

Adaptive Fuzzy Sliding-Mode Control for a Class of Nonlinear Systems with Uncertainties

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Abstract—An approach to adaptive fuzzy sliding-mode control (SMC) for a class of nonlinear dynamical systems with lumped parameter uncertainties accompanied with control input is proposed and analyzed. The approach guarantees that the system state will be uniformly ultimately bounded. Also, discussion on how to alleviate the conservativeness of the approach is provided. Finally, computer simulations illustrate the effectiveness of the control approach proposed in this paper.

Keywords - adaptive fuzzy control; T-S fuzzy model; sliding-mode control; Uncertainties

I. INTRODUCTION

In recent years, there have been significant advances in the study of the stability analysis and controller synthesis for the so-called Takagi-Sugeno (T-S) fuzzy systems, which have been used to represent certain complex nonlinear systems. The overall model of the system is obtained by the fuzzy blending of these local models. The control design is carried out based on the fuzzy model by the so-called parallel distributed compensation (PDC) scheme [1]. For each local linear model, a linear feedback control is designed. The resulting overall controller is again a fuzzy blending of the individual linear controllers. Originally, Tanaka and his colleagues have provided certain conditions that are sufficient for the stability of the T-S fuzzy systems in the sense of Lyapunov [2]. The feedback control gain in each PDC control rule is obtained by solving certain linear matrix inequalities (LMIs).

Although a nonlinear system can be handled by a PDC controller based on the T-S fuzzy model, the control performance depends on the precision of the T-S fuzzy model. That is, if the model uncertainties between the system to be controlled and the T-S fuzzy model are significantly large, then the control performance is no longer to be as good as we expected. To compensate for the uncertainties and/or disturbances, several system design and analysis approaches have been proposed. Among the existing robust control approaches, fuzzy sliding-mode control (SMC) has been recognized as an effective robust control approach for uncertain systems. SMC is a particular type of variable structure control. Variable structure control systems are designed to derive and then constrain the system state to lie within a neighborhood of the sliding surface. There are many advantages to this approach [8]. In particular, the closed-loop response becomes totally insensitive to a particular class of uncertainties. This invariance property clearly makes

the methodology an appropriate candidate for robust control. In addition, the ability to specify performance directly make SMC attractive from the design perspective. The result of Zheng *et al.* [3] is an elaborate example to illustrate that the SMC theory can be successfully applied to robust stabilization of uncertain nonlinear system with T-S fuzzy model, where uncertainties include parameter perturbation and external disturbances for which their upper-bounds are available. The ordinary T-S fuzzy model has been further extended to time-delay systems with uncertainties based on the SCM theory [4], [5]. More recently, by introducing some coordinate transformations to design a reduced-order sliding-mode dynamics, some excellent results on fuzzy sliding-mode control for nonlinear systems with uncertainties, particularly a lumped uncertainties accompanied with control input were reported [6], [7]. However, it should be pointed out that there are some limitations to the existing results. The main drawback comes to the assumption that requires the same control matrix, *i.e.*, $B_1 = B_2 = \dots = B_r$ in the T-S fuzzy model with r rules; however, in practical applications that are a lot of cases where the control matrices are different in the T-S fuzzy model. Second, though all results showed the system state would be uniformly ultimately bounded when using some adaptive laws to deal with unknown parameters in the T-S fuzzy model, the boundedness was not mentioned further, which implies that such results are conservative.

Motivated by the aforementioned observations, in this paper we propose an adaptive SMC approach based on the T-S fuzzy model with uncertainties, trying to remove some strict assumptions on the system and alleviate the conservatism of the control results. Also, computer simulations are provided to illustrate the effectiveness of the proposed control approach.

II. PROBLEM DESCRIPTION

It is now well known that T-S fuzzy models can provide an effective representation of complex nonlinear systems. Therefore, in this study, the nonlinear system under consideration is represented by T-S fuzzy models. Consider the following T-S fuzzy model:

Plant Rule i :

IF $\theta_1(t)$ is M_1^i and $\dots$ and $\theta_p(t)$ is M_p^i ,

THEN $\dot{x}(t) = A_i x(t) + B_i(x(t)) (u(t) + \mathcal{U}_i(x(t)))$,

where $\theta_j^i (i = 1, 2, \dots, r, j = 1, 2, \dots, p)$ are variables in the antecedent; M_j^i , fuzzy sets with membership functions $\mu_{M_j^i}(\theta_j)$; $x(t) \in R^n$, system state; $u(t) \in R^m$, control input; $A_i \in R^{n \times n}$ and $B_i(x(t)) \in R^{n \times m}$, system and input matrices; $\mathcal{U}_i(x(t)) \in R^m$, lumped uncertainties including external disturbance. In this paper, we suppose that $\text{rank}(B) = m < n$. The overall fuzzy model is of the following form accordingly:

$$\dot{x}(t) = \sum_{i=1}^r \alpha_i(t) \{A_i x(t) + B_i(x(t)) (u(t) + \mathcal{U}_i(x(t)))\}, \quad (1)$$

where

$$\alpha_i(t) = \frac{w_i(t)}{\sum_{i=1}^r w_i(t)} \geq 0, \quad \sum_{i=1}^r \alpha_i(t) = 1, \\ w_i(t) = \prod_{j=1}^p \mu_j^i(\theta_j(t)).$$

For the T-S fuzzy model above, we make the following assumptions.

- A1: $B_i(x(t)) = B \cdot G_i(x(t))$, where $B \in R^{n \times m}$, $\text{rank}(B) = m < n$, $G_i(x(t)) \in R^{m \times m}$ is a known matrix function such that $\det G_i(x(t)) \neq 0$, and of known sign; without loss of generality this sign can be taken as positive. Also, any (A_i, B) is a controllable pair.
- A2: The lumped uncertainties $\mathcal{U}_i(x(t))$ are assumed to satisfy the following conditions:

$$\|\mathcal{U}_i(x(t))\| \leq c_i^* + k_i^* \|x(t)\|, \quad (2)$$

where c_i^*, k_i^* are unknown positive constants, $\|\cdot\|$ denotes the Euclidean norm of a vector or a matrix.

It is noted that the T-S fuzzy model considered in this paper is significantly different from the models in this scope (see [6]-[7], for example), in which they took $B_i(x(t)) = B$ for $i = 1, 2, \dots, r$. This implies that there is the same gain for control input $u(t)$ in each local model. Consequently, the overall fuzzy model (1) becomes

$$\dot{x}(t) = \sum_{i=1}^r \alpha_i(t) \{A_i x(t) + B \mathcal{U}_i(x(t))\} + B u(t). \quad (3)$$

Although (3) has the benefit that the corresponding controllers are easier to be designed, apparently, its applications are severely restricted. Therefore, one of challenges in this paper is to design an adaptive fuzzy controller based on the T-S fuzzy model (1) towards more flexibility.

From here, unless confusion arises arguments such as t will be omitted just for notational convenience.

Letting $T_r \in R^{n \times n}$ be an orthogonal matrix such that

$$T_r A_i T_r^T = \begin{bmatrix} A_{11i} & A_{12i} \\ A_{21i} & A_{22i} \end{bmatrix} \triangleq \bar{A}_i, \quad T_r B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad (4)$$

where $A_{11i} \in R^{(n-m) \times (n-m)}$, $A_{12i} \in R^{(n-m) \times m}$, $A_{21i} \in R^{m \times (n-m)}$, $A_{22i} \in R^{m \times m}$, $B_2 \in R^{m \times m}$, and $z = T_r x$, then by pre-multiplying T_r on the both sides of (1), one obtains

the following regular form.

$$\dot{z} = \sum_{i=1}^r \alpha_i \left(\bar{A}_i z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} G_i(u + \mathcal{U}_i) \right) \\ = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} z + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \sum_{i=1}^r \alpha_i G_i(u + \mathcal{U}_i), \quad (5)$$

where

$$\bar{A}_{11} = \sum_{i=1}^r \alpha_i A_{11i}, \quad \bar{A}_{12} = \sum_{i=1}^r \alpha_i A_{12i}, \quad (6)$$

$$\bar{A}_{21} = \sum_{i=1}^r \alpha_i A_{21i}, \quad \bar{A}_{22} = \sum_{i=1}^r \alpha_i A_{22i}, \quad (7)$$

and $\mathcal{U}_i$ is a function with respect to $T_r^{-1}z$ at this point in time. Letting $z = [z_1^T \ z_2^T]^T$, where $z_1 \in R^{n-m}$, $z_2 \in R^m$, (5) can be transformed into

$$\dot{z}_1 = \bar{A}_{11} z_1 + \bar{A}_{12} z_2, \quad (8)$$

$$\dot{z}_2 = \bar{A}_{21} z_1 + \bar{A}_{22} z_2 + B_2 \sum_{i=1}^r \alpha_i G_i(u + \mathcal{U}_i). \quad (9)$$

This transformation can be accomplished for any given controllable pair (A_i, B) , where B is full rank using some MATLAB commands [8].

III. CONTROL SYSTEM DESIGN

SMC system design can be decoupled into two independent tasks. The first one is the selection of an appropriate sliding surface, which should have the property that the desired performance can be achieved while the system state remains on the sliding surface. Another task is to design a control law in order to force the system state to reach the sliding surface in spite of uncertainties in the system.

A. Sliding Surface Design

The corresponding sliding surface is chosen as follows.

$$s(t) = \bar{C} z(t) = 0 \quad (10)$$

where $\bar{C} \triangleq [-C_1 \ C_2]$ with $C_1 \in R^{m \times (n-m)}$, $C_2 = C_2^T \in R^{m \times m}$, which is positive-definite, are parameters to be designed. It follows from (10) that $z_2 = C_2^{-1} C_1 z_1$, and from (8) that

$$\dot{z}_1 = (\bar{A}_{11} + \bar{A}_{12} C_2^{-1} C_1) z_1, \quad (11)$$

which is a reduced-order $(n - m)$ sliding mode dynamics. Thus, the result of designing sliding surface can be provided as follows.

Theorem 1: Consider system (11). If there exists matrices $Q = Q^T \in R^{(n-m) \times (n-m)}$, $C_2 = C_2^T \in R^{m \times m}$, which are positive-definite, $M \in R^{m \times (n-m)}$, such that the following LMIs satisfied:

$$\begin{bmatrix} A_{11i}Q + QA_{11i}^T & M^T & A_{12i} \\ M & -C_2 & 0 \\ A_{12i}^T & 0 & -C_2 \end{bmatrix} < 0, \quad i = 1, 2, \dots, r. \quad (12)$$

then the sliding surface (10) with

$$C_1 = MQ^{-1} \quad (13)$$

will guarantee that the sliding motion in (11) is asymptotically stable.

Proof: Let us define a Lyapunov function candidate as

$$V(t) = z_1(t)^T Q^{-1} z_1(t). \quad (14)$$

Then, taking the time derivative along the state trajectory of system (11), one obtains

$$\begin{aligned} \dot{V}(t) = & z_1^T \left(\bar{A}_{11}^T Q^{-1} + (C_2^{-1} C_1)^T \bar{A}_{12}^T Q^{-1} \right. \\ & \left. + Q^{-1} \bar{A}_{11} + Q^{-1} \bar{A}_{12} C_2^{-1} C_1 \right) z_1. \end{aligned} \quad (15)$$

Obviously, it follows from (15) that the sliding motion (11) is asymptotically stable, if

$$\begin{aligned} & \bar{A}_{11}^T Q^{-1} + C_1^T C_2^{-1} \bar{A}_{12}^T Q^{-1} \\ & + Q^{-1} \bar{A}_{11} + Q^{-1} \bar{A}_{12} C_2^{-1} C_1 < 0. \end{aligned} \quad (16)$$

By pre- and post-multiplying (16) by Q , it follows that

$$Q \bar{A}_{11}^T + M^T C_2^{-1} \bar{A}_{12}^T + \bar{A}_{11} Q + \bar{A}_{12} C_2^{-1} M < 0, \quad (17)$$

where the fact that $M = C_1 Q$ is used. Due to the facts that $C_2^{-1} > 0$ and

$$(M - A_{12}^T)^T C_2^{-1} (M - A_{12}^T) \geq 0,$$

which leads to

$$M^T C_2^{-1} \bar{A}_{12}^T + \bar{A}_{12} C_2^{-1} M \leq M^T C_2^{-1} M + \bar{A}_{12} C_2^{-1} \bar{A}_{12}^T,$$

therefore, the condition for the sliding motion being asymptotically stable (17) can be replaced by the following inequality.

$$Q \bar{A}_{11}^T + \bar{A}_{11} Q + M^T C_2^{-1} M + \bar{A}_{12} C_2^{-1} \bar{A}_{12}^T < 0. \quad (18)$$

By applying the Schur complement formula to (18) and using (6), (7), it leads to LMIs (12). After all, we can conclude that the $(n - m)$ reduced-order sliding mode dynamics (11) is asymptotically stable. This completes the proof. $\square$

Although LMIs (12) can guarantee that the sliding motion (11) is asymptotically stable, *i.e.*,

$$\bar{A}_{11} + \bar{A}_{12} C_2^{-1} C_1 < 0, \quad (19)$$

it is not taken into account how to make the state converge to 0, which is the equilibrium point in this paper, as fast as possible.

In what follows, let us consider the matter. From (11) we have

$$z_2(t) = \exp(A_{11} + \bar{A}_{12} C_2^{-1} C_1) z_2(0),$$

implying that we should find the minimum value of $A_{11} + \bar{A}_{12} C_2^{-1} C_1$ so that z_2 shrinks to 0 most rapidly. Substituting (13) into (19), it follows that

$$\bar{A}_{11} Q + \bar{A}_{12} C_2^{-1} M < 0, \quad (20)$$

which implies that the minimum values of the left side of inequality (17), then LMIs (12) lead to the minimum one of $\bar{A}_{11} + \bar{A}_{12} C_2^{-1} C_1$.

Let $R_i = R_i^T > 0$ be any given matrices, then, the minimum

values of the left sides of LMIs (12) can be found by solving the following LMIs, searching the minimum value of γ .

$$\begin{bmatrix} A_{11i} Q + Q A_{11i}^T & M^T & A_{12i} \\ M & -C_2 & 0 \\ A_{12i}^T & 0 & -C_2 \end{bmatrix} < \gamma R_i, \quad (21)$$

$i = 1, 2, \dots, r.$

Since R_i are fixed, the left sides in (21) become smaller in order to get a smaller γ . Eventually, the minimum values of the left sides can be achieved. Summarizing above, we obtain the following corollary.

Corollary 1: Consider system (11). If there exists matrices $Q = Q^T \in R^{(n-m) \times (n-m)}$, which is positive-definite, $M \in R^{m \times (n-m)}$, $C_2 = C_2^T \in R^{m \times m}$, such that the following LMIs satisfied:

$$\begin{aligned} & \min \gamma \text{ subject to} \\ & \begin{cases} \text{LMIs (12)} \\ \text{LMIs (21)} \end{cases} \end{aligned} \quad (22)$$

where $R_i = R_i^T > 0$ are some given matrices with appropriate dimensions, then the sliding surface (10) with (13) will guarantee that the sliding motion in (11) is asymptotically stable with the maximum convergence rate.

LMIs (22) can be solved by using existing software packages such as mincx solver in the LMI Control Toolbox with MATLAB.

Regarding the LMIs in Theorem 1, we give the following remark.

Remark 1: Since $M^T C_2^{-1} M + \bar{A}_{12} C_2^{-1} \bar{A}_{12}^T > 0$, the condition for the sliding motion being asymptotically stable (17), also, can be transferred into

$$Q \bar{A}_{11}^T + \bar{A}_{11} Q + (M + \bar{A}_{12}^T)^T C_2^{-1} (M + \bar{A}_{12}^T) < 0,$$

which leads to the following LMIs by using the Schur complement formula.

$$\begin{bmatrix} A_{11i} Q + Q A_{11i}^T & M^T + A_{12i} \\ M + A_{12i}^T & -C_2 \end{bmatrix} < 0. \quad (23)$$

That is, LMIs (23) may be another solutions if LMIs (12) are infeasible, though it is difficult to say which of LMIs (12) and (23) is more feasible.

Remark 2: In case neither LMIs (12) nor (23) is feasible, it is not longer to take C_2 as a matrix variable in the LMIs, *i.e.*, we have to provide a prescribed matrix, for example an identity matrix, so that (17) turns out to be an LMI. In this case, therefore, either LMIs (12) or (23) are replaced by

$$\begin{aligned} & A_{11i} Q + Q A_{11i}^T + M^T C_2^{-1} A_{12i}^T + A_{12i} C_2^{-1} M < 0, \\ & i = 1, 2, \dots, r. \end{aligned} \quad (24)$$

B. Control Law

After designing the sliding surface so that the reduced-order sliding mode dynamics has the desired response, the next step of the SMC design procedure is to design a switching feedback control law such that the reachability of the specified sliding surface (10) is ensured.

Theorem 2: Consider the T-S fuzzy model (1) subject to the assumptions A1 and A2 provided in section II. Let us suppose that the LMIs (12), which are associated with the T-S fuzzy model by (4), are feasible and that the sliding surface is given by (10), where C_1 is given by (13) and C_2 is a solution of the LMIs. Then, the all signals involved in the closed-loop system (1) with the following control law are uniformly ultimately bounded.

$$u(t) = -\sum_{i=1}^r \alpha_i \left\{ (C_2 B_2)^{-1} \psi_i(t) \bar{C} \bar{A}_i z(t) + \phi_i(t) \right\} - \frac{(C_2 B_2)^{-1}}{\lambda_{\min}} \mu s(t), \quad (25)$$

where $\mu > 0$, $\lambda_{\min} = \min\{\lambda(G_1), \lambda(G_2), \dots, \lambda(G_r)\}$,

$$\psi_i(t) = \begin{cases} \frac{1}{\lambda_{\min}}, & s(t)^T \bar{C} \bar{A}_i z(t) \geq 0; \\ \frac{1}{\lambda_{\max}}, & \text{otherwise,} \end{cases} \quad (26)$$

$$\phi_i(t) = \begin{cases} \frac{\text{sgn}(B_2^T C_2^T s(t))}{\lambda_{\min}} \hat{\rho}_i(t) \|G_i\|, & \hat{\rho}_i(t) \geq 0; \\ \frac{\text{sgn}(B_2^T C_2^T s(t))}{\lambda_{\max}} \hat{\rho}_i(t) \|G_i\|, & \text{otherwise,} \end{cases} \quad (27)$$

$$\hat{\rho}_i(t) = \hat{c}_i(t) + \hat{k}_i(t) \|z(t)\|, \quad (28)$$

$\lambda_{\max} = \max\{\lambda(G_1), \lambda(G_2), \dots, \lambda(G_r)\}$, and adaptive laws are

$$\dot{\hat{c}}_i(t) = \alpha_i \gamma_{1i} \left(-\epsilon_1 \hat{c}_i(t) + \|G_i\| \|B_2^T C_2^T s(t)\| \right), \quad (29)$$

$$\dot{\hat{k}}_i(t) = \alpha_i \gamma_{2i} \left(-\epsilon_2 \hat{k}_i(t) + \|G_i\| \|B_2^T C_2^T s(t)\| \|z(t)\| \right), \quad (30)$$

where $\gamma_{1i}, \gamma_{2i}, \epsilon_1$, and ϵ_2 are positive design parameters.

Proof: From (5) and (10), we have

$$\dot{s} = \sum_{i=1}^r \alpha_i \left(\bar{C} \bar{A}_i z + C_2 B_2 G_i (u + \mathcal{U}_i) \right). \quad (31)$$

Let us define a Lyapunov function candidate as follows.

$$V(t) = \frac{1}{2} s(t)^T s(t) + \frac{1}{2} \sum_{i=1}^r \left\{ \gamma_{1i}^{-1} \tilde{c}_i^2(t) + \gamma_{2i}^{-1} \tilde{k}_i^2(t) \right\}, \quad (32)$$

where $\tilde{c}_i(t) = c_i^* - \hat{c}_i(t)$, $\tilde{k}_i(t) = k_i^* - \hat{k}_i(t)$.

Using (31), the time derivative of V on any trajectory of the

closed-loop system is

$$\begin{aligned} \dot{V} &= \sum_{i=1}^r \alpha_i \left\{ s^T \bar{C} \bar{A}_i z + s^T C_2 B_2 G_i \mathcal{U}_i \right. \\ &\quad \left. - s^T C_2 B_2 G_i \frac{(C_2 B_2)^{-1}}{\lambda_{\min}} \mu s \right. \\ &\quad \left. - s^T C_2 B_2 G_i \sum_{j=1}^r \alpha_j \left((C_2 B_2)^{-1} \psi_j \bar{C} \bar{A}_j z + \phi_j \right) \right\} \\ &\quad - \sum_{i=1}^r \gamma_{1i}^{-1} \tilde{c}_i \dot{\tilde{c}}_i - \sum_{i=1}^r \gamma_{2i}^{-1} \tilde{k}_i \dot{\tilde{k}}_i \\ &= \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \left\{ s^T \bar{C} \bar{A}_i z + s^T C_2 B_2 G_i \mathcal{U}_i \right. \\ &\quad \left. - s^T C_2 B_2 G_i \frac{(C_2 B_2)^{-1}}{\lambda_{\min}} \mu s \right. \\ &\quad \left. \underbrace{- s^T C_2 B_2 G_i (C_2 B_2)^{-1} \psi_j \bar{C} \bar{A}_j z}_{(I)} \right. \\ &\quad \left. \underbrace{- s^T C_2 B_2 G_i \phi_j}_{(II)} \right\} \\ &\quad - \sum_{i=1}^r \gamma_{1i}^{-1} \tilde{c}_i \dot{\tilde{c}}_i - \sum_{i=1}^r \gamma_{2i}^{-1} \tilde{k}_i \dot{\tilde{k}}_i. \end{aligned} \quad (33)$$

Let us consider (I) – (III) in (33) as follows.

$$\begin{aligned} (I) &\leq -s^T C_2 B_2 \lambda_{\min} \frac{(C_2 B_2)^{-1}}{\lambda_{\min}} \mu s \\ &= -\mu \|s\|, \end{aligned} \quad (34)$$

where $\lambda_{\min} > 0$ due to the assumption A1. As to (II), if $s^T \bar{C} \bar{A}_j z \geq 0$, we have

$$\begin{aligned} (II) &= -s^T C_2 B_2 G_i (C_2 B_2)^{-1} \frac{1}{\lambda_{\min}} \bar{C} \bar{A}_j z \\ &\leq -s^T C_2 B_2 \lambda_{\min} (C_2 B_2)^{-1} \frac{1}{\lambda_{\min}} \bar{C} \bar{A}_j z \\ &= -s^T \bar{C} \bar{A}_j z; \end{aligned} \quad (35)$$

otherwise,

$$\begin{aligned} (II) &= -s^T C_2 B_2 G_i (C_2 B_2)^{-1} \frac{1}{\lambda_{\max}} \bar{C} \bar{A}_j z \\ &\leq -s^T C_2 B_2 \lambda_{\max} (C_2 B_2)^{-1} \frac{1}{\lambda_{\max}} \bar{C} \bar{A}_j z \\ &= -s^T \bar{C} \bar{A}_j z. \end{aligned} \quad (36)$$

That is,

$$(II) \leq -s^T \bar{C} \bar{A}_j z \quad (37)$$

holds in any case by virtue of switching function (26). Similarly, it follows from (27) that

$$\begin{aligned} (III) &\leq -\|B_2^T C_2^T s\|_1 \hat{\rho}_j \|G_j\| \\ &\leq -\|B_2^T C_2^T s\| \hat{\rho}_j \|G_j\|, \end{aligned} \quad (38)$$

where the fact that $\|x\| \leq \|x\|_1$ with x being a vector is used. Substituting (34), (37) and (38) into (33) it follows that

$$\begin{aligned}
\dot{V} &\leq \sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j \left\{ s^T \bar{C} \bar{A}_i z + s^T C_2 B_2 G_i \mathcal{U}_i \right. \\
&\quad \left. - \mu \|s\| - s^T \bar{C} \bar{A}_j z - \|B_2^T C_2^T s\| \hat{\rho}_j \|G_j\| \right\} \\
&\quad - \sum_{i=1}^r \gamma_{1i}^{-1} \tilde{c}_i \dot{\hat{c}}_i - \sum_{i=1}^r \gamma_{2i}^{-1} \tilde{k}_i \dot{\hat{k}}_i \\
&= -\mu \|s\| + \sum_{i=1}^r \alpha_i \left\{ s^T \bar{C} \bar{A}_i z + s^T C_2 B_2 G_i \mathcal{U}_i \right. \\
&\quad \left. - s^T \bar{C} \bar{A}_i z - \|B_2^T C_2^T s\| \hat{\rho}_i \|G_i\| \right\} \\
&\quad - \sum_{i=1}^r \gamma_{1i}^{-1} \tilde{c}_i \dot{\hat{c}}_i - \sum_{i=1}^r \gamma_{2i}^{-1} \tilde{k}_i \dot{\hat{k}}_i \\
&\leq -\mu \|s\| + \sum_{i=1}^r \alpha_i \left\{ \|B_2^T C_2^T s\| \|G_i\| \|\mathcal{U}_i\| \right. \\
&\quad \left. - \|B_2^T C_2^T s\| \hat{\rho}_i \|G_i\| \right\} \\
&\quad - \sum_{i=1}^r \gamma_{1i}^{-1} \tilde{c}_i \dot{\hat{c}}_i - \sum_{i=1}^r \gamma_{2i}^{-1} \tilde{k}_i \dot{\hat{k}}_i. \quad (39)
\end{aligned}$$

Further, substituting (2), (28), and adaptive laws (29), (30) into (39), we obtain

$$\begin{aligned}
\dot{V} &\leq -\mu \|s\| \\
&\quad + \sum_{i=1}^r \alpha_i \left\{ \|B_2^T C_2^T s\| \|G_i\| (c_i^* + k_i^* \|z\|) \right. \\
&\quad \left. - \|B_2^T C_2^T s\| \|G_i\| (\hat{c}_i + \hat{k}_i \|z\|) \right\} \\
&\quad + \sum_{i=1}^r \alpha_i (\epsilon_1 \tilde{c}_i \hat{c}_i - \|G_i\| \|B_2^T C_2^T s\| \tilde{c}_i) \\
&\quad + \sum_{i=1}^r \alpha_i (\epsilon_2 \tilde{k}_i \hat{k}_i - \|G_i\| \|B_2^T C_2^T s\| \tilde{k}_i \|z\|) \\
&= -\mu \|s\| + \sum_{i=1}^r \alpha_i (\epsilon_1 \tilde{c}_i \hat{c}_i + \epsilon_2 \tilde{k}_i \hat{k}_i). \quad (40)
\end{aligned}$$

Since the facts that

$$\begin{aligned}
\epsilon_1 \tilde{c}_i \hat{c}_i &= \frac{\epsilon_1}{2} (2\tilde{c}_i c_i^* - 2\tilde{c}_i^2) \\
&= \frac{\epsilon_1}{2} (-\tilde{c}_i^2 - (\tilde{c}_i - c_i^*)^2 + c_i^{*2}) \\
&\leq -\frac{1}{2} \epsilon_1 \tilde{c}_i^2 + \frac{1}{2} \epsilon_1 c_i^{*2}, \quad (41)
\end{aligned}$$

$$\epsilon_2 \tilde{k}_i \hat{k}_i \leq -\frac{1}{2} \epsilon_2 \tilde{k}_i^2 + \frac{1}{2} \epsilon_2 k_i^{*2} \quad (42)$$

hold, it yields that

$$\begin{aligned}
\dot{V} &\leq -\mu \|s\| - \frac{1}{2} \sum_{i=1}^r \alpha_i (\epsilon_1 \tilde{c}_i^2 + \epsilon_2 \tilde{k}_i^2) + \epsilon \\
&= -\mu \|s\| + \epsilon \\
&\quad - \frac{1}{2} \sum_{i=1}^r \alpha_i (\epsilon_1 \gamma_{1i}^{-1} \gamma_{1i}^{-1} \tilde{c}_i^2 + \epsilon_2 \gamma_{2i}^{-1} \gamma_{2i}^{-1} \tilde{k}_i^2), \quad (43)
\end{aligned}$$

where

$$\epsilon \leq \frac{1}{2} \sum_{i=1}^n (\epsilon_1 c_i^{*2} + \epsilon_2 k_i^{*2}). \quad (44)$$

Let

$$\eta_1 = \min\{\alpha_1 \epsilon_1 \gamma_{11}, \alpha_2 \epsilon_1 \gamma_{12}, \dots, \alpha_r \epsilon_1 \gamma_{1r}\} \quad (45)$$

$$\eta_2 = \min\{\alpha_1 \epsilon_2 \gamma_{21}, \alpha_2 \epsilon_2 \gamma_{22}, \dots, \alpha_r \epsilon_2 \gamma_{2r}\}, \quad (46)$$

then,

$$\begin{aligned}
\dot{V} &\leq -\mu s^T s - \frac{1}{2} \sum_{i=1}^r (\eta_1 \gamma_{1i}^{-1} \tilde{c}_i^2 + \eta_2 \gamma_{2i}^{-1} \tilde{k}_i^2) + \epsilon \\
&\leq -\eta \left\{ \frac{1}{2} s^T s + \frac{1}{2} \sum_{i=1}^r (\gamma_{1i}^{-1} \tilde{c}_i^2 + \gamma_{2i}^{-1} \tilde{k}_i^2) \right\} + \epsilon \\
&= -\eta V + \epsilon, \quad (47)
\end{aligned}$$

where

$$\eta = \min\{2\mu, \eta_1, \eta_2\}. \quad (48)$$

From (47) we have

$$V \leq \frac{\epsilon}{\eta} + \left(V(0) - \frac{\epsilon}{\eta} \right) e^{-\eta t}, \quad (49)$$

where $V(0) = V(z(t))|_{t=0}$. Since that $\eta > 0$, the second block on the right side of (49) tends to 0 when $t \rightarrow \infty$. Therefore, we can get that there exists $\mathcal{T}$ such that for $t \geq \mathcal{T}$ it satisfies

$$\|s\| \leq \tau \quad (50)$$

where

$$\tau = \sqrt{\frac{\epsilon}{\eta}}. \quad (51)$$

This completes the proof. $\square$

Remark 3: In the control law, there are two switching functions, ψ_i and ϕ_i , in which ϕ_i contains a sign function. Such a sign function usually leads to control chattering, which is undesirable in practice because it involves high control activity and further may excite unmodeled high frequency plant dynamics. Fortunately, switching function ψ_i is not around the sliding surface, $s = 0$. This implies that when the system state enters the region $\|s\| \leq \tau$, the chattering phenomenon can be avoided most likely, which can be confirmed in the simulation shown later on. In case the chattering phenomenon occurs, it can be eliminated by replacing all sign functions with the so-called saturation functions [9].

C. Relaxing Conservatism

In theorem 2, it is desirable to have the largest η and the smallest τ so that the sliding mode s can shrink most quickly in a ball centered at the sliding surface $s = 0$ with radius τ . Therefore, the results obtained above are basically conservative since it took no account of the matter.

On the other hand, to our best knowledge, there is not an analytical approach to solve the optimization problem with respect

to η and τ . Therefore, the only way left to us is that we do our best to increase η and reduce τ . From (45), (46) and (48), we observe that a bigger $\mu, \epsilon_j, \gamma_{ji}$ ($j = 1, 2, i = 1, 2, \dots, r$), can help make η larger. At the same time, from (44) and (51) we know that a bigger ϵ_j leads to a bigger ε , further to a bigger τ . That is, there is a trade-off between making τ small and η big in terms of ϵ_j . Therefore, in order to make η big, we can do something on μ, ϵ_j , and γ_{ji} as follows.

- Choosing bigger μ, γ_{ji} . In the adaptive laws (29), (30) since γ_{ji} are free design parameters, basically they can be chosen arbitrarily large. However, there is a limit that large γ_{ji} can lead to big η_j , further to a big η due to the minimum in function in (48).
- Choosing bigger ϵ_j . As γ_{ji} , they are free design parameters in (29), (30) as well and can be chosen arbitrarily large. However, large ϵ_j will lead to a large τ at the same time. That is, we have to make a compromise between making η big and τ small in terms of ϵ_j .

Next, let us consider the problem of making τ small. As mentioned previously, since choosing small ϵ_j to make τ small is in conflict with increasing η , therefore, there is a limit on doing something about ϵ_j .

Again, let us focus on (44). Instead of (29), (30), if the adaptive laws are synthesized by

$$\dot{\hat{c}}_i(t) = \alpha_i \gamma_{1i} \left\{ -\epsilon_1 (\hat{c}_i(t) - c_{0i}) + \|G_i\| \|B_2^T C_2^T s\| \right\}, \quad (52)$$

$$\dot{\hat{k}}_i(t) = \alpha_i \gamma_{2i} \left\{ -\epsilon_2 (\hat{k}_i(t) - k_{0i}) + \|G_i\| \|B_2^T C_2^T s\| \|z\| \right\} \quad (53)$$

where γ_{ji} , and ϵ_{ji} are positive design parameters, c_{0i}, k_{0i} are a priori estimates of c_i^*, k_i^* , respectively, then (44) becomes

$$\varepsilon \leq \frac{1}{2} \sum_{i=1}^n \left\{ \epsilon_1 (c_i^* - c_{0i})^2 + \epsilon_2 (k_i^* - k_{0i})^2 \right\}. \quad (54)$$

From (54), we can see that the better a priori estimates for c_i^* and k_i^* , the smaller ε which further leads to a smaller τ in (51) we can get. Of course, if we do not have such knowledge about c_i^* and k_i^* , we can just put $c_{0i} = 0, k_{0i} = 0$.

IV. SIMULATION

Consider the well-studied example of balancing an inverted pendulum on a cart. The equations of motion for the pendulum are

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = \frac{g \sin(x_1) - \frac{amlx_2^2 \sin(2x_1)}{2} - a \cos(x_1)(u + \mathcal{U})}{\frac{4l}{3} - aml \cos^2(x_1)}, \end{cases} \quad (55)$$

where x_1 denotes the angle of the pendulum from the vertical; x_2 , the angular velocity; u , the force applied to the cart; $g = 9.8m/s^2$, the gravity constant; m , the mass of the pendulum; M , the mass of the cart (in kilo-Newton); $\mathcal{U}$, the lumped uncertainty, which is assumed to be $\mathcal{U} = 0.5 \cos(2\pi) + 1.5 \cos(4\pi)|x|$; $a = \frac{1}{m+M}$. We choose $m = 2.0$ kg, $M = 8.0$ kg, $l = 0.5$ m in this simulation.

The control objective is to balance the inverted pendulum.

Here, Referring to [11] we approximate the system by the following two-rule T-S fuzzy model:

Plant Rule 1 : IF x_1 is $\tilde{0}$

THEN $\dot{x} = A_1 x + B_1 (u + \mathcal{U}_1)$;

Plant Rule 2 : IF x_1 is $\pm \pi/2$

THEN $\dot{x} = A_2 x + B_2 (u + \mathcal{U}_2)$,

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{\frac{4l}{3} - aml} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \frac{-a}{\frac{4l}{3} - aml} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(\frac{4l}{3} - aml\beta^2)} & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{-a\beta}{\frac{4l}{3} - aml\beta^2} \end{bmatrix},$$

$\beta = \cos(88^\circ)$; $\tilde{0}, \pi/2$ are fuzzy sets characterized by the following membership functions:

$$\mu_{\tilde{0}}(x_1) = \begin{cases} \cos(x_1), & -\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\pi/2}(x_1) = \begin{cases} 1 - \cos(x_1), & -\frac{\pi}{2} \leq x_1 \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases};$$

$\mathcal{U}_i$ ($i = 1, 2$) is corresponding with the lumped uncertainty in the plant (55), and satisfies A2 in section II with $c_i^* = 0.5, k_i^* = 1.5$. Regarding their estimates, $\hat{c}_i, \hat{k}_i$, are tuned by adaptive laws (29), and (30) with $\gamma_{1i} = \gamma_{2i} = 0.1, \epsilon_{1i} = \epsilon_{2i} = 0.1$, respectively.

As to the assumption A1, we set $B^T = [0, -1]$ and $G_1 = 0.1765, G_2 = 0.0052$ due to the facts that $B_1^T = [0, -0.1765], B_2^T = [0, -0.0052]$ in this simulation. Since B is already composed of 0 (upper part) and -1 (lower part) in the vector, there is no longer to use an orthogonal matrix T_r for B 's transformation in order to get the form, $[0, B_2]^T$ in (4). By setting $C_2 = I$ and using the software package Robust Control Toolbox (MATLAB), $C_2 = -0.5$ by solving LMIs (24).

Fig. 1 depicts the response of the system state, where the solid line is x_1 and the dotted line is x_2 , with initial values $x_1(0) = 87^\circ, x_2(0) = 0$ under the control law (25) where μ is set to be 0.05. The control input u , and the sliding motion s are shown in Figs. 2, 3, respectively. We observe that when system state enters around the sliding surface, the chattering phenomenon no longer exists just we claimed in Remark 3. In addition, adaptive parameters in (29) and (30) are shown in Fig. 4. From the simulation results, it can be seen that the control approach proposed in this paper works well.

V. CONCLUSIONS

An adaptive fuzzy SMC approach for a class of nonlinear dynamical systems with lumped parameter uncertainties was proposed and analyzed. In particular, the approach could deal with the T-S fuzzy model under consideration with different control matrices in the fuzzy rules. As a future work, other uncertainties including linearization error between the real system to be controlled and its T-S fuzzy model, parameter perturbations, and external disturbances will be considered in the control approach to improve its robustness.

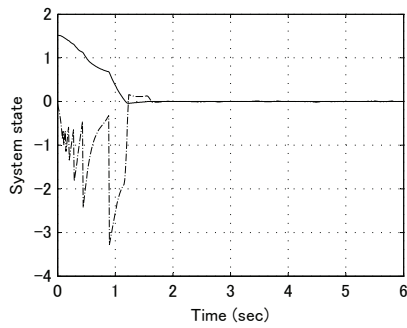


Fig. 1. System state

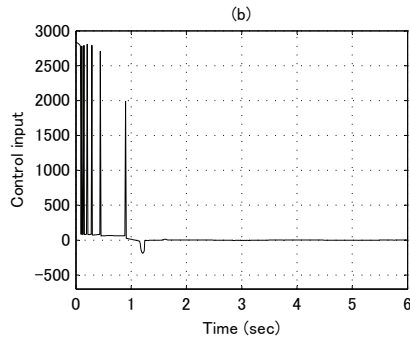


Fig. 2. Control input

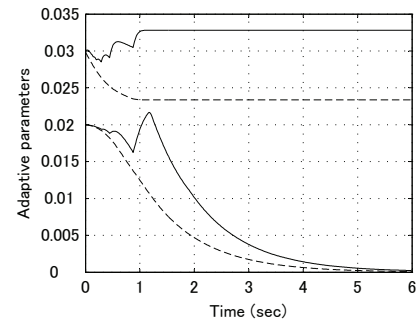


Fig. 4. Adaptive parameters

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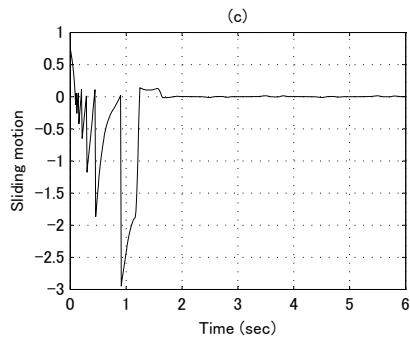


Fig. 3. Sliding motion