

Multi-attribute Decision Making Models under Interval Type-2 Fuzzy Environment

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Abstract—Interval type-2 fuzzy sets, each of which is characterized by the footprint of uncertainty, are a very useful means to depict the decision information in the process of decision making. In this article, we investigate the decision making problems in which all the information provided by the decision maker is expressed as interval type-2 fuzzy decision matrix where each of the elements is characterized by interval type-2 fuzzy set, and the information about attribute weights is partially known, which may be constructed by various forms. We first utilize the average centroid measure to calculate the average centroid of each attribute value and construct the average centroid matrix of the interval type-2 fuzzy decision matrix. Based on the average centroid matrix and the given attribute weight information, we establish some optimization models to determine the weights of attributes. Furthermore, we develop a procedure to identify the best alternative(s). Finally, we give an illustrative example.

Keywords—Interval type-2 fuzzy sets (IT2 FSs); average centroid matrix; multi-attribute decision making (MADM); incomplete information.

I. INTRODUCTION

Multi-attribute decision making (MADM) problems are widespread in real life decision situations. A MADM problem is to find the most desirable alternative(s) from a set of n feasible alternatives $X = \{x_1, x_2, \dots, x_n\}$ according to the decision information about attribute weights and attribute values. First of all, the decision maker generally need to determine the importance degrees of a set of m attributes $G = \{g_1, g_2, \dots, g_m\}$, that is to say, the decision maker need to provide their preference information about attribute weights (for convenience, let the weight vector of attributes be $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$, where $\omega_j \geq 0, j = 1, 2, \dots, m$ and $\sum_{j=1}^m \omega_j = 1$). In general, the provided attribute weight information may be constructed in the following forms [1], [2], for $i = j$,

- 1) A weak ranking: $\{\omega_i \geq \omega_j\}$;
- 2) A strict ranking: $\{\omega_i - \omega_j \geq \delta_i (> 0)\}$;
- 3) A ranking with multiples: $\{\omega_i \geq \delta_i \omega_j\}, 0 \leq \delta_i \leq 1$;
- 4) An interval form: $\{\delta_i \leq \omega_i \leq \delta_i + \varepsilon_i\}, 0 \leq \delta_i < \delta_i + \delta_i \leq 1$;
- 5) A ranking of differences: $\{\omega_i - \omega_j \geq \omega_k - \omega_l\}$, for $j \neq k \neq l$.

The estimation of the attribute values plays an important role in MADM. In the process of decision making, the information about attribute values is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment

and the vagueness of inherent subjective nature of human thinking. Many methods have been presented for handling fuzzy MADM problems based on traditional type-1 fuzzy sets (T1 FSs) [3]. Hwang and Yoon [4] presented the technique for order preference by similarity to ideal solution (TOPSIS) method for handling MADM problems. Chen [5] presented a method for handling fuzzy MADM problems based on the similarity measure between T1 FSs. Chen and Hwang [6] presented some methods for handling fuzzy MADM problems. Chen [7] extended the TOPSIS method to present a method to handle group decision-making problems in a fuzzy environment. Chen [8] presented a method to evaluate the rate of aggregative risk in software development using the T1 FS theory in the fuzzy group decision-making environment. Wang and Lin [9] presented a fuzzy multi-criteria group decision-making approach to select configuration items for software development. Li [10] constructed some novel mathematical programming models for fuzzy MADM problems under uncertainty and proposed corresponding solving methods. Wang and Parkan [11] presented a method for handling MADM problems based on the fuzzy preference information of alternatives. Wang and Parkan [12] presented a general MADM approach for integrating subjective preferences and objective information. Yager and Xu [13] presented a continuous ordered weighted geometric operator and its application to decision making. Tsabadze [14] presented a method for fuzzy aggregation based on group expert evaluations. Wang and Chang [15] presented an application of TOPSIS in evaluating initial aircraft training under a fuzzy environment. Wu and Chen [16] presented a maximizing deviation method for group MADM under the linguistic environment. Li [17] developed a compromise ratio methodology for fuzzy multi-attribute group decision making (MAGDM). Hua et al. [18] presented an approach for MADM problems with incomplete information. Fu [19] presented a fuzzy optimization method for multi-criteria decision making. Lin and Wu [20] presented a causal analytical method for group decision-making in the fuzzy environment. Tsai and Wang [21] presented a computing coordination based fuzzy group decision-making method for web service oriented architectures.

Similar to the transition from ordinary sets to T1 FSs [22], when the circumstances are so fuzzy that we have trouble determining the membership grade even as a crisp number

in $[0, 1]$, we use type-2 fuzzy sets (T2 FSs) [23], [24]. The T2 FSs are now very well established [25] are gaining more and more in popularity [26], [27], [28], [29]. It is known that T2 FSs let us model and minimize the effects of uncertainties in rule-based fuzzy logic systems [22]. The majority of the contributions in the field use interval type-2 fuzzy sets (IT2 FSs) (a particular kind of T2 FSs) which maintain the uncertainty modelling properties of general T2 FSs but reduce the computational efforts that are needed to aggregate them. If we can use IT2 FSs [30] for handling fuzzy decision-making problems, then there is room for more flexibility due to the fact that IT2 FSs provide more flexibility to present uncertainties than traditional T1 FSs. In recent years, some authors have applied the IT2 FS theory to the field of decision making. Wu and Mendel [31], [32] presented a method using the linguistic weighted average and IT2 FSs for handling fuzzy multiple criteria hierarchical group decision-making problems, Wu and Mendel's fuzzy multiple criteria hierarchical group decision making method is to make decisions by means of aggregating the opinions of decision-makers. Wu [33] proposed a novel weighted average as a computing with words engine for MADM, and proposed perceptual reasoning as a computing with words engine for multi-objective decision making. Chen and Lee [34] presented a new method for handling fuzzy multiple criteria hierarchical group decision-making problems based on arithmetic operations and fuzzy preference relations of IT2 FSs. Chen and Lee [35] presented an interval type-2 fuzzy TOPSIS method to handle fuzzy MAGDM problems based on IT2 FSs, and use some examples to illustrate the fuzzy MAGDM process of the proposed method. Chen and Lee [36] presented a new method to handle fuzzy MAGDM problems based on the ranking values and the arithmetic operations of IT2 FSs.

In this paper, we shall focus our attention on the issue of MADM under interval type-2 fuzzy environment where all the information provided by the decision makers is characterized by IT2 FSs and the information about the attribute weights is partially known, which may be constructed by the various forms described above. To do that, the remainder of this paper is organized as follows. In Section II, we give a review of basic concepts and average centroids related to IT2 FSs. Section III presents the considered problem and utilizes the average centroid measure to calculate the average centroid of each attribute value and constructs the average centroid matrix of the interval type-2 fuzzy decision matrix. Based on the average centroid matrix and the given attribute weight information, we establish some optimization models to determine the weights of attributes. Section IV develops an approach to MADM under interval type-2 fuzzy environment, the approach uses the obtained attribute weights and the average centroid matrix of the interval type-2 fuzzy decision matrix to get the overall average centroids of alternatives and the ranking of all the given alternatives. Section V provides a practical example to illustrate the developed approach, and finally, we conclude the paper in Section VI.

II. PRELIMINARIES

Definition 1. [23], [24] A T2 FS $\tilde{A}$ in the universe of discourse X can be represented by a type-2 membership function $\mu_{\tilde{A}}$, shown as follows:

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\} \quad (1)$$

where $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$. The T2 FS $\tilde{A}$ also can be represented as follows:

$$\begin{aligned} \tilde{A} &= \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u) \\ &= \int_{x \in X} \left[\int_{u \in J_x} \mu_{\tilde{A}}(x, u) / u \right] / x \end{aligned} \quad (2)$$

where x is the primary variable, $J_x \subseteq [0, 1]$ is the primary membership of x , u is the secondary variable, and $\int_{u \in J_x} \mu_{\tilde{A}}(x, u) / u$ is the secondary membership function at x . $\int$ denotes the union over all admissible x and u . For discrete universes of discourse, $\int$ is replaced by $\sum$.

Definition 2. [30] Let $\tilde{A}$ be a T2 FS in the universe of discourse X represented by the type-2 membership function $\mu_{\tilde{A}}(x, u)$. If all $\mu_{\tilde{A}}(x, u) = 1$, then $\tilde{A}$ is called an IT2 FS. An IT2 FS can be regarded as a special case of a T2 FS, shown as follows:

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} 1 / (x, u) = \int_{x \in X} \left[\int_{u \in J_x} 1 / u \right] / x \quad (3)$$

where x is the primary variable, $J_x \subseteq [0, 1]$ is the primary membership of x , u is the secondary variable, and $\int_{u \in J_x} 1 / u$ is the secondary membership function at x .

Note that (3) means $\tilde{A} : X \rightarrow \{[a, b] : 0 \leq a \leq b \leq 1\}$. Uncertainty about $\tilde{A}$ is conveyed by the union of all of the primary memberships, called the footprint of uncertainty of $\tilde{A}$ (FOU($\tilde{A}$)), i.e.,

$$\text{FOU}(\tilde{A}) = \bigcup_{x \in X} J_x \quad (4)$$

In this paper, we present a method to use IT2 FSs for handling fuzzy MADM problems, where the reference points and the heights of the upper and the lower membership functions of IT2 FSs are used to characterize IT2 FSs. A trapezoidal IT2 FS $\tilde{A}$ is shown in Fig. 1. The FOU is shown as the shaded region. It is bounded by an upper membership function (UMF) and a lower membership function (LMF), both of which are T1 FSs.

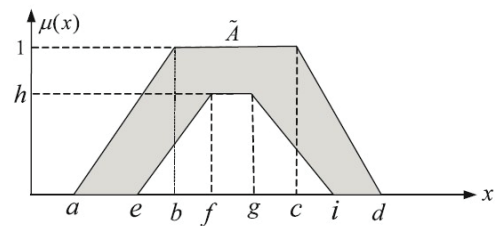


Fig. 1. The nine points to determine a FOU. (a, b, c, d) determines a normal trapezoidal UMF, (e, f, g, i, h) determines a trapezoidal LMF with height h .

Definition 3. [37] The centroid $C(\tilde{A})$ of an IT2 FS $\tilde{A}$ is the union of the centroids of all its embedded T1 FSs A_e , i.e.,

$$C(\tilde{A}) \equiv \bigcup_{\forall A_e} c(A_e) = [c_l(\tilde{A}), c_u(\tilde{A})], \quad (5)$$

where $\bigcup$ is the union operation, and

$$c_l(\tilde{A}) = \min_{\forall A_e} c(A_e) \quad (6)$$

$$c_u(\tilde{A}) = \max_{\forall A_e} c(A_e) \quad (7)$$

$$c(A_e) = \frac{\sum_{i=1}^N x_i \mu_{A_e}(x_i)}{\sum_{i=1}^N \mu_{A_e}(x_i)} \quad (8)$$

It has been shown [38], [37], [39] that $c_l(\tilde{A})$ and $c_u(\tilde{A})$ can be expressed as

$$c_l(\tilde{A}) = \frac{\sum_{i=1}^L x_i \bar{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^N x_i \underline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^L \bar{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^N \underline{\mu}_{\tilde{A}}(x_i)} \quad (9)$$

$$c_u(\tilde{A}) = \frac{\sum_{i=1}^R x_i \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^N x_i \bar{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^R \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^N \bar{\mu}_{\tilde{A}}(x_i)} \quad (10)$$

Switch points L and R , as well as $c_l(\tilde{A})$ and $c_u(\tilde{A})$ are computed by iterative Karnik-Mendel Algorithms [38], [37], [40].

Centroid-based ranking method: First compute the average centroid for each IT2 FS $\tilde{A}_i$:

$$c(\tilde{A}_i) = \frac{c_l(\tilde{A}_i) + c_u(\tilde{A}_i)}{2}, i = 1, 2, \dots, N \quad (11)$$

and then sort $c(\tilde{A}_i)$ to obtain the rank of $\tilde{A}_i$.

The ranking method can be viewed as a generalization of Yager's first ranking method for T1 FSs [41] to IT2 FSs.

III. SOME MODELS FOR DETERMINING ATTRIBUTE WEIGHT

For a MADM problem, let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set of alternatives and $G = \{g_1, g_2, \dots, g_m\}$ be a finite set of attributes, and suppose that the decision maker provides the attribute weight information expressed in the forms 1) – 5) (see Section I). For convenience, we denote by H the set of the known information about attribute weights given by the decision maker. Let $\tilde{R} = (\tilde{A}_{ij})_{n \times m}$ be an interval type-2 fuzzy decision matrix, where $\tilde{A}_{ij}$ is an IT2 FS, provided by the decision maker for the alternative $x_i \in X$ with respect to the attribute g_j , $1 \leq i \leq n$, $1 \leq j \leq m$.

In the situations where the information about attribute weights is completely known, that is, the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ of the attributes g_k ($k = 1, 2, \dots, m$) can be completely determined in advance, then based on the interval type-2 fuzzy decision matrix $\tilde{R} = (\tilde{A}_{ij})_{n \times m}$, Lee and Chen [36] can utilize the fuzzy ranking method and the arithmetic operations between IT2 FSs to handle fuzzy MADM problems.

However, the information about attribute weights provided by the decision maker is usually incomplete because that the increasing complexity of the socio-economic environment makes it less and less possible for the decision maker to

consider all relevant aspects of a problem [1], [2]. How to utilize the interval type-2 fuzzy decision matrix and the known weight information to find the most desirable alternative(s) is an interesting and important issue, which is worth paying attention to.

In the following, we establish some optimization models to determining the weights of attributes.

As interpreted in Section II, the average centroid $c(\tilde{A})$ of an IT FS $\tilde{A}$ can reflect the magnitude of $\tilde{A}$. As a result, we utilize the average centroid function c to define the following concept.

Definition 4. Let $\tilde{R} = (\tilde{A}_{ij})_{n \times m}$ be the interval type-2 fuzzy decision matrix, then the average centroid matrix of $\tilde{R} = (\tilde{A}_{ij})_{n \times m}$ is defined as

$$C = (c_{ij})_{n \times m}$$

where $c_{ij} = c(\tilde{A}_{ij})$ is the average centroid of $\tilde{A}_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq m$.

Based on the average centroid matrix, we express the overall average centroid value of each alternative as:

$$c_i(\omega) = \sum_{j=1}^m \omega_j c_{ij}, i = 1, 2, \dots, n. \quad (12)$$

Obviously, the greater the value $c_i(\omega)$, the better the alternative x_i . When we only consider the alternative x_i , then a reasonable vector of attribute weights $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$ should be determined so as to maximize $c_i(\omega)$, and thus, we establish the following optimization model:

$$(M-1) \max c_i(\omega)$$

$$s.t. \quad \omega = (\omega_1, \omega_2, \dots, \omega_m)^T \in H,$$

$$\omega_j \geq 0, j = 1, 2, \dots, m, \sum_{j=1}^m \omega_j = 1.$$

By solving the model (M-1), we get the optimal solution $\omega^{(i)} = (\omega_1^{(i)}, \omega_2^{(i)}, \dots, \omega_m^{(i)})^T$ corresponding to the alternative x_i . However, in general, in the process of determining the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$, we need to consider all the alternatives x_j ($j = 1, 2, \dots, n$) as a whole. Thus, we construct a combined weight vector as follows:

$$\begin{aligned} \omega &= w_1 \omega^{(1)} + w_2 \omega^{(2)} + \dots + w_n \omega^{(n)} \\ &= \begin{pmatrix} \omega_1^{(1)} & \omega_1^{(2)} & \dots & \omega_1^{(n)} \\ \omega_2^{(1)} & \omega_2^{(2)} & \dots & \omega_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_m^{(1)} & \omega_m^{(2)} & \dots & \omega_m^{(n)} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \\ &= \Omega \omega \end{aligned} \quad (13)$$

where

$$\Omega = \begin{pmatrix} \omega_1^{(1)} & \omega_1^{(2)} & \dots & \omega_1^{(n)} \\ \omega_2^{(1)} & \omega_2^{(2)} & \dots & \omega_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_m^{(1)} & \omega_m^{(2)} & \dots & \omega_m^{(n)} \end{pmatrix}$$

and $w = (w_1, w_2, \dots, w_n)^T$ is an undetermined nonnegative vector satisfying the condition:

$$ww^T = 1 \quad (14)$$

Let $\bar{c}_i = (c_{i1}, c_{i2}, \dots, c_{im})$ ($i = 1, 2, \dots, n$), then the average centroid matrix C can be expressed as $C = (\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n)^T$. By (12) and (13), we have

$$c_i(\omega) = \sum_{j=1}^m \omega_j c_{ij} = \bar{c}_i \omega = \bar{c}_i \Omega w, i = 1, 2, \dots, n \quad (15)$$

To determine the combined weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$, we should make all the overall average centroid values $c_i(\omega)$ ($i = 1, 2, \dots, n$) as greater as possible, which means to maximize the following vector $c(\omega) = (c_1(\omega), c_2(\omega), \dots, c_n(\omega))$ under the condition (14). In order to do that, we establish the following multiple objective optimization model:

$$\begin{aligned} (M-2) \max c(\omega) &= (c_1(\omega), c_2(\omega), \dots, c_n(\omega)) \\ \text{s.t. } ww^T &= 1. \end{aligned}$$

By the equal weighted summation method, the model (M-2) can be transformed into a single objective optimization model:

$$\begin{aligned} (M-3) \max c(\omega)c(\omega)^T \\ \text{s.t. } ww^T &= 1. \end{aligned}$$

Let $f(w) = c(\omega)c(\omega)^T$, then by (15), we have

$$f(w) = c(\omega)c(\omega)^T = w^T (C\Omega)^T (C\Omega) w \quad (16)$$

Let $S = (C\Omega)^T (C\Omega)$, then $S^T = (C\Omega)^T (C\Omega) = S$, i.e., S is a real symmetrical matrix. Moreover, $C \geq 0$. Therefore, S is a nonnegative definite matrix.

Here, we introduce two useful theorems:

Theorem 1. [42] Let $B = (b_{ij})_{n \times n}$ be a real symmetrical matrix, i.e., $B^T = B$, then

$$\max \frac{x^T B x}{x^T x} = \lambda_{\max} \quad (17)$$

where $\lambda_{\max}$ is the largest eigenvalue of B , and x is a nonzero vector.

Theorem 2. [42] Let $B = (b_{ij})_{n \times n}$ be a real irreducible nonnegative matrix, then

- 1) B has a largest eigenvalue $\lambda_{\max}$, which is also a unique eigenvalue of B .
- 2) Let $v = (v_1, v_2, \dots, v_n)^T$ be the eigenvector of $\lambda_{\max}$, then all $v_j > 0$ ($j = 1, 2, \dots, n$), i.e., v is a positive eigenvector.

By Theorems 1 and 2, we know that $f(w)$ has a largest value $\max f(w)$, which is also the largest eigenvalue $\bar{\lambda}_{\max}$ of S . $w = (w_1, w_2, \dots, w_n)^T$ is the eigenvector of $\bar{\lambda}_{\max}$, where $\bar{\lambda}_{\max}$ is unique, and all $w_i > 0$ ($i = 1, 2, \dots, n$). After normalizing the eigenvector w , we can utilize (13) to derive the weight vector ω .

IV. AN APPROACH TO MULTI-ATTRIBUTE DECISION MAKING UNDER INTERVAL TYPE-2 FUZZY ENVIRONMENT

Based on the analysis in Section III, interval type-2 fuzzy MADM with incomplete attribute weight information may be performed via the following steps.

- 1) Obtain the normalized decision matrix. In general, attributes can be classified into two types: benefit attributes and cost attributes. In other words, the attribute set G can be divided into two subsets: G_1 and G_2 , where G_k ($k = 1, 2$) are the subset of benefit attributes and cost attributes, respectively. Furthermore, $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \emptyset$, where $\emptyset$ is empty set. Since the m objectives may be measured in different ways, the decision matrix $\tilde{R}$ needs to be normalized besides all the attributes g_j ($j = 1, 2, \dots, m$) are of the same type. In this paper we choose the following normalization formula to update the decision matrix $\tilde{R}$:

$$\tilde{A}_{ij} = \begin{cases} \tilde{A}_{ij} & j \in G_1 \\ \tilde{A}_{ij}^c & j \in G_2 \end{cases} \quad (18)$$

where $\tilde{A}_{ij}^c$ is the complement of $\tilde{A}_{ij}$. Hence, we obtain the normalized decision matrix $\tilde{R} = (\tilde{A}_{ij})_{n \times m}$.

- 2) Calculate the average centroid matrix $C = (c_{ij})_{n \times m}$ of the normalized decision matrix $\tilde{R} = (\tilde{A}_{ij})_{n \times m}$.
- 3) Utilize the model (M-1) to get the optimal weight vectors $\omega^{(i)} = (\omega_1^{(i)}, \omega_2^{(i)}, \dots, \omega_m^{(i)})^T$ ($i = 1, 2, \dots, n$) corresponding to the alternatives x_i ($i = 1, 2, \dots, n$), and construct the weight matrix Ω .
- 4) Calculate the normalized eigenvector $w = (w_1, w_2, \dots, w_n)^T$ of the matrix $(C\Omega)^T (C\Omega)$.
- 5) Utilize (13) to derive the weight vector $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$.
- 6) Utilize (12) to get the overall average centroid values $c_i(\omega)$ ($i = 1, 2, \dots, n$) of the alternatives x_i ($i = 1, 2, \dots, n$).
- 7) Rank the alternatives x_i ($i = 1, 2, \dots, n$) based on the average centroids $c_i(\omega)$ ($i = 1, 2, \dots, n$), and then select the most desirable one(s).
- 8) End.

V. ILLUSTRATIVE EXAMPLE

Table I shows the linguistic terms “Very Low” (VL), “Low” (L), “Medium Low” (ML), “Medium” (M), “Medium High” (MH), “High” (H), “Very High” (VH) and their corresponding IT2 FSs, respectively. In addition, the corresponding average centroids of IT2 FSs are given in Table I (see Reference [43]).

Assume that there are three cars to be evaluated with four attributes “Safety”, “Price”, “Appearance” and “Performance”. Let X be the set of alternatives, where $X = \{x_1, x_2, x_3\}$, and let G be the set of attributes, where $G = \{g_1, g_2, g_3, g_4\}$. Assume that the decision-maker use the linguistic terms shown in Table I to represent the evaluating values of the alternatives with respect to different attributes, respectively, as shown

TABLE I
LINGUISTIC TERMS AND THEIR CORRESPONDING IT2 FSS.

Linguistic terms	IT2 FSs ($\tilde{A}$)	$c(\tilde{A})$
Very Low (VL)	((0, 0, 0.14, 1.97), (0, 0, 0.05, 0.66, 1))	0.47
Low (L)	((0, 0, 0.64, 2.63), (0, 0, 0.09, 0.99, 1))	0.67
Medium Low (ML)	((0.59, 2.00, 3.25, 4.41), (2.29, 2.70, 2.70, 3.21, 0.42))	2.59
Medium (M)	((3.59, 4.75, 5.50, 6.91), (4.86, 5.03, 5.03, 5.14, 0.27))	5.19
Medium High (MH)	((5.38, 7.50, 8.75, 9.81), (7.79, 8.22, 8.22, 8.81, 0.45))	7.9
High (H)	((7.37, 9.41, 10, 10), (8.72, 9.91, 10, 10, 1))	9.3
Very High (VH)	((8.68, 9.91, 10, 10), (9.61, 9.97, 10, 10, 1))	9.69

TABLE II
THE LINGUISTIC TERMS OF IT2 FSS $\tilde{A}$ AND $\tilde{A}^c$.

The linguistic terms of $\tilde{A}$	The linguistic terms of $\tilde{A}^c$
VL	VH
L	H
ML	MH
M	M
MH	ML
H	L
VH	VL

follows:

$$\tilde{R} = \begin{pmatrix} \text{MH} & \text{H} & \text{VH} & \text{VH} \\ \text{H} & \text{MH} & \text{H} & \text{H} \\ \text{VH} & \text{VH} & \text{M} & \text{H} \end{pmatrix}$$

If the information about attribute weights is completely known in advance, here, we suppose that the weight vector $\omega = (0.25, 0.15, 0.25, 0.35)^T$, then we first utilize the average centroid measure [43] and the weighted mean to handle the above problem, which needs the following steps:

- 1) Considering that the attributes are the benefit attributes except to the attribute "Price" (g_2), then based on the formula (18) and Table II, the decision matrix $\tilde{R}$ can be updated to the following normalized matrix.

$$\tilde{R} = \begin{pmatrix} \text{MH} & \text{L} & \text{VH} & \text{VH} \\ \text{H} & \text{ML} & \text{H} & \text{H} \\ \text{VH} & \text{VL} & \text{M} & \text{H} \end{pmatrix} \quad (19)$$

- 2) Based on Table I, we can get the average centroid matrix $C = (c_{ij})_{n \times m}$ of the normalized decision matrix $\tilde{R}$ as follows:

$$C = \begin{pmatrix} 7.9 & 0.67 & 9.69 & 9.69 \\ 9.3 & 2.59 & 9.3 & 9.3 \\ 9.69 & 0.47 & 5.19 & 9.3 \end{pmatrix} \quad (20)$$

- 3) Utilize the weight vector $\omega = (0.25, 0.15, 0.25, 0.35)^T$ and the i th line values c_{ij} ($j = 1, 2, 3, 4$) of the average centroid matrix C to derive the overall average centroids c_i of the alternatives x_i ($i = 1, 2, 3$):
 $c_1 = \sum_{j=1}^4 \omega_j c_{1j} = 7.8895$,

$$c_2 = \sum_{j=1}^4 \omega_j c_{2j} = 8.2835,$$

$$c_3 = \sum_{j=1}^4 \omega_j c_{3j} = 7.0455.$$

- 4) Rank the alternatives x_i ($i = 1, 2, 3$) based on the average centroids c_i ($i = 1, 2, 3$): $x_2 \succ x_1 \succ x_3$, and then the most desirable car is x_2 .

The decision maker may, however, provide the information about attribute weights with value ranges or order relations as described in Section I because that a decision may be made under time pressure and lack of knowledge, or the decision maker has limited attention and information processing capabilities. For example, the information about attribute weights, given by the decision maker, may be as follows:

$$H = \{0.2 \leq \omega_3 \leq 0.5, \omega_3 - \omega_2 \geq \omega_4 - \omega_1, \\ \omega_3 - \omega_2 \geq 0.1, \omega_4 \geq \omega_1\}$$

As presented in Section I, many approaches have been developed to dealing with the decision making problems based on IT2 FS theory, however, all of them are not suitable to handle this case, in the following, we will select the best car following the procedure proposed in this paper:

- 1) Obtain the normalized decision matrix $\tilde{R}$, i.e., (19).
- 2) Based on Table I, we can get the average centroid matrix $C = (c_{ij})_{n \times m}$ of the normalized decision matrix $\tilde{R}$, i.e., (20).
- 3) Utilize the model (M-1) to get the optimal weight vectors $\omega^{(i)} = (\omega_1^{(i)}, \omega_2^{(i)}, \omega_3^{(i)}, \omega_4^{(i)})^T$ ($i = 1, 2, 3$) corresponding to the alternatives x_i ($i = 1, 2, 3$):
 $\omega^{(1)} = (0.05, 0.05, 0.5, 0.4)^T$,
 $\omega^{(2)} = (0.225, 0.05, 0.5, 0.225)^T$,
 $\omega^{(3)} = (0.2, 0.05, 0.3, 0.45)^T$,
and construct the weight matrix

$$\Omega = \begin{pmatrix} 0.05 & 0.225 & 0.2 \\ 0.05 & 0.05 & 0.05 \\ 0.5 & 0.5 & 0.3 \\ 0.4 & 0.225 & 0.45 \end{pmatrix}$$

then

$$(C\Omega)^T(C\Omega) = \begin{pmatrix} 210.6289 & 208.2285 & 214.1800 \\ 208.2285 & 205.9309 & 211.9237 \\ 214.1800 & 211.9237 & 218.5783 \end{pmatrix}$$

- 4) Calculate the normalized eigenvector w of the matrix $(C\Omega)^T(C\Omega)$: $w = (0.3325, 0.3289, 0.3386)^T$.
- 5) Utilize (13) to derive the weight vector ω :

$$\omega = \Omega w$$

$$= \begin{pmatrix} 0.05 & 0.225 & 0.2 \\ 0.05 & 0.05 & 0.05 \\ 0.5 & 0.5 & 0.3 \\ 0.4 & 0.225 & 0.45 \end{pmatrix} \begin{pmatrix} 0.3325 \\ 0.3289 \\ 0.3386 \end{pmatrix}$$

$$= (0.1583, 0.0500, 0.4323, 0.3594)^T.$$

- 6) Utilize (12) to get the overall average centroid values c_i ($i = 1, 2, 3$) of the alternatives x_i ($i = 1, 2, 3$):
 $c_1 = \sum_{j=1}^4 \omega_j c_{1j} = 8.9556$,

$$c_2 = \sum_{j=1}^4 \omega_j c_{2j} = 8.9645,$$

$$c_3 = \sum_{j=1}^4 \omega_j c_{3j} = 7.1435.$$

- 7) Rank the alternatives $x_i (i = 1, 2, 3)$ based on the average centroids $c_i (i = 1, 2, 3) : x_2 \succ x_1 \succ x_3$, and then the most desirable car is also x_2 .

VI. CONCLUSION

We have investigated the MADM problems under interval type-2 fuzzy environment, and developed an approach to handling the situations where the attribute values are characterized by IT2 FSs, and the information about attribute weights is partially known. We have constructed the average centroid matrix of the interval type-2 fuzzy decision matrix, and established some optimization models to determine the attribute weights. Then we have utilized the obtained attribute weights to get the overall average centroid values of the alternatives, and then find their ranking and select the most desirable one. All these procedures have been illustrated by a decision making problem regarding the selection of cars. In the future, we will apply the developed procedures to some other fields such as medical diagnosis, pattern recognition, machine learning and market prediction, supply chain management, etc.

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