mu \wedge U_1)$ and $(\mu \wedge U_2)$ in μ with $K \leq (\mu \wedge U_1)$, $L \leq (\mu \wedge U_2)$ and $(\mu \wedge U_1) \leq \mu - (\mu \wedge U_2)$ in the subspace μ . $\square$

V. CONCLUSION

In this paper, we introduced fuzzy δ -separation axioms in fuzzy topological spaces, and proved that

$$\delta-T_4 \Rightarrow \delta-T_3 \Rightarrow \delta-T_2 \Rightarrow \delta-T_1 \Rightarrow \delta-T_0.$$

Also we defined fuzzy δ -separation axioms in the fuzzy subspace in the sense of [22]. We have shown the hereditariness of fuzzy δ -separation in the fuzzy regular subspaces. However we cannot find counterexamples for the reverse directions of the above implications except $\delta-T_0 \not\Rightarrow \delta-T_1$. In the subsequential research, we will find these examples and investigate other properties of fuzzy separation axioms.

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