

Output Feedback Control for Discrete-Time Takagi-Sugeno Fuzzy Systems

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Abstract—This paper is concerned with output feedback control design for a discrete-time fuzzy system with immeasurable premise variables. It is well known that Takagi-Sugeno fuzzy model describes a wide class of nonlinear systems especially when its premise variables include immeasurable functions. However, when it comes to control design of such a fuzzy system with immeasurable premise variables, a conventional parallel distributed compensator(PDC) is not feasible because it shares the same premise variables as those of a fuzzy system. In this paper, we introduce an output feedback controller with the estimate of the premise variables of an original fuzzy system. We then formulate the stabilization problem for a discrete-time fuzzy system with immeasurable premise variables. Our control design method is based on a set of strict LMI conditions. No tuning parameter is necessary a priori to solve LMI conditions. Our method includes tuning matrices for control gains in a controller and hence they can be chosen to optimize the control performance of the system. Furthermore, we extend our results to a class of discrete-time fuzzy systems with uncertain parameters. A design method of robust controller for such uncertain systems is also given. Numerical examples are finally given to illustrate our control design method.

keywords—Takagi-Sugeno fuzzy systems, Stabilization, Output feedback control

I. INTRODUCTION

It is well-known that the Takagi-Sugeno fuzzy model is one of the efficient mathematical models that describe a wide class of nonlinear systems([6], [10]). Hence, its analysis and control synthesis are important. In the past years, sufficient stability conditions were given, and control design methods were proposed([7], [11], [12], [13]). A conventional approach to control design for a fuzzy system is to employ a parallel distributed compensator(PDC). It is shown that a PDC works well for a fuzzy system because it shares the same premise variables as those of the original fuzzy system. The papers [7] and [13] gave stability conditions and proposed a control design method by a PDC. Furthermore, relaxed stability conditions have been investigated in [11] and [12]. These results were only concerned with the state feedback control design, and the premise variables were assumed to be available. However, this is not always the case in many situations. Especially, in practical situations, the state of the system is not measurable and we need an output feedback control design. In this case, the above mentioned results are infeasible. In fact, if the premise variable is the state, a conventional PDC,

which feeds back the state variables, is infeasible. Furthermore, when the premise variable is the state, the Takagi-Sugeno model describes the widest class of nonlinear systems. In other words, a conventional PDC can work only for a limited class of nonlinear systems([16]).

In this paper, we propose a design method of a stabilizing output feedback controller for a discrete-time fuzzy system with immeasurable premise variables. Such a fuzzy system describes a large class of nonlinear systems. The results in [3], [8], [9], [16] do not take care of such a class of fuzzy systems. Yoneyama et al. [16] gave an approximate stabilizing output feedback controller. Assawinchaichote et al. [1] and Yoneyama [15] proposed an uncertain fuzzy system approach. Guerra et al. [2] gave a design method based on non-strict LMI conditions. These results are conservative and are not easy to solve controller design conditions. The paper by Yoneyama [17] proposed a design method based on strict LMI conditions. It gives a two-step procedure to solve the design problem. The paper [17] considers only a class of continuous-time fuzzy systems with immeasurable premise variables. This paper is concerned with a discrete-time counterpart. A control design procedure is also based on strict LMI conditions. No tuning parameter is required in the control design, which is much easier than others in the literature. In addition, we have free scaling matrices that can tune control gain matrices.

We also consider the robust stabilization for fuzzy systems with uncertain parameters. Our method can easily be extended to such a robust stabilization problem. For this robust stabilization problem, we also give a similar two-step procedure with strict LMI conditions, which can easily be calculated via numerical software. Finally, some numerical examples are given to illustrate our design procedure and the effectiveness of our design method.

II. FUZZY SYSTEMS

In this section, we introduce discrete-time Takagi-Sugeno fuzzy systems. Consider the discrete-time Takagi-Sugeno fuzzy model, described by the following IF-THEN rules:

IF ξ_1 is M_{i1} and $\dots$ and ξ_p is M_{ip} ,
THEN $x(k+1) = A_i x(k) + B_i u(k)$,
 $y(k) = C_i x(k)$, $i = 1, \dots, r$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input and $y(k) \in \mathbb{R}^q$ is the measurement output. The matrices A_i, B_i

and C_i are constant matrices of appropriate dimensions. r is the number of IF-THEN rules. M_{ij} are fuzzy sets and $\xi_1, \dots, \xi_p$ are premise variables. We set $\xi = [\xi_1 \ \dots \ \xi_p]^T$. The premise variable $\xi(k)$ is assumed to be a function of the state, which is immeasurable.

For simplicity, we define the following notations:

$$\begin{aligned} X_\xi &= \sum_{i=1}^r \lambda_i(\xi(k)) X_i, \\ X_{\hat{\xi}} &= \sum_{i=1}^r \lambda_i(\hat{\xi}(k)) X_i, \\ X_{\xi\hat{\xi}} &= \sum_{i=1}^r \sum_{j=1}^r \lambda_i(\xi(k)) \lambda_j(\hat{\xi}(k)) X_{ij}. \end{aligned}$$

and so on.

Then, the state equation and the output equation are described, respectively, by

$$\begin{aligned} x(k+1) &= \sum_{i=1}^r \lambda_i(\xi(k)) \{A_i x(k) + B_i u(k)\} \\ &\triangleq \sum_{i=1}^r \lambda_i(\xi(k)) A_i x(k) + B_\xi u(k), \\ y(k) &= \sum_{i=1}^r \lambda_i(\xi(k)) C_i x(k) \\ &\triangleq C_\xi x(k), \end{aligned} \quad (1)$$

where $\lambda_i(\xi(k))$

$$\lambda_i(\xi(k)) = \frac{\beta_i(\xi(k))}{\sum_{i=1}^r \beta_i(\xi(k))}, \quad \beta_i(\xi(k)) = \prod_{j=1}^p M_{ij}(\xi_j(k))$$

and $M_{ij}(\cdot)$ is the grade of the membership function of M_{ij} . We assume

$$\beta_i(\xi(k)) \geq 0, \quad i = 1, \dots, r, \quad \sum_{i=1}^r \beta_i(\xi(k)) > 0$$

for any $\xi(k)$. Hence $\lambda_i(\xi(k))$ satisfy

$$\lambda_i(\xi(k)) \geq 0, \quad i = 1, \dots, r, \quad \sum_{i=1}^r \lambda_i(\xi(k)) = 1$$

for any $\xi(k)$.

As stated in [15], [16], a fuzzy system covers a wider class of nonlinear systems when its premise variables are the state variables than the measurement output. In order to design a stabilizing controller for a fuzzy system, a PDC is traditionally used for stabilization because it shares the same premise variables as those of a fuzzy system([7], [13]). However, a PDC is feasible only if the premise variables are measurable or available functions. Thus, most existing results have assumed that the premise variables are measurable([3], [8], [9], [16]). In this paper, we consider an output feedback stabilization where the state is not directly measurable. A fuzzy system is supposed to have the premise variables that are functions of the states, which are immeasurable. In this case, actual values of the premise variables are not available. Hence, we employ

the following rules for an output feedback controller with the premise variables being their estimates:

$$\begin{aligned} \text{IF} \quad & \hat{\xi}_1 \text{ is } M_{i1} \text{ and } \dots \text{ and } \hat{\xi}_p \text{ is } M_{ip}, \\ \text{THEN} \quad & \hat{x}(k+1) = \hat{A}_i \hat{x}(k) + \hat{B}_i y(k) \\ & u(k) = \hat{C}_i \hat{x}(k), \quad i, j = 1, \dots, r. \end{aligned}$$

where $\hat{A}_i = \sum_{j=1}^r \lambda_j(\hat{\xi}(k)) \hat{A}_{ij}$, and $\hat{A}_{ij}$, $\hat{B}_i$ and $\hat{C}_i$ are constant matrices to be determined. Then the natural choice of a controller is given by

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^r \sum_{j=1}^r \lambda_i(\hat{\xi}(k)) \lambda_j(\hat{\xi}(k)) \{ \hat{A}_{ij} \hat{x}(k) + \hat{B}_j y(k) \} \\ &\triangleq \hat{A}_{\hat{\xi}\hat{\xi}} \hat{x}(k) + \hat{B}_{\hat{\xi}} y(k), \\ u(k) &= \sum_{i=1}^r \lambda_i(\hat{\xi}(k)) \hat{C}_i \hat{x}(k) \triangleq \hat{C}_{\hat{\xi}} \hat{x}(k). \end{aligned} \quad (2)$$

Remark 2.1: One might consider the following controller:

$$\begin{aligned} \hat{x}(k+1) &= \sum_{i=1}^r \lambda_i(\hat{\xi}(k)) \{ \hat{A}_i \hat{x}(k) + \hat{B}_i y(k) \}, \\ u(k) &= \sum_{i=1}^r \lambda_i(\hat{\xi}(k)) \hat{C}_i \hat{x}(k). \end{aligned}$$

However, this class of controllers is limited, and a general class of controller is given as in (2)(see [10] for more detail.).

The closed-loop system (1) with (2) becomes

$$x_e(k+1) = \bar{A}_{\xi\hat{\xi}\hat{\xi}} x_e(k), \quad (3)$$

where $x_e(k)$ and $\bar{A}_{\xi\hat{\xi}\hat{\xi}}$ are given by

$$x_e(k) = \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}, \quad \bar{A}_{\xi\hat{\xi}\hat{\xi}} = \begin{bmatrix} A_\xi & B_\xi \hat{C}_{\hat{\xi}} \\ \hat{B}_{\hat{\xi}} C_\xi & \hat{A}_{\hat{\xi}\hat{\xi}} \end{bmatrix}$$

Our problem is to find an output feedback controller (2) such that the closed-loop system (3) is asymptotically stable.

The following lemma is useful to prove our main results.

Lemma 2.1: ([12])

$$\sum_{i=1}^r \sum_{j=1}^r \lambda_i(\xi(k)) \lambda_j(\xi(k)) \Phi_{ij} < 0$$

holds if the followings are satisfied:

$$\begin{aligned} \Phi_{ii} &< 0, \quad i = 1, \dots, r, \\ \frac{2}{r-1} \Phi_{ii} + \Phi_{ij} + \Phi_{ji} &< 0, \quad i, j = 1, \dots, r, \quad i \neq j \end{aligned}$$

III. STABILITY AND STABILIZATION

Here, we consider an output feedback control of a discrete-time fuzzy system. We first give sufficient conditions for the closed-loop system (3) to be asymptotically stable. Then, we propose a design method of an output feedback controller for a discrete-time fuzzy system.

A. Stabilization Analysis

We make the stability analysis of the closed-loop system (3).

Theorem 3.1: An output feedback controller of the form (2) stabilizes the fuzzy system (1) if there exist matrices $X > 0$, $Y > 0$, $L_{\hat{\xi}\hat{\xi}}$, $W_{\hat{\xi}}$, $Z_{\hat{\xi}}$ such that

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \quad (4)$$

$$\Psi_{\xi\xi\xi} = \begin{bmatrix} -X & -I & \Psi_{13\xi\xi} & A_{\xi} \\ -I & -Y & \Psi_{23\xi\xi\xi} & \Psi_{24\xi\xi} \\ \Psi_{13\xi\xi}^T & \Psi_{23\xi\xi\xi}^T & -X & -I \\ A_{\xi}^T & \Psi_{24\xi\xi}^T & -I & -Y \end{bmatrix} < 0. \quad (5)$$

where

$$\begin{aligned} \Psi_{13\xi\xi} &= A_{\xi}X + B_{\xi}Z_{\hat{\xi}}, \\ \Psi_{23\xi\xi\xi} &= YA_{\xi}X + YB_{\xi}Z_{\hat{\xi}} + W_{\hat{\xi}}C_{\xi}X + L_{\hat{\xi}\hat{\xi}}, \\ \Psi_{24\xi\xi} &= YA_{\xi} + W_{\hat{\xi}}C_{\xi}. \end{aligned}$$

Proof: We investigate the stability of the closed-loop system (3). A condition for the system (3) to be asymptotically stable is that there exists a matrix $P > 0$ such that

$$\bar{A}_{\xi\xi\xi}^T P \bar{A}_{\xi\xi\xi} - P < 0$$

or equivalently

$$\Lambda = \begin{bmatrix} -P^{-1} & \bar{A}_{\xi\xi\xi} \\ \bar{A}_{\xi\xi\xi}^T & -P \end{bmatrix} < 0. \quad (6)$$

To solve the matrix inequality (6), we follow the idea of [5] and partition P and P^{-1} as

$$P = \begin{bmatrix} Y & N \\ N^T & V \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} X & M \\ M^T & U \end{bmatrix}. \quad (7)$$

Define a transformation matrix Π as

$$\Pi = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}. \quad (8)$$

Then, it can be inferred from $PP^{-1} = I$ that

$$YX + NM^T = I. \quad (9)$$

We take a congruence transformation with Π for P to obtain

$$\Pi^T P \Pi = \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0,$$

which is exactly the same as (4).

Pre- and post-multiplying Λ by $S = \text{diag}[P\Pi \Pi]$, we obtain

$$\Psi_{\xi\xi\xi} = S^T \Lambda S = \begin{bmatrix} -\Pi^T P \Pi & \Pi^T P \bar{A}_{\xi\xi\xi} \Pi \\ \Pi^T \bar{A}_{\xi\xi\xi}^T P \Pi & -\Pi^T P \Pi \end{bmatrix} < 0,$$

or more exactly we get (5) where we define

$$L_{\hat{\xi}\hat{\xi}} = N \hat{A}_{\xi\xi} M^T, \quad W_{\hat{\xi}} = N \hat{B}_{\xi}, \quad \text{and} \quad Z_{\hat{\xi}} = \hat{C}_{\xi} M^T. \quad (10)$$

B. Stabilization

This section gives sufficient conditions for the closed-loop system (3) to be asymptotically stable, and a method of controller design base on such stability conditions.

Theorem 3.2: An output feedback controller of the form (2) stabilizes the fuzzy system (1) if there exist matrices $X > 0$, $Y > 0$, L_{jk} , W_j , Z_j , $j, k = 1, \dots, r$ such that (4) holds and

$$\begin{aligned} \Psi_{ijj} &< 0, \quad i, j = 1, \dots, r, \\ \frac{2}{r-1} \Psi_{ijj} + \Psi_{ijk} + \Psi_{ikj} &< 0, \quad i, j, k = 1, \dots, r, \\ &j \neq k. \end{aligned} \quad (11)$$

where

$$\begin{aligned} \Psi_{ijk} &= \begin{bmatrix} -X & \Psi_{13ij}^T & \Psi_{23ijk}^T & -I \\ \Psi_{13ij} & -X & -I & A_i \\ \Psi_{23ijk} & -I & -Y & \Psi_{24ij} \\ -I & A_i^T & \Psi_{24ij}^T & -Y \end{bmatrix}, \\ \Psi_{13ij} &= A_i X + B_i Z_j, \\ \Psi_{23ijk} &= Y A_i X + Y B_i Z_j + W_j C_i X + L_{jk}, \\ \Psi_{24ij} &= Y A_i + W_j C_i. \end{aligned}$$

In this case, control gain matrices in (2) are given by

$$\begin{aligned} \hat{A}_{jk} &= N^{-1} L_{jk} M^{-T}, \\ \hat{B}_j &= N^{-1} W_j, \\ \hat{C}_j &= Z_j M^{-T}, \quad j, k = 1, \dots, r \end{aligned} \quad (12)$$

where M and N can be found from the equation $NM^T = I - YX$.

Proof: If conditions (11) are satisfied, it follows from Lemma 2.1 that

$$\sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r \lambda_i(\xi(k)) \lambda_j(\hat{\xi}(k)) \lambda_k(\hat{\xi}(k)) \Psi_{ijk} = \Psi_{\xi\xi\xi} < 0.$$

Taking a congruence transformation with the following T :

$$T = \begin{bmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and calculating $T^T \Psi_{\xi\xi\xi} T < 0$, we obtain (5).

By Theorem 3.1, the stability is established. Control gain matrices (12) follow from the relations (10).

Theorem 3.2 does not give LMI conditions. However, the following algorithm provides an efficient LMI-based approach to solve for (4) and (11).

Algorithm 3.1: Step 1: Solve $Y > 0$, W_j , $j = 1, \dots, r$ for the following LMI conditions

$$\begin{bmatrix} -Y & \Psi_{24ij} \\ \Psi_{24ij}^T & -Y \end{bmatrix} < 0, \quad i, j = 1, \dots, r.$$

Step 2: Insert matrices Y , W_j , $j = 1, \dots, r$, which are obtained in **Step 1**, into Ψ_{ijk} . Solve $X > 0$, Z_j , L_{jk} , $j, k = 1, \dots, r$ for (4) and (11).

Step 3: If there exists a solution, obtain gain matrices as in (12).

Remark 3.1: It follows from (4) that $Y - X^{-1} > 0$ and hence $NM^T = I - YX$ is nonsingular. M and N can tune control gain matrices in (2), which reflect trajectories of the states of the controlled system. One way to find M is to choose $M = X$. In this case, $\hat{x}$ in (2) becomes an estimate of the state x (see [4] for more detail.), and $N = X^{-1} - Y$.

Remark 3.2: Assawinchaichote et al. [1] and Yoneyama [15] considered the grade $\lambda_i(\xi)$ as uncertainty, and then gave stability conditions. Since their results depend on the maximum upper bound of $\lambda(\xi)$, they are rather conservative. The result of Guerra et al. [2] is based on non-LMI conditions with four scaling parameters, which must be chosen a priori. Algorithm 3.1 is involved with only strict LMI conditions, which are much easier to be solved than other methods.

C. Robust Stabilization

Now, we extend the results in the previous sections to a class of uncertain discrete-time fuzzy systems. Consider the uncertain discrete-time Takagi-Sugeno fuzzy model, described by the following IF-THEN rules:

IF ξ_1 is M_{i1} and $\dots$ and ξ_p is M_{ip} ,
THEN $x(k+1) = (A_i + \Delta A_i)x(k)$
 $\quad + (B_i + \Delta B_i)u(k)$,
 $y(k) = (C_i + \Delta C_i)x(k)$, $i = 1, \dots, r$

where ΔA_i , ΔB_i and ΔC_i , $i = 1, \dots, r$ are uncertain matrices that have the following structures:

$$\begin{bmatrix} \Delta A_i & \Delta B_i \\ \Delta C_i & * \end{bmatrix} = \begin{bmatrix} H_{1i} \\ H_{2i} \end{bmatrix} F_i(k) \begin{bmatrix} E_{1i} & E_{2i} \end{bmatrix}, \quad i = 1, \dots, r. \quad (13)$$

and $F_i(k)$ is an uncertain matrix satisfying

$$F_i^T(k)F_i(k) \leq I, \quad i = 1, \dots, r,$$

and H_{1i} , H_{2i} , E_{1i} and E_{2i} are constant matrices of appropriate dimensions.

The state and output equations are described by

$$\begin{aligned} x(k+1) &= \sum_{i=1}^r \lambda_i(\xi) \{ (A_i + H_{1i}F_i(k)E_{1i})x(k) \\ &\quad + (B_i + H_{1i}F_i(k)E_{2i})u(k) \}, \\ &\triangleq A_{\Delta\xi}x(k) + B_{\Delta\xi}u(k), \\ y(k) &= \sum_{i=1}^r \lambda_i(\xi) \{ (C_i + H_{2i}F_i(k)E_{1i})x(k) \} \\ &\triangleq C_{\Delta\xi}x(k). \end{aligned} \quad (14)$$

If there exists a controller of the form (2) that stabilizes the system (14) for all admissible uncertain matrices ΔA_i , ΔB_i and ΔC_i satisfying (13), then it is called a robust stabilizing controller.

For the system (14), we will find a robust stabilizing controller. To this end, we use the following lemma.

Lemma 3.1: ([14]) Given matrices $Q = Q^T$, H , E and $R = R^T > 0$ of appropriate dimensions

$$Q + HF(k)E + E^T F^T(k)H^T < 0$$

for all $F(k)$ satisfying $F^T(k)F(k) \leq R$ if and only if there exists a scalar $\varepsilon > 0$ such that

$$Q + \frac{1}{\varepsilon}HH^T + \varepsilon E^T R E < 0.$$

The closed-loop system (14) with (2) is given by

$$x_e(k+1) = \bar{A}_{\xi\xi\xi}x_e(k) \quad (15)$$

where $x_e(k)$ and $\bar{A}_{\xi\xi\xi}$ is given by

$$x_e(k) = \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}, \quad \bar{A}_{\xi\xi\xi} = \begin{bmatrix} A_{\Delta\xi} & B_{\Delta\xi}\hat{C}_{\xi} \\ \hat{B}_{\xi}C_{\Delta\xi} & \hat{A}_{\xi\xi} \end{bmatrix}.$$

Theorem 3.3: An output feedback controller of the form (2) robustly stabilizes the uncertain fuzzy system (14) if there exist matrices $X > 0$, $Y > 0$, L_{jk} , W_j , Z_j , $j, k = 1, \dots, r$ and scalars $\varepsilon_i > 0$ such that (4) holds and

$$\begin{bmatrix} -\frac{1}{\varepsilon_{ij}}I & 0 & \bar{E}_{1ij} & 0 & 0 \\ 0 & -\frac{1}{\varepsilon_{ij}}I & \bar{E}_{2i} & 0 & 0 \\ \bar{E}_{1ij}^T & \bar{E}_{2i}^T & \Xi_{ijj} & \bar{H}_{1i} & \bar{H}_{2ij} \\ 0 & 0 & \bar{H}_{1i}^T & -\varepsilon_{ij}I & 0 \\ 0 & 0 & \bar{H}_{2ij}^T & 0 & -\varepsilon_{ij}I \end{bmatrix} < 0, \quad i, j = 1, \dots, r, \quad (16)$$

$$\begin{bmatrix} -\frac{1}{\varepsilon_{ij}}I & 0 & \hat{E}_{1ijk} & 0 & 0 \\ 0 & -\frac{1}{\varepsilon_{ij}}I & \hat{E}_{2i} & 0 & 0 \\ \hat{E}_{1ijk}^T & \hat{E}_{2i}^T & \hat{\Xi}_{ijk} & \bar{H}_{1i} & \hat{H}_{2ijk} \\ 0 & 0 & \bar{H}_{1i}^T & -\varepsilon_{ij}I & 0 \\ 0 & 0 & \hat{H}_{2ijk}^T & 0 & -\varepsilon_{ij}I \end{bmatrix} < 0, \quad i, j, k = 1, \dots, r, \quad j \neq k, \quad (17)$$

where

$$\begin{aligned} \Xi_{ijk} &= \begin{bmatrix} -X & \Xi_{13ij}^T & \Xi_{23ijk}^T & -I \\ \Xi_{13ij} & -X & -I & A_i \\ \Xi_{23ijk} & -I & -Y & \Xi_{24ij} \\ -I & A_i^T & \Xi_{24ij}^T & -Y \end{bmatrix}, \\ \Xi_{14ij} &= A_i X + B_i Z_j, \\ \Xi_{24ijk} &= Y A_i X + Y B_i Z_j + W_j C_i X + L_{jk}, \\ \Xi_{25ij} &= Y A_i + W_j C_i, \\ \hat{\Xi}_{ijk} &= \frac{2}{r-1} \Xi_{ijj} + \Xi_{ijk} + \Xi_{ikj}, \\ \bar{H}_{1i} &= \begin{bmatrix} 0 & H_{1i}^T & H_{1i}^T Y & 0 \end{bmatrix}^T, \\ \bar{H}_{2ij} &= \begin{bmatrix} 0 & 0 & H_{2i}^T W_j^T & 0 \end{bmatrix}^T, \\ \hat{H}_{2ijk} &= \frac{r+1}{r-1} \bar{H}_{2ij} + \bar{H}_{2ik}, \\ \bar{E}_{1ij} &= \begin{bmatrix} E_{1i}X + E_{2i}Z_j & 0 & 0 & E_{1i} \end{bmatrix}^T, \\ \bar{E}_{2i} &= \begin{bmatrix} E_{1i}X & 0 & 0 & E_{1i} \end{bmatrix}^T, \\ \hat{E}_{1ijk} &= \frac{r+1}{r-1} \bar{E}_{1ij} + \bar{E}_{1ik}. \end{aligned}$$

In this case, control gain matrices in (2) are given by (12).

Proof: Replacing A_i , B_i and C_i by $A_i + H_{1i}F_iE_{1i}$, $B_i + H_{1i}F_iE_{2i}$ and $C_i + H_{2i}F_iE_{1i}$, respectively in the first LMI (11), we get

$$\begin{aligned} \Psi_{ijj} &+ \bar{H}_{1i}F_i\bar{E}_{1ij} + \bar{E}_{1ij}^T F_i^T \bar{H}_{1i}^T \\ &+ \bar{H}_{2ij}F_i\bar{E}_{2i} + \bar{E}_{2i}^T F_i^T \bar{H}_{2ij}^T < 0, \quad i, j = 1, \dots, r. \end{aligned}$$

We apply Lemma 3.1, and obtain an equivalent condition that there exists $\varepsilon_i > 0$ such that

$$\Psi_{ijj} + \varepsilon_i \bar{E}_{1ij}^T \bar{E}_{1ij} + \varepsilon_i \bar{E}_{2i}^T \bar{E}_{2i} + \frac{1}{\varepsilon_i} \bar{H}_{1i} \bar{H}_{1i}^T + \frac{1}{\varepsilon_i} \bar{H}_{2ij} \bar{H}_{2ij}^T < 0, \quad i, j = 1, \dots, r.$$

By Schur complement formula, we finally obtain the LMI in (16).

The LMI in (17) can be obtained in a similar manner. In fact, it follows from (17) that

$$\begin{aligned} & \frac{2}{r-1} \Psi_{ijj} + \Psi_{ijk} + \Psi_{ikj} + \frac{\varepsilon_i(r+1)}{r-1} \bar{E}_{1ij}^T \bar{E}_{1ij} \\ & + \varepsilon_i \bar{E}_{1ik}^T \bar{E}_{1ik} + \frac{2r\varepsilon_i}{r-1} \bar{E}_{2i}^T \bar{E}_{2i} + \frac{2r}{\varepsilon_i(r-1)} \bar{H}_{1i} \bar{H}_{1i}^T \\ & + \frac{r+1}{\varepsilon_i(r-1)} \bar{H}_{2ij} \bar{H}_{2ij}^T + \frac{1}{\varepsilon_i} \bar{H}_{2ik} \bar{H}_{2ik}^T < 0, \\ & i = 1, \dots, r, \quad j \neq k = 1, \dots, r. \end{aligned}$$

Then, the results follow.

The following algorithm is used for the control design.

Algorithm 3.2: **Step 1:** Solve $Y > 0$, W_j , $j = 1, \dots, r$ and $\varepsilon_i > 0$, $i = 1, \dots, r$ for the following LMI conditions

$$\begin{bmatrix} -Y & \Xi_{24ij} & YH_{1i} & W_j H_{2i} \\ \Xi_{24ij}^T & -Y & 0 & 0 \\ H_{1i}^T Y & 0 & -\varepsilon_{ij} I & 0 \\ H_{2i}^T W_j^T & 0 & 0 & -\varepsilon_{ij} I \end{bmatrix} < 0, \quad i, j = 1, \dots, r,$$

$$\begin{bmatrix} -\frac{2r}{r-1} Y & \hat{\Xi}_{24ijk} & YH_{1i} & \tilde{H}_{ijk} \\ \hat{\Xi}_{24ijk}^T & -\frac{2r}{r-1} Y & 0 & 0 \\ H_{1i}^T Y & 0 & -\varepsilon_{ij} I & 0 \\ \tilde{H}_{ijk}^T & 0 & 0 & -\varepsilon_{ij} I \end{bmatrix} < 0, \quad i, j, k = 1, \dots, r, \quad j \neq k$$

where

$$\begin{aligned} \hat{\Xi}_{24ijk} &= \frac{r+1}{r-1} \Xi_{24ij} + \Xi_{24ik}, \\ \tilde{H}_{ijk} &= \frac{r+1}{r-1} W_j H_{2i} + W_k H_{2i}, \end{aligned}$$

Step 2: Insert matrices Y , W_j , $j = 1, \dots, r$ and $\varepsilon_i > 0$, $i = 1, \dots, r$, which are obtained in **Step 1**, into Ψ_{ijk} . Then, solve $X > 0$, L_{jk} , Z_j , $j, k = 1, \dots, r$ for (4), (16) and (17).

Step 3: If there exists a solution, obtain gain matrices as in (12).

Remark 3.3: Algorithm 3.2 also gives LMI approach to solving an output feedback control problem.

IV. EXAMPLES

We give two examples. The first one illustrates stabilization for a nominal fuzzy system. The second example is concerned with an uncertain fuzzy system. We note that the results in [3], [8], [9], [16] cannot be applied to these problems because these papers assume the measurable premise variables.

Example 4.1 Consider the fuzzy system (1) with following matrices:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.2 & -0.2 \\ 0.1 & -0.4 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 1.2 & -0.2 \\ 0.2 & 0 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1.3 \\ 0.7 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}, \\ C_1 = C_2 &= \begin{bmatrix} 10 & -2 \end{bmatrix}, \\ \lambda_1(x_1) &= \frac{1}{1 + e^{-x_1}}, \\ \lambda_2(x_1) &= 1 - \lambda_1(x_1). \end{aligned}$$

The system is descretized with sampling time $t_s = 0.01$. We attempt to obtain a stabilizing output feedback controller of the form (2). In this problem, the output y is the only measurable and premise variable x_1 is immeasurable. Algorithm 3.1 guarantees the existence of a stabilizing output feedback controller. According to the result in [4], we choose $M = X$. In this case, then a stabilizing controller (2) is given with

$$\begin{aligned} \hat{A}_{11} &= \begin{bmatrix} -0.7815 & 0.0775 \\ -0.4765 & -0.1442 \end{bmatrix}, \\ \hat{A}_{12} &= \begin{bmatrix} -1.4191 & 0.7697 \\ 0.4761 & -3.4518 \end{bmatrix}, \\ \hat{A}_{21} &= \begin{bmatrix} -0.4473 & -0.5188 \\ -1.3826 & 3.2341 \end{bmatrix}, \\ \hat{A}_{22} &= \begin{bmatrix} -0.9619 & 0.0924 \\ -0.5902 & -0.1377 \end{bmatrix}, \\ \hat{B}_1 &= \begin{bmatrix} 0.1251 \\ 0.0130 \end{bmatrix}, \\ \hat{B}_2 &= \begin{bmatrix} 0.1471 \\ 0.0274 \end{bmatrix}, \\ \hat{C}_1 &= \begin{bmatrix} -0.5381 & 0.0250 \end{bmatrix}, \\ \hat{C}_2 &= \begin{bmatrix} -0.5253 & 0.0088 \end{bmatrix}. \end{aligned}$$

Simulation results of x_1 and $\hat{x}_1$ with the initial conditions

$$x(0) = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \quad \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

are depicted in Fig. 1. The solid lines show the state x and the dotted lines the estimate $\hat{x}$. The controller stabilizes the system. No tuning parameter is necessary for solving LMI conditions in Algorithm 3.1, while several tuning parameters are required to be given to solve design conditions in [2].

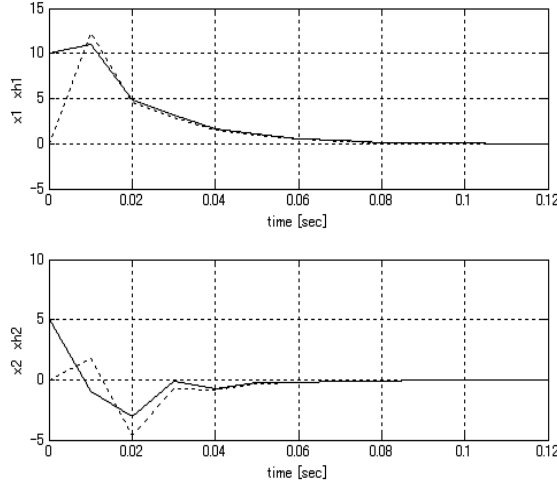


Fig. 1. States of the system

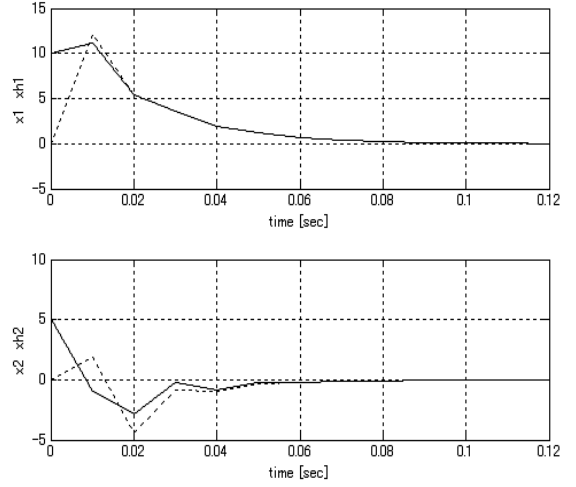


Fig. 2. States of the system

Example 4.2 We consider the system (14) with the following matrices:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.2 & -0.2 \\ 0.1 & -0.4 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 1.2 & -0.2 \\ 0.2 & 0 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1.3 \\ 0.7 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}, \\ C_1 &= C_2 = \begin{bmatrix} 10 & -2 \end{bmatrix}, \\ \lambda_1(x_1) &= \frac{1}{1 + e^{-x_1}}, \\ \lambda_2(x_1) &= 1 - \lambda_1(x_1), \\ H_{11} &= H_{12} = 0.1I, \\ H_{21} &= H_{22} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \\ E_{11} &= E_{12} = 0.1I, \\ E_{21} &= E_{22} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}. \end{aligned}$$

The system is discretized with sampling time $t_s = 0.01$.

Algorithm 3.2 gives a stabilizing output feedback controller (2) with $\hat{\xi} = \hat{x}_1$. Choosing $M = X$, we obtain control gains:

$$\begin{aligned} \hat{A}_{11} &= \begin{bmatrix} -0.7862 & 0.0180 \\ -0.5132 & -0.1761 \end{bmatrix}, \\ \hat{A}_{12} &= \begin{bmatrix} -1.0024 & -0.6752 \\ -1.1031 & -2.2682 \end{bmatrix}, \\ \hat{A}_{21} &= \begin{bmatrix} -0.8287 & 0.8416 \\ 0.2020 & 2.0217 \end{bmatrix}, \\ \hat{A}_{22} &= \begin{bmatrix} -0.8223 & 0.0216 \\ -0.5389 & -0.1751 \end{bmatrix}, \\ \hat{B}_1 &= \begin{bmatrix} 0.1328 \\ 0.0195 \end{bmatrix}, \\ \hat{B}_2 &= \begin{bmatrix} 0.1371 \\ 0.0227 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \hat{C}_1 &= \begin{bmatrix} -0.5091 & -0.0145 \end{bmatrix}, \\ \hat{C}_2 &= \begin{bmatrix} -0.5104 & -0.0184 \end{bmatrix}. \end{aligned}$$

Simulation results with the initial conditions and uncertain parameter:

$$x(0) = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \quad \hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad F(k) = \cos(10k).$$

are shown in Fig. 2. The solid lines show the state x and the dotted lines the estimate $\hat{x}$. The controller robustly stabilizes the system.

V. CONCLUSIONS

We have proposed a stabilizing output feedback controller for discrete-time fuzzy systems with immeasurable premise variables. Our design method was based on LMI approach. No tuning parameter is necessary to solve the output feedback controller design problem. Furthermore, we extended the results to the case of robust output feedback controller for a class of uncertain fuzzy systems. Finally, some numerical examples are given to illustrate our design methods.

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