

# A Robust Portfolio Selection Problem based on a Confidence Interval with Investor's Subjectivity

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**Abstract**—This paper considers a robust portfolio selection problem considering a confidence interval with subjectivity to the weight for the standard deviation. Since the proposed model is formulated as an ill-defined problem due to fuzziness and bi-object derived from maximizing both robustness and investor's satisfaction level to the total profit, it is hard to solve it directly. Therefore, introducing fuzzy goals for the bi-objective functions, the proposed model is transformed into the deterministic equivalent problem. Furthermore, in order to obtain the exact optimal portfolio analytically, the solution method is developed introducing a parameter and doing the equivalent transformations.

**Keywords**—component; portfolio selection problem; robust programming; confidential interval; fuzzy goal;

## I. INTRODUCTION

In recent years, not only big companies and institutional investors but also many individual investors take part in investment. Therefore, the role of investment theory, which is generally called portfolio theory, becomes increasingly important. Of course, investors confront several uncertainties derived from changes in social conditions such as price escalation and burst of the economic bubble which have a great influence on the future return, and from insufficiency of available information. Under such conditions, investors must find ways to reduce risks as well as maximizing the total profit.

Until now, various studies concerning portfolio selection problems have been performed. Regarding the research history on the mathematical approach, Markowitz [21] proposed the mean-variance model. Since then, it has been the centre of research activities in the real financial field, and numerous researchers have contributed to the development of modern portfolio theory. Then, many researchers have proposed models of portfolio selection problems that have extended the Markowitz model; mean-absolute-deviation model (Konno et al. [18, 19]), semi-variance model (Bawa and Lindenberg [1]), safety-first model (Elton and Gruber [7]), Value-at-Risk (VaR) and conditional Value-at-Risk model (CVaR) (Rockafellar and Uryasev [22]) etc..

On the other hand, investors may receive a lot of data or information in the current market, but it is almost impossible to

determine strict random distributions of future returns. These distributions may be statistically determined as a confidence interval to the total profit involving the mean value and variance. Therefore, using these statistical distributions, it is more important to considering that decision makers optimize the problem in the worst case; i.e. robust optimization problem.

Recently, the robust optimization problem becomes a more active area of research, and there exists various studies (For example, Ben-Tal and Nemirovski [2, 3], Ghaoui and Lebreton [8]). Particularly, there are some studies of robust portfolio selection problems determining optimal investment strategy using the robust approach (For example, Goldfarb and Iyengar [9], Lobo [20]). The expected return and variance of each asset are mainly estimated from historical data based on the statistical analysis. However, in the case that we consider the value of received information, the investor's subjectivity to the trend of current market, and the satisfaction function to the achievement of objects such as maximizing the total profit, we need to consider that statistical distribution or confidential interval considering these conditions includes some ambiguity and is involved some flexibility. In this paper, we propose an extended model of robust portfolio selection problems including fuzziness.

Until now, there are some basic researches under various uncertainty conditions with respect to portfolio selection problems (Bilbao-Terol et al. [4], Carlsson et al. [5], Guo and Tanaka [10], Hasuike et al. [11], Huang [12, 13], Inuiguchi et al. [14, 15], Katagiri et al. [17, 18], Tanaka et al. [23, 24], Watada [25]). However, there are few models considering the confidential interval with investor's subjectivity. Furthermore, there are no researches which are analytically extended and explicitly solved these types of portfolio selection problems. Therefore, we develop the exact solution algorithm for the proposed robust portfolio selection problem. Since our model is a multi-objective programming problem with investor's subjectivity, in this paper, we transform main problems into the deterministic equivalent problems, and obtain the exact optimal portfolio.

This paper is organized as follows. In Section II, we introduce a robust portfolio selection problem maximizing the region of confidential interval to the total profit. Furthermore, we propose the bi-objective programming problem maximizing

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the total profit as well as the region of confidential interval. In Section III, introducing fuzzy goals to both objective functions, we propose the maximin robust portfolio selection problem and develop the strict solution algorithm. In the solution algorithm, we need to solve a convex programming problem, repeatedly. Therefore, in order to solve the problem analytically and explicitly, in Section IV, we perform the deterministic equivalent transformations, and obtain the exact optimal portfolio. Finally, in Section V, we conclude this paper and discuss future researches.

## II. FORMULATION OF A ROBUST PORTFOLIO SELECTION PROBLEM WITH CONFIDENTIAL INTERVAL

In this section, we consider a robust portfolio selection problem with a confidential interval. First of all, we set parameters in portfolio optimization. The expected return of total future profit  $E(\mathbf{r})$  and the total variance  $\mathbf{Var}(\mathbf{r})$  is defined as follows:

$$E(\mathbf{r}) = \bar{\mathbf{r}}' \phi, \quad \mathbf{Var}(\mathbf{r}) = \phi' \mathbf{V} \phi \quad (1)$$

where the notation means as follows:

$\mathbf{r}$  : Future return vector assumed to be a random variable

$\bar{\mathbf{r}}$  : Mean value vector of random variable  $\mathbf{r}$

$\mathbf{V}$  : Variance-covariance matrix of random variable  $\mathbf{r}$

$\phi$  : Portfolio with respect to each asset  $j$ , ( $j = 1, 2, \dots, n$ )

In general, a confidential interval is represented by mean value and standard deviation of each random variable such as normal distribution, student's t-distribution, etc.. Therefore, in this paper, we define  $d$ -confidential interval using  $E(\mathbf{r})$  and  $\mathbf{Var}(\mathbf{r})$  based on the study (Watada et al. [26]):

$$\left[ E(\mathbf{r}) - d\sqrt{\mathbf{Var}(\mathbf{r})}, E(\mathbf{r}) + d\sqrt{\mathbf{Var}(\mathbf{r})} \right] \quad (2)$$

With respect to this confidential interval, in this case that constant value of parameter  $d$  is larger, the considerable region is also wide, and it urges the investor to deal with more robust case than the small value of  $d$ . In this paper, in terms of considering such robust case by maximizing parameter  $d$ , the following robust portfolio selection problem is formulated:

$$\begin{aligned} & \text{Maximize } d \\ & \text{subject to } E(\mathbf{r}) - d\sqrt{\mathbf{Var}(\mathbf{r})} \geq f, \\ & \quad E(\mathbf{r}) \geq f', \\ & \quad \mathbf{1}'\phi = 1 \end{aligned} \quad (3)$$

where  $f$  and  $f'$  are the worst and standard target values of total future return, respectively. In this paper, we assume  $f \leq f'$  and these parameters are initially determined by the investor. Since the first constraint of problem (3) is equivalently transformed into the following form;

$$E(\mathbf{r}) - d\sqrt{\mathbf{Var}(\mathbf{r})} \geq f \Leftrightarrow \frac{E(\mathbf{r}) - f}{\sqrt{\mathbf{Var}(\mathbf{r})}} \geq d,$$

problem (3) is equivalently transformed into the following problem:

$$\begin{aligned} & \text{Maximize } \frac{E(\mathbf{r}) - f}{\sqrt{\mathbf{Var}(\mathbf{r})}} \\ & \text{subject to } E(\mathbf{r}) \geq f', \\ & \quad \mathbf{1}'\phi = 1 \end{aligned} \quad (4)$$

This problem is a convex programming problem similar to the probability maximization model, and so we obtain the optimal solution using the convex programming approach.

However, in practical investment situations, it is natural for the investor to consider maximizing the target total profit as well as minimizing the total risk, i.e., maximizing the region of confidential interval in terms of robustness. Therefore, the proposed model is reformulated as the following problem:

$$\begin{aligned} & \text{Maximize } d \\ & \text{Maximize } f \\ & \text{subject to } E(\mathbf{r}) - d\sqrt{\mathbf{Var}(\mathbf{r})} \geq f, \\ & \quad E(\mathbf{r}) \geq f', \\ & \quad \mathbf{1}'\phi = 1 \end{aligned} \quad (5)$$

## III. FUZZY PROGRAMMING APPROACH TO THE PROPOSED ROBUST PORTFOLIO SELECTION PROBLEM

In addition to bi-criteria in Section 2, taking account of subjectivity of investment decision and flexibility for the execution of portfolio management, the investor often has satisfaction functions for target parameters  $d$  and  $f$  called fuzzy goals. In this paper, we defined each fuzzy goal as the fuzzy set characterized by a linear membership function:

$$\mu_{\hat{d}}(\omega) = \begin{cases} 1 & (d_U \leq \omega) \\ \frac{\omega - d_L}{d_U - d_L} & (d_L < \omega < d_U) \\ 0 & (\omega \leq d_L) \end{cases} \quad (6)$$

$$\mu_{\hat{f}}(f) = \begin{cases} 1 & (f_U \leq f) \\ \frac{f - f_L}{f_U - f_L} & (f_L < f < f_U) \\ 0 & (f \leq f_L) \end{cases}$$

where  $d_L, d_U$  and  $f_L, f_U$  are constant values determined by

the investor. Then, substituting  $z_d(\mathbf{x}) = \frac{E(\mathbf{r}) - f}{\sqrt{\mathbf{Var}(\mathbf{r})}}$  into

fuzzy goal  $\mu_{\hat{d}}(\omega)$ , the following formula is obtained:

$$\mu_{g(d)}(z_d(\mathbf{x})) = \begin{cases} 1 & (\omega \leq g_U(d)) \\ \frac{g_L(d) - \omega}{(d_U - d_L)\sqrt{\mathbf{Var}(\mathbf{r})}} & (g_U(d) < \omega < g_L(d)) \\ 0 & (g_L(d) \leq \omega) \end{cases} \quad (7)$$

$$g_L(d) = E(\mathbf{r}) - d_L\sqrt{\mathbf{Var}(\mathbf{r})}, \quad g_U(d) = E(\mathbf{r}) - d_U\sqrt{\mathbf{Var}(\mathbf{r})}$$

With respect to problem (5), this problem is a bi-objective programming problem, and so without introducing some criterion of optimality for the bi-objective functions, it is hard to solve this problem directly. Therefore, considering fuzzy goals (6) and (7) for the bi-object and  $f' = f_U$  for the standard target profit, we transform problem (5) into the following maximin programming problem:

$$\begin{aligned} & \text{Maximize } \min \left\{ \mu_{\tilde{g}(d)}(z_d(\mathbf{x})), \mu_{\tilde{G}}(f) \right\} \\ & \text{subject to } E(\mathbf{r}) \geq f_U, \\ & \mathbf{1}'\phi = 1 \end{aligned} \quad (8)$$

This problem is also equivalently transformed into the following single-objective programming problem introducing parameter  $h$  which means the common target satisfaction level:

$$\begin{aligned} & \text{Maximize } h \\ & \text{subject to } \min \left\{ \mu_{\tilde{g}(d)}(z_d(\mathbf{x})), \mu_{\tilde{G}}(f) \right\} \geq h, \\ & E(\mathbf{r}) \geq f_U, \\ & \mathbf{1}'\phi = 1 \end{aligned} \quad (9)$$

With respect to the first constraint of problem (9), we obtain the following deterministic equivalent transformations:

$$\begin{aligned} & \min \left\{ \mu_{\tilde{g}(d)}(z_d(\mathbf{x})), \mu_{\tilde{G}}(f) \right\} \geq h, \\ \Leftrightarrow & \mu_{\tilde{g}(d)}(z_d(\mathbf{x})) \geq h, \mu_{\tilde{G}}(f) \geq h \\ \Leftrightarrow & \exists f: \frac{g_L(d) - f}{((d_U - d_L)\sqrt{\mathbf{Var}(\mathbf{r})})} \geq h, \frac{f - f_L}{f_U - f_L} \geq h \\ \Leftrightarrow & \exists f: g_L(d) + h(d_L - d_U)\sqrt{\mathbf{Var}(\mathbf{r})} \geq f, f \geq (1-h)f_L + hf_U \quad (10) \\ \Leftrightarrow & E(\mathbf{r}) - d_L\sqrt{\mathbf{Var}(\mathbf{r})} + h(d_L - d_U)\sqrt{\mathbf{Var}(\mathbf{r})} \geq (1-h)f_L + hf_U \\ \Leftrightarrow & E(\mathbf{r}) - d_L\sqrt{\mathbf{Var}(\mathbf{r})} - f_L \geq h((d_U - d_L)\sqrt{\mathbf{Var}(\mathbf{r})} + (f_U - f_L)) \\ \Leftrightarrow & \frac{E(\mathbf{r}) - d_L\sqrt{\mathbf{Var}(\mathbf{r})} - f_L}{(d_U - d_L)\sqrt{\mathbf{Var}(\mathbf{r})} + (f_U - f_L)} \geq h \end{aligned}$$

Consequently, the proposed model is transformed into the following problem:

$$\begin{aligned} & \text{Maximize } h \\ & \text{subject to } \frac{E(\mathbf{r}) - d_L\sqrt{\mathbf{Var}(\mathbf{r})} - f_L}{(d_U - d_L)\sqrt{\mathbf{Var}(\mathbf{r})} + (f_U - f_L)} \geq h, \\ & E(\mathbf{r}) \geq f_U, \\ & \mathbf{1}'\phi = 1 \end{aligned} \quad (11)$$

i.e.,

$$\begin{aligned} & \text{Maximize } \frac{E(\mathbf{r}) - d_L\sqrt{\mathbf{Var}(\mathbf{r})} - f_L}{(d_U - d_L)\sqrt{\mathbf{Var}(\mathbf{r})} + (f_U - f_L)} \\ & \text{subject to } E(\mathbf{r}) \geq f_U, \\ & \mathbf{1}'\phi = 1 \end{aligned} \quad (12)$$

This objective function is fractional, but the numerator  $E(\mathbf{r}) - d_L\sqrt{\mathbf{Var}(\mathbf{r})} - f_L$  is a concave function and the

denominator  $(d_U - d_L)\sqrt{\mathbf{Var}(\mathbf{r})} + (f_U - f_L)$  is a convex function, and so the objective function is convex. Then, all constraints are linear functions, and so problem (12) is a convex programming problem. Furthermore, this problem is similar to the model proposed by Dinkelbach [6].

In Dinkelbach [6], the following parametric programming problem  $\mathbf{P}(q)$  introducing parameter  $q$  has been considered:

$$\begin{aligned} & \text{Maximize } E(\mathbf{r}) - d_L\sqrt{\mathbf{Var}(\mathbf{r})} - f_L \\ & \quad - q((d_U - d_L)\sqrt{\mathbf{Var}(\mathbf{r})} + (f_U - f_L)) \\ & \text{subject to } E(\mathbf{r}) \geq f_U, \\ & \mathbf{1}'\phi = 1 \\ \Leftrightarrow & \text{Maximize } E(\mathbf{r}) - (1-q)f_L - qf_U \\ & \quad - ((1-q)d_L + qd_U)\sqrt{\mathbf{Var}(\mathbf{r})} \\ & \text{subject to } E(\mathbf{r}) \geq f_U, \\ & \mathbf{1}'\phi = 1 \end{aligned} \quad (13)$$

With respect to this problem  $\mathbf{P}(q)$ , the following theorem holds.

*Theorem 1* (Dinkelbach [6]): Let the objective function of  $\mathbf{P}(q)$  be  $F(q, \phi)$ , and optimal solution of problem (12) be  $\phi^*$ . Then,

$$q^* = \frac{\bar{\mathbf{r}}'\phi^* - d_L\sqrt{\phi^{*'}\mathbf{V}\phi^*} - f_L}{(d_U - d_L)\sqrt{\phi^{*'}\mathbf{V}\phi^*} + (f_U - f_L)}$$

if, and only if,  $F(q^*, \phi^*) = 0$ .

Consequently, we develop the following exact solution algorithm for the proposed robust portfolio selection problem based on Dinkelbach [6].

#### Solution algorithm.

STEP1: Set  $q_0 = 0$ , and solve  $\mathbf{P}(q_0)$ . If there is no feasible

solution, then terminate this algorithm. In this case, there is no feasible solution of main problem and it is necessary to reset fuzzy goals. Else if there is an optimal portfolio, let the optimal portfolio of  $\mathbf{P}(q_0)$

be  $\phi_1$ , and set  $k = 1$ . Go to STEP2.

STEP2: Evaluate

$$q_k = \frac{\bar{\mathbf{r}}'\phi_k - d_L\sqrt{\phi_k'\mathbf{V}\phi_k} - f_L}{(d_U - d_L)\sqrt{\phi_k'\mathbf{V}\phi_k} + (f_U - f_L)},$$

and go to STEP3.

STEP3: Solve  $\mathbf{P}(q_k)$ , and denote the optimal solution by  $\phi_{k+1}$ . Go to STEP4.

STEP4: With respect to sufficiently small number  $\varepsilon$ , if  $F(q_k, \phi_{k+1}) < \varepsilon$ , then terminate this algorithm. In

this case,  $\phi_{k+1}$  is the optimal portfolio. Else if  $F(q_k, \phi_{k+1}) \geq \varepsilon$ ,  $k \leftarrow k+1$  and return to STEP2.

#### IV. EXACT OPTIMAL PORTFOLIO OF $\mathbf{P}(q)$

In the solution algorithm in Section 3, we need to solve problem  $\mathbf{P}(q_k)$  on each  $q_k$ , repeatedly. Therefore, it is also important to consider the solution algorithm for  $\mathbf{P}(q)$ . In this section, we develop the strict solution algorithm to obtain the exact optimal portfolio of  $\mathbf{P}(q)$ .

We consider the following problem which is equivalent to be  $\mathbf{P}(q)$ :

$$\begin{aligned} & \text{Minimize } -\bar{\mathbf{r}}' \phi + f_q + d_q \sqrt{\phi' \mathbf{V} \phi} \\ & \text{subject to } \bar{\mathbf{r}}' \phi \geq f_U, \\ & \mathbf{1}' \phi = 1 \end{aligned} \quad (14)$$

where  $f_q = (1-q)f_L + qf_U$  and  $d_q = (1-q)d_L + qd_U$ . Then,  $f_q$  and  $d_q$  are positive values. With respect to this problem, since  $\mathbf{V}$  is a variance-covariance matrix which is positive definite, we perform the following equivalent transformations of variables:

$$\begin{aligned} & \phi' \mathbf{V} \phi = \mathbf{x}' \mathbf{x} \\ & \mathbf{V} = \mathbf{Q}' \mathbf{\Lambda} \mathbf{Q}, \quad \left( \begin{array}{l} \mathbf{Q}: \text{unitary matrix} \\ \text{consisting of eigen vectors of } \mathbf{V} \end{array} \right) \\ & \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{pmatrix}, \quad (\lambda_i: \text{eigen value of } \mathbf{V}) \end{aligned} \quad (15)$$

$$\mathbf{\Lambda} = (\mathbf{\Lambda}^{1/2})' \mathbf{\Lambda}^{1/2}, \quad \mathbf{\Lambda}^{1/2} = (\mathbf{\Lambda}^{1/2})' = \begin{pmatrix} \sqrt{\lambda_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \sqrt{\lambda_n} \end{pmatrix}$$

$$\mathbf{x} = \mathbf{\Lambda}^{1/2} \mathbf{Q} \phi, \quad \bar{\mathbf{r}}' = (\mathbf{\Lambda}^{1/2})^{-1} \mathbf{Q} \bar{\mathbf{r}}, \quad \mathbf{a} = (\mathbf{\Lambda}^{1/2})^{-1} \mathbf{Q} \mathbf{1}$$

Using these transformations of variables, problem (15) is equivalently transformed into the following problem:

$$\begin{aligned} & \text{Minimize } -\bar{\mathbf{r}}' \mathbf{x} + f_q + d_q \sqrt{\mathbf{x}' \mathbf{x}} \\ & \text{subject to } \bar{\mathbf{r}}' \mathbf{x} \geq f_U, \\ & \mathbf{a}' \mathbf{x} = 1 \end{aligned} \quad (16)$$

This problem still includes a square root term as well as problem (14). Therefore, in order to obtain the exact optimal portfolio to problem (16), we introduce the following auxiliary problem using a parameter  $R$  not including the square root term:

$$\begin{aligned} & \text{Minimize } R(-\bar{\mathbf{r}}' \mathbf{x} + f_q) + \frac{d_q}{2} (\mathbf{x}' \mathbf{x}) \\ & \text{subject to } \bar{\mathbf{r}}' \mathbf{x} \geq f_U, \\ & \mathbf{a}' \mathbf{x} = 1 \end{aligned} \quad (17)$$

Since this problem is a quadratic convex programming problem, we solve problem (17) more easily than problem (16). Furthermore, with respect to the relation between problem (16) and its auxiliary problem (17), the following theorem holds.

*Theorem 2:* Let the optimal solution of auxiliary problem (17) be  $\mathbf{x}(R)$ . If  $R = \sqrt{\sum_{j=1}^n x_j(R)^2}$  is satisfied,  $\mathbf{x}(R)$  is also the optimal solution of problem (16).

*Proof:* We compare Karush-Kuhn-Tucker (KKT) conditions of problem (16) with that of problem (17). the KKT condition of problem (16) is as follows:

$$\begin{aligned} L &= -\sum_{j=1}^n \bar{r}_j x_j + f_q + d_q \sqrt{\sum_{j=1}^n x_j^2} \\ &+ u \left( f_U - \sum_{j=1}^n \bar{r}_j x_j \right) + v \left( 1 - \sum_{j=1}^n a_j x_j \right) \\ \frac{\partial L}{\partial x_j} &= -\bar{r}_j + d_q \frac{x_j}{\sqrt{\sum_{j=1}^n x_j^2}} - u \bar{r}_j - v a_j = 0 \end{aligned} \quad (18)$$

the KKT condition of auxiliary problem (17) is also obtained as follows:

$$\begin{aligned} L' &= -R \sum_{j=1}^n \bar{r}_j x_j + R f_q + \frac{d_q}{2} \left( \sum_{j=1}^n x_j^2 \right) \\ &+ u' \left( f_U - \sum_{j=1}^n \bar{r}_j x_j \right) + v' \left( 1 - \sum_{j=1}^n a_j x_j \right) \\ \frac{\partial L'}{\partial x_j} &= -R \bar{r}_j + d_q x_j - u' \bar{r}_j - v' a_j = 0 \end{aligned} \quad (19)$$

With respect to these KKT conditions, we set

$u = u'/R$ ,  $v = v'/R$ . Then, satisfying  $R = \sqrt{\sum_{j=1}^n x_j(R)^2}$ , the

KKT condition of problem (17) is equal to that of problem (16). Therefore, this theorem holds.  $\square$

This theorem means that we can obtain the optimal solution of problem (14) by solving the auxiliary quadratic programming problem (17) setting the appropriate parameter  $R$ .

With respect to problem (17) and the KKT condition (19), we consider the following two cases. We reconsider the following KKT condition of problem (17) including complementary conditions:

$$\begin{cases} \frac{\partial L}{\partial \mathbf{x}} = -R\bar{\mathbf{r}}' + d_q \mathbf{x} - u\bar{\mathbf{r}}' - v\mathbf{a} = 0 \\ u(f_U - \bar{\mathbf{r}}'^t \mathbf{x}) = 0 \\ \mathbf{a}' \mathbf{x} = 1 \end{cases} \quad (20)$$

First, in the case of  $f_U - \bar{\mathbf{r}}'^t \mathbf{x} = 0$ , KKT condition (20) is degenerated as follows:

$$\begin{cases} -R\bar{\mathbf{r}}' + d_q \mathbf{x} - u\bar{\mathbf{r}}' - v\mathbf{a} = 0 \\ f_U - \bar{\mathbf{r}}'^t \mathbf{x} = 0 \\ \mathbf{a}' \mathbf{x} = 1 \end{cases} \quad (21)$$

Since we obtain  $\mathbf{x} = \frac{(R+u)\bar{\mathbf{r}}' + v\mathbf{a}}{d_q}$  from the first equation

of (21), we calculate the following solution for each parameter:

$$\begin{cases} v = \frac{d_q (\|\bar{\mathbf{r}}'\|^2 - f_U (\bar{\mathbf{r}}'^t \mathbf{a}))}{\|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a})^2} \\ (R+u) = \frac{d_q (f_U \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a}))}{\|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a})^2} \end{cases} \quad (22)$$

Consequently, we obtain the optimal portfolio of problem (14) as follows:

$$\begin{aligned} \mathbf{x}^* &= \frac{\bar{\mathbf{r}}'}{d_q} \left( \frac{d_q (f_U \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a}))}{\|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a})^2} \right) + \frac{\mathbf{a}}{d_q} \left( \frac{d_q (\|\bar{\mathbf{r}}'\|^2 - f_U (\bar{\mathbf{r}}'^t \mathbf{a}))}{\|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a})^2} \right) \\ \Leftrightarrow \mathbf{x}^* &= \frac{(f_U \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a}))}{\|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a})^2} \bar{\mathbf{r}}' + \frac{(\|\bar{\mathbf{r}}'\|^2 - f_U (\bar{\mathbf{r}}'^t \mathbf{a}))}{\|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a})^2} \mathbf{a} \\ \Leftrightarrow \Lambda^{1/2} \mathbf{Q} \phi^* &= \frac{\left( f_U \left( (\Lambda^{1/2})^{-1} \mathbf{Q} \bar{\mathbf{r}}' \right)^2 - (\bar{\mathbf{r}}'^t \mathbf{Q}' \Lambda^{-1} \mathbf{Q} \mathbf{1}) \right)}{\left\| (\Lambda^{1/2})^{-1} \mathbf{Q} \bar{\mathbf{r}}' \right\|^2 \left\| (\Lambda^{1/2})^{-1} \mathbf{Q} \mathbf{1} \right\|^2 - (\bar{\mathbf{r}}'^t \mathbf{Q}' \Lambda^{-1} \mathbf{Q} \mathbf{1})^2} \bar{\mathbf{r}}' \\ &\quad + \frac{\left( \left\| (\Lambda^{1/2})^{-1} \mathbf{Q} \bar{\mathbf{r}}' \right\|^2 - f_U (\bar{\mathbf{r}}'^t \mathbf{Q}' \Lambda^{-1} \mathbf{Q} \mathbf{1}) \right)}{\left\| (\Lambda^{1/2})^{-1} \mathbf{Q} \bar{\mathbf{r}}' \right\|^2 \left\| (\Lambda^{1/2})^{-1} \mathbf{Q} \mathbf{1} \right\|^2 - (\bar{\mathbf{r}}'^t \mathbf{Q}' \Lambda^{-1} \mathbf{Q} \mathbf{1})^2} \mathbf{a} \\ \Leftrightarrow \phi^* &= \frac{(f_U (\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}) - (\bar{\mathbf{r}}'^t \mathbf{V}^{-1} \mathbf{1})) \bar{\mathbf{r}}' + ((\bar{\mathbf{r}}'^t \mathbf{V}^{-1} \bar{\mathbf{r}}) - f_U (\bar{\mathbf{r}}'^t \mathbf{V}^{-1} \mathbf{1})) \mathbf{a}}{(\bar{\mathbf{r}}'^t \mathbf{V}^{-1} \mathbf{1})(\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}) - (\bar{\mathbf{r}}'^t \mathbf{V}^{-1} \mathbf{1})^2} \end{aligned} \quad (23)$$

From solution (23), by calculating only inverse matrix  $\mathbf{V}^{-1}$  of variance-covariance matrix  $\mathbf{V}$ , we obtain optimal portfolio  $\phi^*$ . Furthermore, we find that the optimal portfolio of problem  $\mathbf{P}(q)$  in the case  $f_U - \bar{\mathbf{r}}'^t \mathbf{x} = 0$  does not depend on parameter  $d_q$  to represent robustness.

Second, in the case of  $f_U - \bar{\mathbf{r}}'^t \mathbf{x} > 0$ , i.e.,  $u = 0$ , KKT condition (20) is degenerated as follows:

$$\begin{cases} \frac{\partial L}{\partial \mathbf{x}} = -R\bar{\mathbf{r}}' + d_q \mathbf{x} - v\mathbf{a} = 0 \\ \mathbf{a}' \mathbf{x} = 1 \end{cases} \quad (24)$$

Since we obtain  $\mathbf{x} = \frac{R\bar{\mathbf{r}}' + v\mathbf{a}}{d_q}$  from the first equation of (24),

we calculate the following solution for  $v$ :

$$\begin{aligned} \mathbf{a}' \mathbf{x} &= 1 \\ \Leftrightarrow \mathbf{a}' \left( \frac{R\bar{\mathbf{r}}' + v\mathbf{a}}{d_q} \right) &= 1 \\ \Leftrightarrow v &= \frac{d_q - R(\bar{\mathbf{r}}'^t \mathbf{a})}{\|\mathbf{a}\|^2} \end{aligned} \quad (25)$$

Therefore, we also calculate solution  $\mathbf{x}$  including parameter  $R$  as follows:

$$\begin{aligned} \mathbf{x} &= \frac{R}{d_q} \bar{\mathbf{r}}' + \frac{v}{d_q} \mathbf{a} = \frac{R}{d_q} \bar{\mathbf{r}}' + \frac{d_q - R(\bar{\mathbf{r}}'^t \mathbf{a})}{d_q \|\mathbf{a}\|^2} \mathbf{a} \\ \Leftrightarrow \mathbf{x}(R) &= \frac{R}{d_q} \left( \bar{\mathbf{r}}' - \frac{(\bar{\mathbf{r}}'^t \mathbf{a})}{\|\mathbf{a}\|^2} \mathbf{a} \right) + \frac{\mathbf{a}}{\|\mathbf{a}\|^2} \end{aligned} \quad (26)$$

By substituting  $\mathbf{x}(R)$  of (26) into optimal condition for

parameter  $R$ :  $R = \sqrt{\sum_{j=1}^n x_j(R)^2}$ , we solve the following equation:

$$\begin{aligned} R^2 &= \left( \frac{R}{d_q} \left( \bar{\mathbf{r}}' - \frac{(\bar{\mathbf{r}}'^t \mathbf{a})}{\|\mathbf{a}\|^2} \mathbf{a} \right) + \frac{\mathbf{a}}{\|\mathbf{a}\|^2} \right)' \left( \frac{R}{d_q} \left( \bar{\mathbf{r}}' - \frac{(\bar{\mathbf{r}}'^t \mathbf{a})}{\|\mathbf{a}\|^2} \mathbf{a} \right) + \frac{\mathbf{a}}{\|\mathbf{a}\|^2} \right) \\ R^2 &= \frac{R^2}{d_q^2} \left( \bar{\mathbf{r}}' - \frac{(\bar{\mathbf{r}}'^t \mathbf{a})}{\|\mathbf{a}\|^2} \mathbf{a} \right)' \left( \bar{\mathbf{r}}' - \frac{(\bar{\mathbf{r}}'^t \mathbf{a})}{\|\mathbf{a}\|^2} \mathbf{a} \right) \\ &\quad + \frac{2R}{d_q} \left( \bar{\mathbf{r}}' - \frac{(\bar{\mathbf{r}}'^t \mathbf{a})}{\|\mathbf{a}\|^2} \mathbf{a} \right)' \frac{\mathbf{a}}{\|\mathbf{a}\|^2} + \frac{1}{\|\mathbf{a}\|^2} \\ R^2 &= \frac{R^2}{d_q^2} \left( \|\bar{\mathbf{r}}'\|^2 - \frac{(\bar{\mathbf{r}}'^t \mathbf{a})^2}{\|\mathbf{a}\|^2} \right) + \frac{1}{\|\mathbf{a}\|^2} \\ d_q^2 \|\mathbf{a}\|^2 R^2 &= R^2 (\|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2 - (\bar{\mathbf{r}}'^t \mathbf{a})^2) + 1 \\ \left( d_q^2 \|\mathbf{a}\|^2 + (\bar{\mathbf{r}}'^t \mathbf{a})^2 - \|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2 \right) R^2 &= 1 \\ R^* &= \sqrt{\frac{1}{d_q^2 \|\mathbf{a}\|^2 + (\bar{\mathbf{r}}'^t \mathbf{a})^2 - \|\bar{\mathbf{r}}'\|^2 \|\mathbf{a}\|^2}} \quad (\because R \geq 0) \end{aligned} \quad (27)$$

Consequently, we obtain the exact optimal solution of  $\mathbf{x}$  as

$$\mathbf{x}^* = \frac{R^*}{d_q} \left( \bar{\mathbf{r}}' - \frac{(\bar{\mathbf{r}}'^t \mathbf{a})}{\|\mathbf{a}\|^2} \mathbf{a} \right) + \frac{\mathbf{a}}{\|\mathbf{a}\|^2},$$

and so the exact optimal solution of problem (14) is represented as the following form:

$$\begin{aligned}\phi^* &= \mathbf{V}^{-1} \left( \frac{R^*}{d_q} \left( \bar{\mathbf{r}} - \frac{(\bar{\mathbf{r}}' \mathbf{V}^{-1} \mathbf{1})}{\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}} \mathbf{1} \right) + \frac{1}{\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}} \mathbf{1} \right) \\ R^* &= \sqrt{\frac{1}{(d_q^2 - (\bar{\mathbf{r}}' \mathbf{V}^{-1} \bar{\mathbf{r}}))(\mathbf{1}' \mathbf{V}^{-1} \mathbf{1}) + (\bar{\mathbf{r}}' \mathbf{V}^{-1} \mathbf{1})}}\end{aligned}\quad (28)$$

From solution (28), by calculating only inverse matrix  $\mathbf{V}^{-1}$  of variance-covariance matrix  $\mathbf{V}$ , we obtain optimal parameter  $R^*$  and portfolio  $\phi^*$ . Furthermore, we find that the optimal portfolio of problem  $\mathbf{P}(q)$  in the case  $f_U - \bar{\mathbf{r}}' \mathbf{x} > 0$  does not depend on parameter  $f_U$  to represent the standard target profit.

## V. CONCLUSION

In this paper, we have proposed a robust portfolio selection problem considering the confidential interval to the total profit with investor's subjectivity. Since the proposed model is a bi-objective programming problem with fuzziness, we have introduced fuzzy goals for both objective functions, and transformed it into the deterministic equivalent problems. Furthermore, in order to obtain the explicit optimal portfolio, we have developed the exact solution algorithm based on the Dinkelbach's algorithm and auxiliary quadratic programming problem without loss of optimality. Our proposed model includes some existing robust portfolio selection problems, and specific random distributions are not assumed in the model. Therefore, we may apply our models to the more flexible and complex portfolio selection problems in real investment markets than the previous models.

As future studies, we need to consider the application of our proposed model and solution algorithm to large-scale or multi-period portfolio selection problems. Then, we are now attacking the sensitivity analysis for parameters of the proposed robust portfolio model.

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