

The Derivation of the Analytical Structure of a class of Interval Type-2 Fuzzy PD and PI Controllers

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Abstract—Analytical structure of a fuzzy logic system (FLS) is to establish its mathematical input-output relationship. In this paper, the analytical structure of a class of Interval type-2 (IT2) fuzzy PD and PI controllers are derived by dividing the input space and identifying the input-output relationship for each region. Unlike the symmetric fuzzy PD and PI controller [12] which have three different consequent sets $H_4 < H_3 = H_2 < H_1$, the IT2 fuzzy PD and PI controllers considered in this paper have four different consequent sets $H_4 < H_3 < H_2 < H_1$. Compared with the symmetric IT2 fuzzy controllers investigated in [12], the relatively large number of divisions of the input space indicates the potential of this class of IT2 fuzzy controllers in achieving better performance. The equations describing the input-output relationship for each region provide mathematical tools for further investigations of these IT2 fuzzy PD controllers.

Index Terms—Interval type-2 fuzzy logic system, analytical structure, fuzzy PD and PI controller, Karnik-Mendel type-reduction method.

I. INTRODUCTION

Numerical studies have shown that the IT2 fuzzy logic controller (FLC) may outperform the type-1 (T1) FLC in complex control systems, particularly in faces of complicated imprecision and uncertainty. One possible reason is that the 3-dimensional structure of an IT2 FS allows additional design parameters to be introduced. These additional design freedom may enable the IT2 FLC to produce satisfactory performance. But most of the efforts to demonstrate the advantages of the IT2 FLC over the T1 FLC have been made through case studies, and their conclusions may not be general enough to provide guidelines for the design of IT2 FLCs for other complex plants. As a result, the IT2 FLCs are generally treated as black-box systems whose parameters are optimized through genetic algorithms or other optimization algorithms only. Such FLCs may not be amenable for analysis and interpretation by engineers.

As an efficient theoretical tool, the analytical structure of a T1 fuzzy controller may be described by mathematical equations relating the output of the fuzzy controller and its inputs [3]-[10]. This analytical tool can overcome the limitation of fuzzy logic system—lack of explicit model, and reveal how the design parameters affect the input-output relationship. More importantly, this analytical tool can provide guidelines for the design of T1 FLCs in choosing a t-norm operation, the shapes of antecedent sets and tuning the design parameters. An early effort to derive the analytical structure of IT2 fuzzy controllers

has been made in [11]. In this work, the authors approximated the output of the controller by averaging 16 embedded T1 FLSs, instead of the widely used Karnik-Mendel (KM) type-reduction method.

The KM type-reduction method is established on wavy-slice representation theorem: an IT2 fuzzy set (FS) contains an infinite number of embedded T1 FSs whose centroids will form an interval, the type-reduced set. In an IT2 FLS, the type-reduced set is formed by the centroid of all possible embedded T1 FLSs. The smallest and largest centroid which are the left and right endpoints of the interval respectively, can be searched using the Karnik-Mendel iterative algorithm [2]. There lacks closed-form expressions relating the output and inputs of an IT2 FLS. The left and right endpoints have been proven to be generated by the centroids of the output sets weighted by the upper or lower bound of the corresponding firing sets. Therefore, the challenge that impedes the derivation of analytical structure for IT2 fuzzy controllers is to figure out the situations when the upper or lower bound of firing sets should be applied. An effort has been made to derive the analytical structure for a class of symmetrical IT2 fuzzy PD or PI controllers [12]. Although this work has shed some lights on how to derive the analytical structure for IT2 fuzzy controllers, it is limited to symmetrical IT2 FLSs whose consequent sets H_1, H_2, H_3, H_4 satisfy $H_1 > H_2 = H_3 > H_4$. In this paper, another class of IT2 fuzzy controllers whose consequent sets satisfy $H_1 > H_2 > H_3 > H_4$ is considered. The main contribution of this paper is to propose an algorithm for deriving the analytical structure of the considered IT2 FLCs and to derive the analytical structure by following the proposed algorithm. The following parts of this paper are organized as: Section II describes the configurations of IT2 fuzzy PD and PI controllers; Section III presents the algorithm to derive their analytical structure, followed by the detailed derivation of the analytical structure of IT2 fuzzy controllers In Section IV. A conclusion is given in Section V.

II. CONFIGURATION OF INTERVAL T2 FUZZY PD AND PI CONTROLLER

Fig. 1 contains the schematic diagram of an IT2 FLS. A two-inputs single-output fuzzy system commonly used for feedback control may be defined as

$$\triangle U(n) = f(E(n), R(n)) \quad (1)$$

where $E(n) = K_e e(n) = K_e(SP(n) - y(n))$, $R(n) = K_r r(n) = K_r(e(n) - e(n-1))$, $y(n)$ is the output of the closed-loop system, $SP(n)$ is the reference signal, K_e and K_r are the scaling constants. The output $\Delta U(n)$ may be interpreted directly as the control signal or as the rate of change in the actuation signal. Depending how the output of the fuzzy controller is defined, it may be interpreted as a fuzzy PD or a fuzzy PI controller.

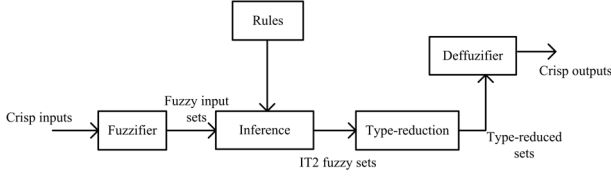


Fig. 1. The structure of IT2 FLS

For this study, two IT2 FSs are used to partition the space of each input: EN and EP for $E(n)$, RN and RP for $R(n)$. By shifting the membership of two symmetrical T1 FSs vertically by the amount of θ_1 , the upper and lower membership of EN and EP , $\overline{EN}$, $\underline{EN}$, $\overline{EP}$ and $\underline{EP}$ in Fig. 2, can be obtained. In the same way, θ_2 represents the amount by which the upper and lower membership of RN and RP are shifted to obtain $\overline{RN}$, $\underline{RN}$, $\overline{RP}$ and $\underline{RP}$ in Fig. 3. Based on whether its value depends on the inputs, the lower bound or upper bound of any antecedent set can be decomposed into dependent part or independent segment (0 or 1). For example,

$$\overline{EN} = \begin{cases} \frac{1}{2L_1} E(n) + 0.5 + \theta_1 & \text{for } -L_1 + P_1 \leq E(n) \leq L_1 + P_1 \\ 0 & \text{for } E(n) \geq L_1 + P_1 \\ 1 & \text{for } E(n) \leq -L_1 + P_1 \end{cases} \quad (2)$$

where $\overline{EN}$ is the segment of the membership function of $\overline{EN}$ that is dependent on the input, while the other part are independent of the input. (2) reveals the relationship between the membership grade and the input.

For fuzzy systems that partition the input space using IT2 fuzzy sets shown in Fig. 2 and 3, a commonly used rule base comprises of the following IF-THEN statements:

- Rule 1: IF $E(n)$ is Positive AND $R(n)$ is Positive THEN $\Delta U(n)$ is H_1
- Rule 2: IF $E(n)$ is Positive AND $R(n)$ is Negative THEN $\Delta U(n)$ is H_2
- Rule 3: IF $E(n)$ is Negative AND $R(n)$ is Positive THEN $\Delta U(n)$ is H_3
- Rule 4: IF $E(n)$ is Negative AND $R(n)$ is Negative THEN $\Delta U(n)$ is H_4

where H_1 , H_2 , H_3 and H_4 are the four singleton consequent fuzzy sets.

The result of the input and antecedent operations is an IT1 set called the firing set. Using Zadeh fuzzy AND operator as

the t-norm operator, the firing set for each rule are as follows:

$$R_1 = [\underline{R}_1, \overline{R}_1] = [\min(\underline{EP}, \underline{RP}), \min(\overline{EP}, \overline{RP})] \quad \text{for } \Delta U(n) = H_1 \quad (3)$$

$$R_2 = [\underline{R}_2, \overline{R}_2] = [\min(\underline{EP}, \underline{RN}), \min(\overline{EP}, \overline{RN})] \quad \text{for } \Delta U(n) = H_2 \quad (4)$$

$$R_3 = [\underline{R}_3, \overline{R}_3] = [\min(\underline{EN}, \underline{RP}), \min(\overline{EN}, \overline{RP})] \quad \text{for } \Delta U(n) = H_3 \quad (5)$$

$$R_4 = [\underline{R}_4, \overline{R}_4] = [\min(\underline{EN}, \underline{RN}), \min(\overline{EN}, \overline{RN})] \quad \text{for } \Delta U(n) = H_4 \quad (6)$$

The output of the fuzzy inference engine, an interval type-2 fuzzy set, is obtained by combining the fired output consequent sets, where each fired output consequent set is the union of the firing set and the output consequent sets. Using the Mendel-John Representation theorem [1] (Theorem 1), the IT2 FS formed by the fuzzy inference engine may be viewed as the collection of all of its embedded IT1 FSs. Hence, the output of the inference engine may be type-reduced into an IT1 set comprising of the centroids of all embedded T1 FSs.

$$\Delta U_j = \frac{R_1^* * H_1 + R_2^* * H_2 + R_3^* * H_3 + R_4^* * H_4}{R_1^* + R_2^* + R_3^* + R_4^*} \quad (7)$$

where R_i^* is a value within the lower and upper bound of the firing set for the i th rule, R_i . In summary, the type-reduced set $\Delta U_{TR}(n)$ may be expressed mathematically as $[\Delta U_j^{min}, \Delta U_j^{max}]$, ΔU_j^{min} and ΔU_j^{max} are respectively the smallest and largest centroid of all the possible embedded T1 FS. Lastly, using the height defuzzification, the crisp output of the IT2 FLC is

$$\Delta U(n) = \frac{\Delta U_j^{min} + \Delta U_j^{max}}{2} \quad (8)$$

The four singleton consequent sets H_4, H_3, H_2, H_1 are equally spaced. Further, it is assumed that $H_4 < H_3 < H_2 < H_1$, which is shown in Fig. 4.

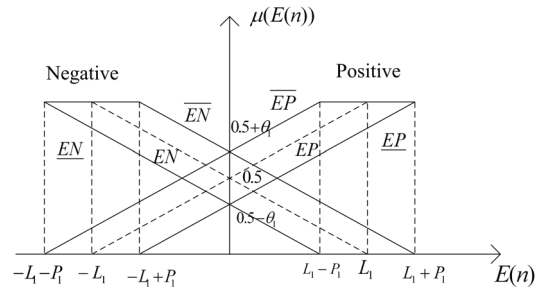


Fig. 2. IT2 FSs EN and EP as antecedent sets for the input $E(n)$ ($P_1 = 2L_1\theta_1$)

III. ALGORITHMS TO DERIVE THE ANALYTICAL STRUCTURE OF IT2 FUZZY PD CONTROLLERS

This section will first establish the mathematical model of the left and right endpoints ΔU_j^{min} and ΔU_j^{max} , and then present the algorithm to identify the mathematical expression for each firing strength in the IT2 fuzzy PD controller.

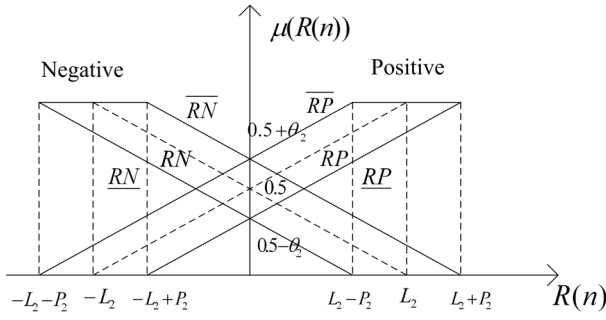


Fig. 3. IT2 FLSs RN and RP as antecedent sets for the input $R(n)$ ($P_2 = 2L_2\theta_2$)

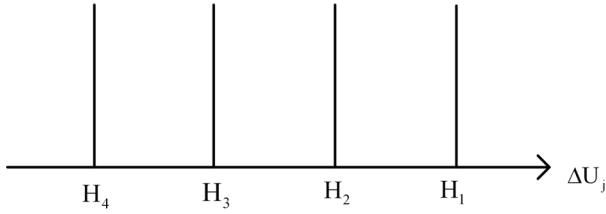


Fig. 4. Singleton consequent fuzzy sets of IT2 fuzzy PD controller

A. Mathematical model for the left endpoint ΔU_j^{min}

From the Karnik-Mendel type-reduction method [2], the left and right endpoint ΔU_j^{min} and ΔU_j^{max} can be re-expressed as

$$\Delta U_j^{min} = \frac{\sum_{i=1}^{L-1} \underline{R}_i H_i + \sum_{i=L}^4 \bar{R}_i H_i}{\sum_{i=1}^{L-1} \underline{R}_i + \sum_{i=L}^4 \bar{R}_i} \quad (9)$$

$$\Delta U_j^{max} = \frac{\sum_{i=1}^{R-1} \bar{R}_i H_i + \sum_{i=R}^4 \underline{R}_i H_i}{\sum_{i=1}^{R-1} \bar{R}_i + \sum_{i=R}^4 \underline{R}_i} \quad (10)$$

where the switch point L and R coincide with the location of one of the 3 singleton consequent sets (H_1, H_2, H_3, H_4). From the derivation of the Karnik-Mendel type-reduction method [2], the following can be observed:

$$H_L \leq \Delta U_j^{min} < H_{L-1} \quad (11)$$

$$H_R \leq \Delta U_j^{max} < H_{R-1} \quad (12)$$

Due to the constraint that the switch point L and R must be positioned at one of the three singleton consequent sets and the inequalities in (11) and (12), it may be concluded that L and R can assume one of the three values i.e. $L = \{4, 3, 2\}$ and $R = \{4, 3, 2\}$. Hence, the only three possible expressions

for ΔU_j^{min} may be written as

$$\begin{aligned} \text{Mode 1 :} \quad & \text{When } H_4 \leq \Delta U_j^{min} \leq H_3 \Leftrightarrow L = 4, \\ & \Delta U_j^{min} = \Delta U_{j1}^{min} \\ & = \frac{\bar{R}_4 * H_4 + \bar{R}_3 * H_3 + \underline{R}_2 * H_2 + \underline{R}_1 * H_1}{\bar{R}_4 + \bar{R}_3 + \underline{R}_2 + \underline{R}_1} \end{aligned} \quad (13)$$

$$\begin{aligned} \text{Mode 2 :} \quad & \text{When } H_3 \leq \Delta U_j^{min} \leq H_2 \Leftrightarrow L = 3 \\ & \Delta U_j^{min} = \Delta U_{j2}^{min} \\ & = \frac{\bar{R}_4 * H_4 + \bar{R}_3 * H_3 + \underline{R}_2 * H_2 + \underline{R}_1 * H_1}{\bar{R}_4 + \bar{R}_3 + \underline{R}_2 + \underline{R}_1} \end{aligned} \quad (14)$$

$$\begin{aligned} \text{Mode 3 :} \quad & \text{When } H_2 \leq \Delta U_j^{min} \leq H_1 \Leftrightarrow L = 2 \\ & \Delta U_j^{min} = \Delta U_{j3}^{min} \\ & = \frac{\bar{R}_4 * H_4 + \bar{R}_3 * H_3 + \bar{R}_2 * H_2 + \underline{R}_1 * H_1}{\bar{R}_4 + \bar{R}_3 + \bar{R}_2 + \underline{R}_1} \end{aligned} \quad (15)$$

Based on the output conditions for Mode 1 ($H_4 \leq \Delta U_j^{min} \leq H_3$), Mode 2 ($H_3 \leq \Delta U_j^{min} \leq H_2$) and Mode 3 ($H_2 \leq \Delta U_j^{min} \leq H_1$), the input conditions when the three embedded T1 FLSs in (13)-(15) are respectively used as the left endpoint ΔU_j^{min} may be established as follows:

- 1) The condition when the switch point changes from the position of $L = 4$ to $L = 3$ and vice versa can be established as

$$\Delta U_j^{min} = \Delta U_{j1}^{min} = \Delta U_{j2}^{min} = H_3 \quad (16)$$

By replacing ΔU_{j1}^{min} and ΔU_{j2}^{min} with the expressions (13) and (14), the above equation may be written as

$$\bar{R}_4(H_4 - H_3) = \underline{R}_2(H_3 - H_2) + 2\underline{R}_1(H_3 - H_1)$$

Because of the assumption that the four consequent sets are equally spaced and $H_4 < H_2 < H_3 < H_1$, the condition established above may be reduced to

$$\bar{R}_4 = \underline{R}_2 + 2\underline{R}_1 \quad (17)$$

Further, the input subregions where (13) is used to calculate the left endpoint of the type-reduced set (Mode 1) can be found using its corresponding output condition:

$$\Delta U_j^{min} = \Delta U_{j1}^{min} \leq H_3 \Leftrightarrow \bar{R}_4 \geq \underline{R}_2 + 2\underline{R}_1 \quad (18)$$

and the areas where the IT2 fuzzy PD controller operate in Mode 2, i.e. the output is defined by (14), should satisfy the condition:

$$\Delta U_j^{min} = \Delta U_{j2}^{min} \geq H_3 \Leftrightarrow \bar{R}_4 \leq \underline{R}_2 + 2\underline{R}_1 \quad (19)$$

- 2) The condition when the switch point changes from the position of $L = 3$ to $L = 2$ and vice versa can be established as

$$\Delta U_j^{min} = \Delta U_{j2}^{min} = \Delta U_{j3}^{min} = H_2 \quad (20)$$

By replacing ΔU_{j2}^{min} and ΔU_{j3}^{min} with the corresponding expression in (14)-(15), the above equation may be written as

$$2\bar{R}_4 + \bar{R}_3 = \underline{R}_1 \quad (21)$$

Further, the subregions where (14) is used to calculate the left endpoint of the type-reduced set (Mode 2) can be found using the following equality

$$\Delta U_j^{min} = \Delta U_{j1}^{min} \leq H_2 \Leftrightarrow 2\bar{R}_4 + \bar{R}_3 \geq \underline{R}_1 \quad (22)$$

and the areas where the IT2 fuzzy PD controller operates in Mode 3, i.e. the output is defined by (15), should satisfy the condition:

$$\Delta U_j^{min} = \Delta U_{j3}^{min} \geq H_2 \Leftrightarrow 2\bar{R}_4 + \bar{R}_3 \leq \underline{R}_1 \quad (23)$$

B. Methodology for deriving analytical structure of IT2 fuzzy PD controller

An investigation of the conditions (18)-(19) and (22)-(23) reveals that

- 1) As stated above, the regions in the input space where the left endpoint ΔU_j^{min} is operating at Mode 1 and Mode 2 satisfy the condition (18) and (19) respectively. For the embedded T1 FLS of Mode 3, its corresponding region in the input space also satisfies (19) because its output condition $\Delta U_j^{min} \geq H_2$ and the assumption $H_2 > H_3$ may lead to $\Delta U_j^{min} > H_3$. Due to the complementarities between (18) and (19), the inequality (18) can be used to identify the corresponding region for the embedded T1 FLS of Mode 1.
- 2) Similarly, the regions in the input space which correspond to Mode 2 and Mode 3 satisfy (22) and (23) respectively. The corresponding region of Mode 1 in the input space also satisfies the inequality (22) because its output condition $\Delta U_j^{min} \leq H_3$ and the assumption $H_3 < H_2$ may result in $\Delta U_j^{min} < H_2$. Consequently, the region for the embedded T1 FLS of Mode 3 can be identified using the inequality (23).

On the other hand, by comparing the expressions of the three embedded T1 FLSs in (13)-(15), their properties can be generalized as

- 1) The weight associated with H_1 is always the lower bound of the firing set for Rule 1. Similarly, H_4 is weighted by the upper bound of the firing set for Rule 4. Consequently, the weight on H_1 and H_4 is independent of the switch points L and R .
- 2) In Mode 1, H_3 is weighted by the lower bound $\underline{R}_3$, while the corresponding upper bounds $\bar{R}_3$ is used to weight H_3 in Mode 2 and 3. On the other hand, the lower bound $\underline{R}_2$ weights H_2 in Mode 1 and 2, while in Mode 3 H_2 is weighted by the corresponding upper bound $\bar{R}_2$.

The first property shows that the firing strength of Rule 1 and Rule 4 used to calculate ΔU_j^{min} is independent of the switch point L , and the firing strength of Rule 1 and Rule 4 is always governed by the lower and upper bound of the firing set i.e. $\underline{R}_1$ and $\bar{R}_4$. Based on this observation, the first step in partitioning the input space in order to derive closed form firing level is performed by considering the outcomes of Zadeh AND operations (minimum operator) for Rule 1 and Rule 4. As stated in the second property, the firing strength for Rule 2 and Rule 3 can be the lower or upper bound of their corresponding

firing set, depending on which embedded T1 FLS in (13)-(15) dominates the sub-region. Hence, the specification of the firing strength for Rule 2 and Rule 3 can be done only after the regions for the three embedded T1 FLSs in (13)-(15) are identified.

Because the conditions (18) and (23) contains three of R_1 , R_2 , R_3 and R_4 , the challenge is how to identify the sub-regions in the input space for the three embedded T1 FLSs when only the expressions of R_1 and R_4 are known. To achieve the goal of identifying the regions for Mode 1 and Mode 3, the following technique will be used in the second step:

- Assuming that $\underline{R}_2$ weights H_2 in the whole input space, the firing strength of Rule 2 can be specified by considering the following Zadeh AND operation:

$$\underline{R}_2 = \min(\underline{\mathbf{EP}}, \underline{\mathbf{RN}}) \quad (24)$$

Then the relative value of $\underline{R}_1$, $\underline{R}_2$ and $\bar{R}_4$ can be examined via (18) to identify the region for Mode 1.

This assumption does not enlarge the area of the region for Mode 1 because in the region where H_2 is weighted by its upper bound $\bar{R}_2$, the property $2\bar{R}_4 + \bar{R}_3 \leq \underline{R}_1$ holds which leads to $\bar{R}_4 < \underline{R}_2 + 2\underline{R}_1$ and thus contradicts the inequality (18).

- Under the assumption that H_3 is weighted by its corresponding upper bound $\bar{R}_3$, the firing strength of Rule 3 can be mathematically expressed by considering the following Zadeh operation:

$$\bar{R}_3 = \min(\bar{\mathbf{EN}}, \bar{\mathbf{RP}}) \quad (25)$$

According to (23), the relative value of $\underline{R}_1$, $\bar{R}_3$ and $\bar{R}_4$ may be examined to determine the sub-regions for Mode 3.

The assumption made above does not affect the size of the region for Mode 3. This is because the regions where the lower bound $\underline{R}_3$ weights H_3 satisfies $\bar{R}_4 \geq \underline{R}_2 + 2\underline{R}_1$ which contradicts the inequality (23).

Once the regions for Mode 1 and Mode 3 are respectively identified, the firing strength for Rule 2 and Rule 3 may also be mathematically expressed by considering the corresponding Zadeh AND operations in the subregions. In summary, the proposed algorithm to derive the mathematical expression of the firing strength in ΔU_j^{min} may be presented as:

Step 1 : The firing strength of Rule 1 ($\underline{R}_1$) and Rule 4 ($\bar{R}_4$) can be specified by dividing the input space via the outcomes of Zadeh AND operations (minimum operator) for Rule 1 and Rule 4 i.e.

$$\underline{R}_1 = \min\{\underline{\mathbf{EP}}, \underline{\mathbf{RP}}\} \quad (26)$$

$$\bar{R}_4 = \min\{\bar{\mathbf{EN}}, \bar{\mathbf{RN}}\} \quad (27)$$

Step 2 : This step is to identify the regions in the input space that correspond to Mode 1 and Mode 3 respectively. Then the other region must correspond to Mode 2.

- 1) *Step 2.1* : Assuming that H_2 is weighted by its corresponding lower bound $\underline{R}_2$ in the whole input space, $\underline{R}_2$

can be specified as

$$\underline{R}_2 = \min(\underline{EP}, \underline{RN}) \quad (28)$$

The partitions obtained using $\underline{R}_1$, $\bar{R}_4$ in Step 1 and $\underline{R}_2$ are further examined to find out the group for Mode 1 (M_{L1}) satisfying the following inequality (18):

$$\text{Mode 1 : } \bar{R}_4 \geq \underline{R}_2 + 2\underline{R}_1$$

- 2) *Step 2.2* : Assume that in the input space H_3 is always weighted by its upper bound $\bar{R}_3$ and thus specify $\bar{R}_3$ as

$$\bar{R}_3 = \min(\bar{EN}, \bar{RP}) \quad (29)$$

The relative values of $\bar{R}_4$, $\underline{R}_1$ obtained in Step 1 and $\bar{R}_3$ are examined to find out the group for Mode 3 (M_{L3}) which satisfies the inequality (23):

$$\text{Mode 3: } 2\bar{R}_4 + \bar{R}_3 \leq \underline{R}_1$$

Step 3 : In Step 3, the firing strength associated with Rule 2 and 3 can be specified via the Zadeh AND operation.

- 1) *Step 3.1* : Since Step 2.1 divides the whole input space into two parts: the group M_{L1} for Mode 1 and the other region for Mode 2 and 3, the firing strength associated with Rule 3 can be respectively specified as follows:
In the group M_{L1} , H_3 is weighted by its lower bound $\underline{R}_3$. By considering the Zadeh AND operation the firing strength for Rule 3 may be specified as

$$\underline{R}_3 = \min\{\underline{EN}, \underline{RP}\} \quad (30)$$

In the other regions for Mode 2 and Mode 3, the corresponding upper bound $\bar{R}_3$ weights H_3 and thus the firing strength for Rule 3 can be specified as

$$\bar{R}_3 = \min\{\bar{EN}, \bar{RP}\} \quad (31)$$

- 2) *Step 3.2* : The whole input space is divided into two parts in Step 2.2: the group M_{L3} for Mode 3 and the other region for Mode 1 and Mode 2. Therefore, the firing strength for Rule 2 can be specified by using Zadeh AND operation to divide the input space as follows:
The upper bound $\bar{R}_2$ weights H_2 in the region M_{L3} . The regions M_{L3} is divided by the Zadeh AND operation as

$$\bar{R}_2 = \min\{\bar{EP}, \bar{RN}\} \quad (32)$$

In the other regions, H_2 is weighted by the corresponding lower bound $\underline{R}_2$. The Zadeh AND operation divides the regions as

$$\underline{R}_2 = \min\{\underline{EP}, \underline{RN}\} \quad (33)$$

Step 4 : Superimpose all the partitions obtained by considering the mode switch and Zadeh AND operations.

Similarly, the firing strength for the right endpoint ΔU_j^{max} can be specified.

IV. DERIVATION OF THE ANALYTICAL STRUCTURE OF IT2 FUZZY PD CONTROLLER

By following the proposed algorithm presented in the last section, the analytical structure of the fuzzy PD controller will be derived in this section.

A. The specification of the firing strength

By following the proposed algorithm, the specification of the firing strength for the left endpoint ΔU_j^{min} will be derived as an example to show the procedures:

Step 1 is to mathematically express the firing strength for Rule 1 ($\underline{R}_1$) and Rule 4 ($\bar{R}_4$) by dividing the input space into regions.

Step 2 is to sperate the input space into the three groups where the three embedded T1 FLSs in (13)-(15) are respectively used as the left endpoint ΔU_j^{min} .

- 1) By superimposing the partitions of $\bar{R}_1$, $\underline{R}_3$ and $\underline{R}_4$, nine regions in Fig. 5(a) are generated: S_{L1}^1 - S_{L9}^1 . Whether the IT2 fuzzy PD controller operates in Mode 1 (M_{L1}) depends on the relative values of $\bar{R}_4$, $\underline{R}_1$ and $\underline{R}_2$.

- Region S_{L1}^1 : $\underline{R}_2 = \underline{R}_1 = 0$ and $\bar{R}_4 > 0$. Hence, $\bar{R}_4 > \underline{R}_2 + 2\underline{R}_1$ and thus for any input pair in Region S_{L1}^1 , the IT2 fuzzy PD controller is operating in Mode 1. ($S_{L1}^1 \in M_{L1}$)
- Region S_{L2}^1 : $\bar{R}_4 = \bar{RN}$, $\underline{R}_2 = 0$, $\underline{R}_1 = \underline{EP}$. The inequality (18) can be simplified into $\bar{RN} \geq 2\underline{EP}$. As shown in Fig. 5(b), the condition $\bar{RN} = 2\underline{EP}$ is denoted by the red line EG .
- Region S_{L3}^1 : $\bar{R}_4 = \bar{EN}$, $\underline{R}_2 = 0$, $\underline{R}_1 = \underline{RP}$. The fact $\bar{EN} < \bar{RN}$ and $\bar{RN} < \underline{RP}$ in Region S_{L3}^1 lead to $\bar{EN} < \underline{RP}$. Hence, in Region S_{L3}^1 , the property $\bar{R}_4 < \underline{R}_2 + 2\underline{R}_1$ holds and the IT2 fuzzy PD controller operates in Mode 2 or 3.
- Region S_{L4}^1 : $\bar{R}_4 = \bar{RN}$, $\underline{R}_2 = \underline{RN}$, $\underline{R}_1 = \underline{EP}$. The inequality (18) leads to $\bar{RN} \geq \underline{RN} + 2\underline{EP}$ which is the left side of the red line IJ in Fig.5(b).
- Region S_{L5}^1 : $\bar{R}_4 = \bar{RN}$, $\underline{R}_2 = \underline{EP}$, $\underline{R}_1 = \underline{EP}$ and then the condition (18) can be simplified into $\bar{RN} > 3\underline{EP}$. The line EF includes all input pairs where $\bar{RN} = 3\underline{EP}$ holds in Region S_{L5}^1 .
- Region S_{L6}^1 : $\bar{R}_4 = \bar{EN}$, $\underline{R}_2 = \underline{EP}$, $\underline{R}_1 = \underline{RP}$. The condition in (18) leads to $\bar{EN} \geq \underline{EP} + 2\underline{RP}$. The boundary $\bar{EN} = \underline{EP} + 2\underline{RP}$ is denoted by the red line MG in Fig.5(b).
- Region S_{L7}^1 : $\bar{R}_4 = \bar{EN}$, $\underline{R}_2 = \underline{RN}$, $\underline{R}_1 = \underline{RP}$. Due to the fact $\bar{R}_4 < \underline{R}_2 + 2\underline{R}_1$ in Region S_{L7}^1 , the IT2 fuzzy PD controller is operating at Mode 2 or 3.
- Region S_{L8}^1 : $\bar{R}_4 = \bar{EN}$, $\underline{R}_2 = \underline{EP}$, $\underline{R}_1 = 0$. The condition (18) can be reduced to $\bar{EN}_U \geq \underline{EP}_L$, which is the left side of the red line GN in Fig.5(b).
- Region S_{L9}^1 : $\bar{R}_4 = \bar{EN}$, $\underline{R}_2 = \underline{RN}$, $\underline{R}_1 = 0$. In Region S_{L9}^1 , there exists $\bar{R}_4 < \underline{R}_2 + 2\underline{R}_1$ and thus Region S_{L9}^1 does not belong to M_{L1} .

- 2) *Step 2.2* seeks to identify the region M_{L3} where the IT2 fuzzy PD controller is operating at Mode 3. As stated in the proposed algorithm, the superimposition of the partitions of $\underline{R}_1$, $\bar{R}_3$ and $\bar{R}_4$ creates 5 regions in Fig. 6(a): S_{L1}^2 - S_{L5}^2 . As shown in Fig. 6(b), in the regions above the line QSV , the IT2 fuzzy controller is operating at Mode 3 while in the other regions, the fuzzy controller works at Mode 1 or Mode 2.

In this manner, the whole input space is divided into the three parts: M_{L1} , M_{L2} and M_{L3} , where the fuzzy controller is operating at Mode 1, Mode 2 or Mode 3, respectively.

Step 3 This step is to specify the firing strength for Rule 2 and Rule 3 by considering the corresponding Zadeh AND operation in the subregions for Mode 1-3.

- 1) In the region M_{L1} for Mode 1, the firing strength for Rule 3 can be specified by considering the following Zadeh AND operation: $\underline{R}_3 = \min\{\underline{EN}, \underline{RP}\}$ while in the other regions M_{L2} and M_{L3} , the following Zadeh AND operation is considered to specify the firing strength for Rule 3: $\overline{R}_3 = \min\{\overline{EN}, \overline{RP}\}$. The red and blue lines in Fig. 7 show the divisions of the input space for Rule 3.
- 2) As shown in Fig. 7, the red lines and yellow lines show the partition for Rule 2. In the subregion M_{L1} and M_{L2} , H_2 is weighted by the lower bound $\underline{R}_2$. Hence, the firing strength for Rule 2 in these regions can be specified by considering $\underline{R}_2 = \min\{\underline{EP}, \underline{RN}\}$. In the subregion M_{L3} , the firing strength for Rule 2 can be specified using $\overline{R}_2 = \min\{\overline{EP}, \overline{RN}\}$.

Step 4 Similarly, the partitions by the right endpoint ΔU_j^{max} can be derived in Fig. 8. At last, the partition of the input space for the IT2 fuzzy controllers in Fig. 9 can be obtained by superimposing all the results derived in Step 1-3.

B. The expressions of IT2 fuzzy PD controller

Fig. 10 shows the partitions of the input space when $\theta_1 = \theta_2$. The mathematical expressions of the firing strengths for 34 regions in Fig. 10 has been listed in Table I. By replacing the firing strength in (9) and (10) with their mathematical expressions, the mathematical equations relating the output of the fuzzy controller and its inputs can be derived. The region IC 1 is taken as an example to show the procedure. Table I shows that in IC1, $R_4 = R_2 = \overline{RN} = -\frac{1}{2L_2}R(n) + 0.5 + \theta_2$, $R_3 = \overline{EN} = -\frac{1}{2L_1}E(n) + 0.5 + \theta_1$, $R_1 = \underline{EP} = \frac{1}{2L_1}E(n) + 0.5 - \theta_1$. After simple manipulation, ΔU_j^{min} for IC 1 can be expressed into functions of $E(n)$ and $R(n)$ in (34).

C. Discussion

Comparing with the IT2 FLSs in [12], the introduction of the additional consequent set results in three possible embedded T1 FLSs for each endpoint ΔU_j^{min} or ΔU_j^{max} , instead of the two in [12]. An observation of Fig. 7 and 8 indicates that the input space is divided into three parts that corresponds to the embedded T1 FLSs which produce the left or right endpoint ΔU_j^{min} or ΔU_j^{max} . A comparative study of the partitions in Fig. 9 and that in [12] shows the introduced parameters produces 14 more regions, which indicates the potential of the considered IT2 fuzzy controllers to produce better performance.

V. CONCLUSION

In this paper, the analytical structure of the IT2 fuzzy controller has been derived by following the proposed algorithm. The relative large number of divisions provides the potential to

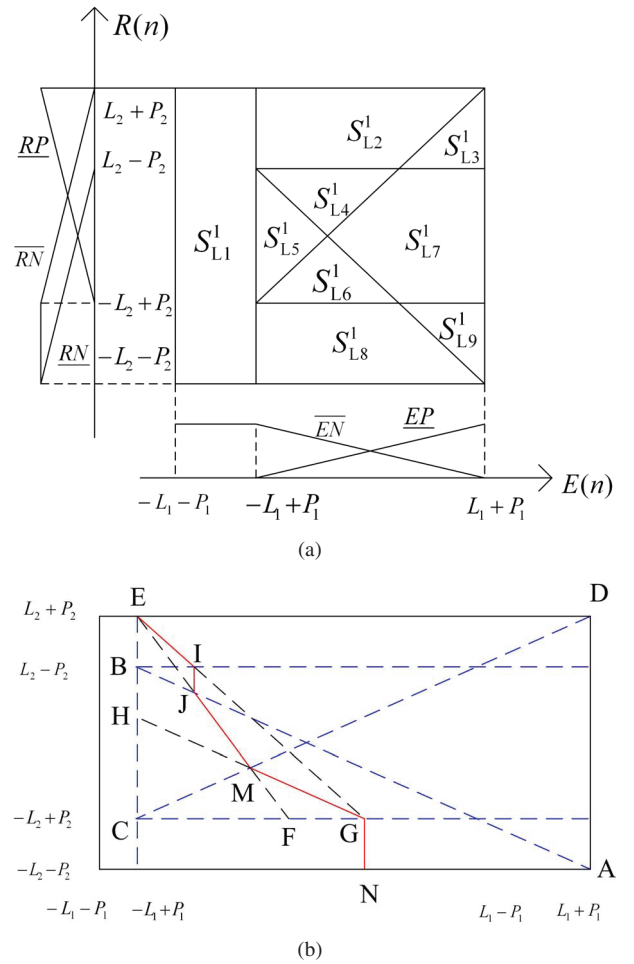


Fig. 5. Partition of the input space by ΔU_j^{min} (a) The superimposition of $\overline{R}_4$, $\underline{R}_1$ and $\underline{R}_2$. (b) The boundary between subregions for Mode 1 and the other modes: $R_4 = \underline{R}_2 + 2\underline{R}_1$

produce better performance. The unique divisions may assist the research community to understand the configurations of the considered IT2 fuzzy controllers. The equations relating the output and the input of the IT2 fuzzy controller provide tools for further investigations of their characteristics.

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$$\begin{aligned}
\Delta U_{jIC1}^{min} &= \frac{R_4^* * H_4 + R_3^* * H_3 + R_2^* * H_2 + R_1^* * H_1}{R_4^* + R_3^* + R_2^* + R_1^*} \\
&= \frac{\overline{RN} * H_4 + \overline{EN} * H_3 + \overline{RN} * H_2 + \underline{EP} * H_1}{\overline{RN} + \overline{EN} + \overline{RN} + \underline{EP}} \\
&= \frac{[-\frac{1}{2L_2}R(n) + 0.5 + \theta_1](H_2 + H_4) + [-\frac{1}{2L_1}E(n) + 0.5 + \theta_1]H_3 + [\frac{1}{2L_1}E(n) + 0.5 - \theta_1]H_1}{2[-\frac{1}{2L_2}R(n) + 0.5 + \theta_2] + [-\frac{1}{2L_1}E(n) + 0.5 + \theta_1] + [\frac{1}{2L_1}E(n) + 0.5 - \theta_1]} \\
&= \frac{-L_1(H_2 + H_4)R(n) + L_2(H_1 - H_3)E(n)}{4L_1L_2(1 + \theta_2) - 2L_1R(n)} \\
&\quad - \frac{2L_1L_2[(0.5 + \theta_1)(H_2 + H_4) + (0.5 - \theta_1)H_1 + (0.5 + \theta_1)H_3]}{4L_1L_2(1 + \theta_2) - 2L_1R(n)} \\
&= K_p^1 E(n) + K_d^1 R(n) + \delta^1
\end{aligned} \tag{34}$$

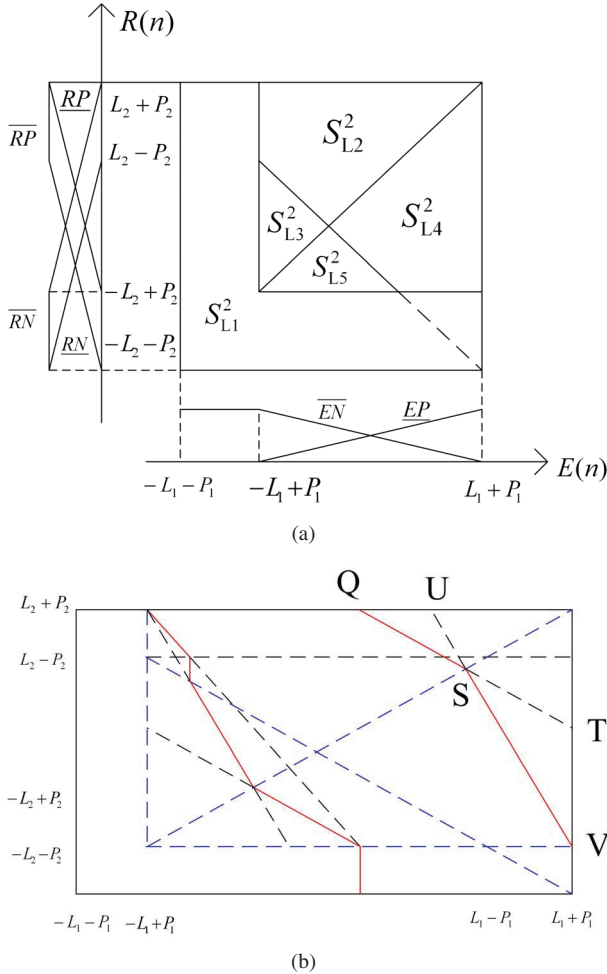


Fig. 6. Partition of the input space by ΔU_j^{min} : (a) The superimposition of $\overline{R}_4$, $\underline{R}_1$ and $\overline{R}_3$. (b) The boundary between subregions for Mode 3 and the other modes: $2\overline{R}_4 + \overline{R}_3 = \underline{R}_1$

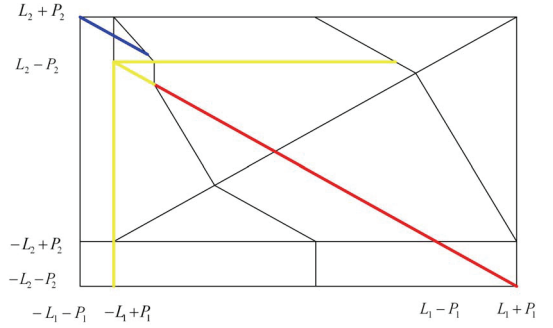


Fig. 7. The partition of the input space for the left endpoint ΔU_j^{min}

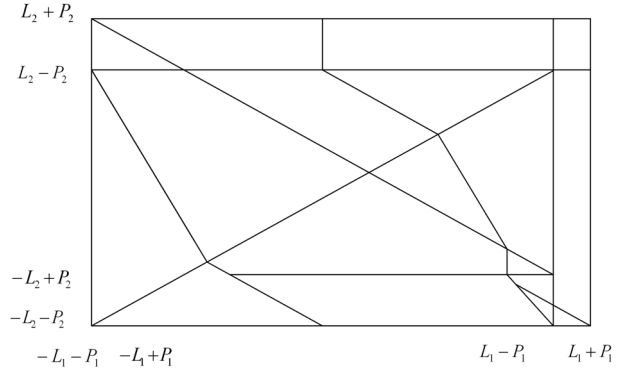


Fig. 8. The partition of the input space for the right endpoint ΔU_j^{max}

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TABLE I

THE FIRING STRENGTHS OF FOUR RULES IN ΔU_j^{min}

IC No.	ΔU_j^{min}			
	Rule 4	Rule 3	Rule 2	Rule 1
1	$\overline{RN}$	$\overline{EN}$	$\overline{RN}$	$\overline{EP}$
2	$\overline{RN}$	$\overline{EN}$	$\overline{RN}$	$\overline{EP}$
3	$\overline{RN}$	$\overline{EN}$	$\overline{RN}$	$\overline{EP}$
4	$\overline{RN}$	$\overline{RP}$	$\overline{EP}$	$\overline{EP}$
5	$\overline{RN}$	$\overline{EN}$	$\overline{RN}$	$\overline{EP}$
6	$\overline{RN}$	$\overline{RP}$	$\overline{EP}$	$\overline{EP}$
7	$\overline{RN}$	$\overline{RP}$	$\overline{EP}$	$\overline{EP}$
8	$\overline{EN}$	$\overline{RP}$	$\overline{EP}$	$\overline{RP}$
9	$\overline{EN}$	$\overline{RP}$	$\overline{EP}$	$\overline{RP}$
10	$\overline{EN}$	$\overline{RP}$	$\overline{EP}$	$\overline{RP}$
11	$\overline{EN}$	$\overline{EN}$	$\overline{RN}$	$\overline{RP}$
12	$\overline{EN}$	$\overline{EN}$	$\overline{RN}$	$\overline{RP}$
13	$\overline{EN}$	$\overline{EN}$	$\overline{RN}$	$\overline{RP}$
14	$\overline{EN}$	$\overline{RP}$	$\overline{EP}$	$\overline{RP}$
15	$\overline{EN}$	$\overline{EN}$	$\overline{RN}$	$\overline{RP}$
16	$\overline{RN}$	$\overline{EN}$	$\overline{RN}$	$\overline{EP}$
17	$\overline{RN}$	$\overline{EN}$	$\overline{RN}$	$\overline{EP}$
18	$\overline{RN}$	$\overline{EN}$	0	$\overline{EP}$
19	$\overline{RN}$	$\overline{EN}$	0	$\overline{EP}$
20	$\overline{RN}$	$\overline{EN}$	0	$\overline{EP}$
21	$\overline{RN}$	$\overline{EN}$	0	0
22	$\overline{RN}$	$\overline{RP}$	0	0
23	$\overline{RN}$	$\overline{RP}$	0	0
24	$\overline{RN}$	$\overline{RP}$	0	0
25	1	0	0	0
26	1	0	0	0
27	$\overline{EN}$	0	$\overline{EP}$	0
28	$\overline{EN}$	0	$\overline{EP}$	0
29	$\overline{EN}$	$\overline{RP}$	$\overline{EP}$	0
30	$\overline{EN}$	$\overline{RP}$	$\overline{EP}$	0
31	$\overline{EN}$	$\overline{RP}$	$\overline{EP}$	0
32	$\overline{EN}$	$\overline{EN}$	$\overline{RN}$	0
33	$\overline{EN}$	$\overline{EN}$	$\overline{RN}$	$\overline{RP}$
34	$\overline{EN}$	$\overline{EN}$	$\overline{RN}$	$\overline{RP}$

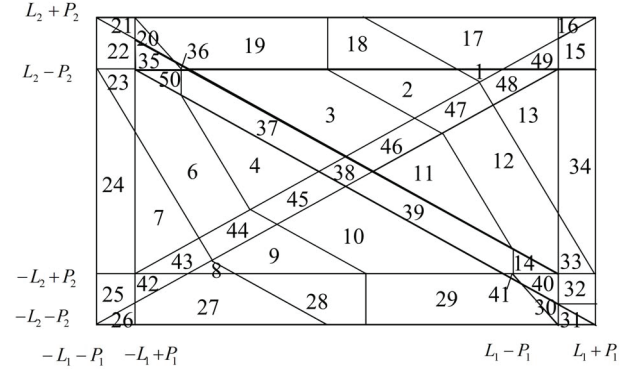
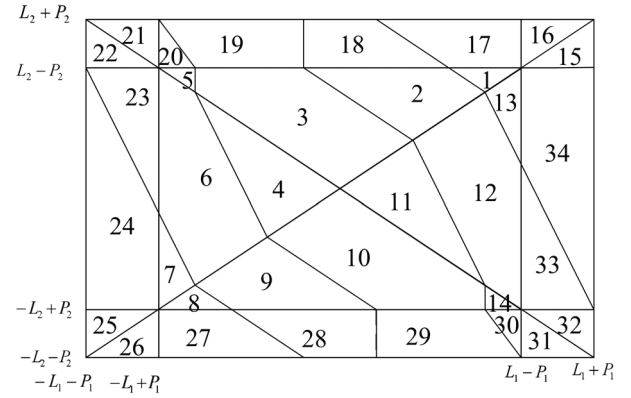


Fig. 9. The partition of the input space for the IT2 FLC

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Fig. 10. The partition of the input space for the IT2 FLC when $\theta_1 = \theta_2$