

Estimation of the Asymptotic Stability Region of Uncertain Fuzzy Systems with Bounded Controllers Using Variable Structure System Design Approach

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Abstract—In this paper, the problem of estimating the asymptotic stability region (ASR) of uncertain nonlinear systems is considered. Our approach is based on Takagi-Sugeno (T-S) fuzzy model and variable structure control (VSC) technique. To simplify the problem, we use a state transformation to reduce the system order of the T-S fuzzy systems. Then we design the sliding surface for the transformed system. By the Lyapunov stability theorem, the ASR can be estimated. A simulation example is presented to verify the effectiveness of the proposed scheme.

Keywords—component; asymptotic stability region; Takagi-Sugeno fuzzy model; variable structure control; sliding mode controller

I. INTRODUCTION

As we know that, design a sliding mode control system[1-5], only need to use a rough mathematical model to describe the physical system. Design of the controller and analysis of the complex nonlinear systems is very difficult task. Thus, the fuzzy technique has been widely adopted to model complex nonlinear plants [6]. Among the many types of fuzzy modules, the Takagi-Sugeno (T-S) fuzzy model [7] is a popular and convenient tool for nonlinear system. T-S type fuzzy models which are the local dynamics in different state-space regions are represented by linear models and the nonlinear system is approximated by the overall fuzzy linear models. In recent years, the problem of fuzzy control systems based on T-S model for complex nonlinear system has become more and more popular [8-17].

Although the complex nonlinear system can be represented by T-S model, there are many uncertainties in actual situations for the nonlinear system. The uncertainty may include parameter variations, modeling errors, extraneous disturbances, and fuzzy approximation errors, etc. To solve this problem, some authors [5, 18-21], use the variable structure control (VSC) theory to deal with the problem of an uncertainty, it can more effectively achieve the control objective.

The VSC technique has been used to control complex nonlinear systems with their T-S fuzzy models [19-23]. The robust variable structure controller is designed to ensure the existence of a sliding mode and to overcome the effect of the uncertainty. If the VSC theory and T-S fuzzy model are

combined to apply to the nonlinear systems, the dynamic performance of nonlinear systems can be improved by taking advantage of their respective merit [24-27].

In VSS, due to physical constraints, the control input magnitude must be bounded. The problem of estimating the asymptotic stability region (ASR) for uncertain systems with bounded sliding mode controllers is proposed in [28]. In [28], the conception of combining different Lyapunov function along with a state transformation is used to estimate ASR. By using an example and under the same conditions, the method of estimating the ASR in [29] is better than in [28]. These methods [28, 29] of using state transformation matrices are indirect and complex. So, they do not seem to be useful in designing a sliding surface. Thus, the method of using linear matrix inequalities (LMIs) to deal with the problem is presented [30], and its method of estimating the ASR is better than the methods presented in [28, 29]. But, these cases are limited to the explicit linear system [28 - 30], their examples are only the numerical example, it does not seem to be useful in estimating the ASR of the real model. In this paper, the method of estimating the ASR via T-S fuzzy modeling method and VSC technique is presented. Our approach is to use the T-S fuzzy model to approximately a class of the complex nonlinear systems, and design the sliding mode controllers for the T-S fuzzy model.

II. SYSTEM DESCRIPTION

The fuzzy model is described by fuzzy If-Then rules and will be employed here to deal with the control design problem for nonlinear systems. In this section, we consider the following uncertain T-S fuzzy control systems with r plant rules that can be represented as:

Plant Rule i :

If ϕ_1 is F_1^i and $\dots$ and ϕ_n is F_n^i

Then

$$\dot{\mathbf{x}}(t) = (\mathbf{A}_i + \Delta\mathbf{A}_i(t))\mathbf{x}(t) + (\mathbf{B}_i + \Delta\mathbf{B}_i(t))\mathbf{u}(t) + \mathbf{g}_i(t, \mathbf{x}) \quad (1)$$

where $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]^T \in \mathfrak{X}^n$ is the state vector,

$\mathbf{u}(t) \in \mathfrak{U}^m$ is the input vector, $\phi_j (j = 1, 2, \dots, n)$ are the state

variables, $F_1^i, \dots, F_n^i$ are the fuzzy sets ($i = 1, 2, \dots, r$),

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$\mathbf{A}_i \in \mathfrak{R}^{n \times n}$ and $\mathbf{B}_i \in \mathfrak{R}^{n \times m}$ are system matrices, $\Delta \mathbf{A}_i(t)$ and $\Delta \mathbf{B}_i(t)$ stand for the uncertainties in $\mathbf{A}_i$ and $\mathbf{B}_i$, respectively, $\mathbf{g}_i(t, \mathbf{x})$ is the external disturbances.

Given the a pair $(\mathbf{x}, \mathbf{u})$, using the singleton fuzzifier, product inference, and center average defuzzifier [7], the final state of the uncertain T-S fuzzy control systems is obtained as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^r h_i(\phi_j) [(\mathbf{A}_i + \Delta \mathbf{A}_i(t)) \mathbf{x}(t) + (\mathbf{B}_i + \Delta \mathbf{B}_i(t)) \mathbf{u}(t) + \mathbf{g}_i(t, \mathbf{x})] \quad (2)$$

where $h_i(\phi_j)$ is the fuzzy basis functions given as

$$h_i(\phi_j) = \frac{\prod_{j=1}^n \mu_{ij}(\phi_j)}{\sum_{i=1}^r \prod_{j=1}^n \mu_{ij}(\phi_j)}, i=1, 2, \dots, r \quad (3)$$

in which $\mu_{ij}(x_j(t))$ is the grade of membership function of $x_j(t)$ in F_j^i , $h_i(\phi_j)$ can be regarded as the normalized weight of each If-Then rule and satisfies

$$h_i(\phi_j) \geq 0, i=1, \dots, r \quad (4)$$

and

$$\sum_{i=1}^r h_i(\phi_j) = 1 \quad (5)$$

To simplify the formula (2), we make the following assumptions:

Assumption 1 [21]: There exist $\beta_A(t) \in \mathfrak{R}^{m \times n}$, $\beta_B(t) \in \mathfrak{R}^{m \times n}$ and $\mathbf{f}(t, \mathbf{x}) \in \mathfrak{R}^{m \times 1}$ such that $\Delta \mathbf{A}_i(t) = \mathbf{B}_i \beta_A(t)$, $\Delta \mathbf{B}_i(t) = \mathbf{B}_i \beta_B(t)$ and $\mathbf{g}_i(t) = \mathbf{B}_i \mathbf{f}(t, \mathbf{x})$.

Assumption 2 [21]: There exist known continuous positive scalar-valued functions $\rho_A(t, \mathbf{x})$, $\rho_f(t, \mathbf{x})$ and known constant $0 \leq \varphi < 1$ such that $\|\beta_A(t) \mathbf{x}(t)\| \leq \rho_A(t, \mathbf{x})$, $\|\mathbf{f}(t, \mathbf{x})\| \leq \rho_f(t)$, and $\|\beta_B(t)\| \leq \varphi$.

Assumption 3 [21]: The $n \times m$ matrix $\mathbf{B}$ has full rank m defined by

$$\mathbf{B} = \frac{1}{r} \sum_{i=1}^r \mathbf{B}_i \quad (6)$$

According to assumption 1, assumption 2 and assumption 3, the uncertain T-S fuzzy control systems (2) can be rewritten as follows:

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^r h_i(\phi_j) \mathbf{A}_i \mathbf{x}(t) + \bar{\mathbf{B}}[\mathbf{u}(t) + \mathbf{h}(t, \mathbf{u}, \mathbf{x})] \quad (7)$$

where $\bar{\mathbf{B}} = \mathbf{B} + \mathbf{KFR}$, $\mathbf{K} = \frac{1}{2}[(\mathbf{B} - \mathbf{B}_1), \dots, (\mathbf{B} - \mathbf{B}_r)] \in \mathfrak{R}^{m \times rm}$,

$\mathbf{F} = \text{Diag}[(1 - 2h_1)\mathbf{I}, \dots, (1 - 2h_r)\mathbf{I}] \in \mathfrak{R}^{n \times rm}$, $\mathbf{R} = [\mathbf{I}, \dots, \mathbf{I}]^T \in \mathfrak{R}^{rm \times m}$, and $\mathbf{h}(t, \mathbf{u}, \mathbf{x})$ is the lumped uncertainty given as $\mathbf{h}(t, \mathbf{u}, \mathbf{x}) = \beta_A(t) \mathbf{x}(t) + \beta_B(t) \mathbf{u}(t) + \mathbf{f}(t, \mathbf{x})$.

III. ESTIMATION OF ASYMPTOTIC STABILITY REGION OF SYSTEM STATE

A variable structure system (VSS) is a control system implementing different control laws in different regions of the state space. The regions of the VSS are divided by a set of boundary manifolds which represent the checking conditions of the VSS. Thus, we consider the uncertain T-S fuzzy control systems (7) and define the following linear switching surface [2].

$$\Omega = \{\mathbf{x} : \sigma(\mathbf{x}) = \mathbf{S}\mathbf{x}(t) = 0\} \quad (8)$$

where $\mathbf{S} \in \mathfrak{R}^{m \times n}$ with full rank m . We assume that the switching surface parameter matrix $\mathbf{S}$ satisfies the following properties [21, 30].

Property 1: $\mathbf{S}\bar{\mathbf{B}} > 0$ for any $h_i(\phi_j)$ satisfying $h_i(\phi_j) \geq 0$

and $\sum_{i=1}^r h_i(\phi_j) = 1$.

Property 2: By using an appropriately chosen stabilizing feedback control law, the system trajectory is already stably restricted to the linear switching surface $\mathbf{S}\mathbf{x}(t) = 0$, the reduced $(n-m)$ th order sliding mode dynamics is asymptotically stable.

Our problem can be formulated as follows: 1) Design a sliding mode control for the complex nonlinear systems with matched uncertainties so that the systems are asymptotically stable. 2) Estimate the ASR of (7) for a given $\mathbf{S}$ satisfying *Property 1* and *Property 2*.

Referring to [30], we define nonsingular transformation matrix $\mathbf{M}$ as follows:

$$\mathbf{M} = \begin{bmatrix} \Phi^T \\ (\bar{\mathbf{B}}^T \bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T \end{bmatrix} \text{ and } \mathbf{M}^{-1} = \begin{bmatrix} \Phi & \bar{\mathbf{B}} \end{bmatrix}. \quad (9)$$

And we have the associated state $\mathbf{v}(t)$ by (9) as follows:

$$\mathbf{v}(t) = \begin{bmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{bmatrix} = \mathbf{M}\mathbf{x}(t) = \begin{bmatrix} \Phi^T \mathbf{x}(t) \\ (\bar{\mathbf{B}}^T \bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T \mathbf{x}(t) \end{bmatrix} \quad (10)$$

where $\mathbf{v}_1(t) \in \mathfrak{R}^{(n-m) \times 1}$ and $\mathbf{v}_2(t) \in \mathfrak{R}^{m \times 1}$. From (8), (9) and (10), we obtain

$$\mathbf{S}\mathbf{x}(t) = \mathbf{S}\mathbf{M}^{-1}\mathbf{v}(t) = \mathbf{S}\Phi \mathbf{v}_1(t) + \mathbf{S}\bar{\mathbf{B}} \mathbf{v}_2(t) = \mathbf{0}. \quad (11)$$

Under a sliding condition, the trajectory asymptotically goes to the zero of the phase plant. The dynamics can be formulated as (11). Thus, from the equivalent control method [1], we can design the equivalent controllers as follows:

$$\mathbf{u}_{eq}(t) = -\sum_{i=1}^r h_i(\phi_j) (\mathbf{S}\bar{\mathbf{B}})^{-1} \mathbf{S}\mathbf{A}_i \mathbf{x}(t) - \mathbf{h}(t, \mathbf{u}, \mathbf{x}). \quad (12)$$

Proposition 1: If there exists positive scalar $\varepsilon \in \mathfrak{R}$ and positive-definite matrix $\mathbf{P}$ satisfy the following conditions

$$\mathbf{P} > 0, \Phi^T (\mathbf{A}_i \mathbf{P} + \mathbf{P} \mathbf{A}_i^T) \Phi < 0, i=1, 2, \dots, r \quad (13)$$

$$\varepsilon > 0, \mathbf{S}\mathbf{P} = \varepsilon \bar{\mathbf{B}}^T \quad (14)$$

then the switching surface parameter matrix $\mathbf{S}$ satisfying *Property 1* and *Property 2* for the uncertain T-S fuzzy control systems (2.5) where $\Phi \in \mathfrak{R}^{n \times (n-m)}$ is any full rank matrix such that $\bar{\mathbf{B}}^T \Phi = \mathbf{0}$ and $\Phi^T \Phi = \mathbf{I}$.

Proof:

Our aim is to choose an appropriate $\mathbf{P}$ such that the systems motion on the sliding surface is asymptotically stable. In order to obtain the dynamics on the sliding surface, we must do a transformation for the state $\mathbf{x}(t)$.

From the above transformation and the time derivative of (10), the dynamics as follows:

$$\begin{aligned}\dot{\mathbf{v}}(t) &= \mathbf{M} \left(\sum_{i=1}^r h_i(\phi_j) \mathbf{A}_i \mathbf{x}(t) + \bar{\mathbf{B}}[\mathbf{u}(t) + \mathbf{h}(t, \mathbf{u}, \mathbf{x})] \right) \\ &= \sum_{i=1}^r h_i(\phi_j) \mathbf{M} \mathbf{A}_i \mathbf{M}^{-1} \mathbf{v}(t) + \mathbf{M} \bar{\mathbf{B}}[\mathbf{u}(t) + \mathbf{h}(t, \mathbf{u}, \mathbf{x})] \\ &= \sum_{i=1}^r h_i(\phi_j) [\Phi \bar{\mathbf{B}}(\bar{\mathbf{B}}^T \bar{\mathbf{B}})^{-1}]^T \mathbf{A}_i [\Phi \bar{\mathbf{B}}(\bar{\mathbf{B}}^T \bar{\mathbf{B}})^{-1}]^{-T} \mathbf{v}(t) \\ &\quad + [\Phi \bar{\mathbf{B}}(\bar{\mathbf{B}}^T \bar{\mathbf{B}})^{-1}]^T \bar{\mathbf{B}}[\mathbf{u}(t) + \mathbf{h}(t, \mathbf{u}, \mathbf{x})] \\ &= \sum_{i=1}^r h_i(\phi_j) \begin{bmatrix} \Phi^T \mathbf{A}_i \Phi & \Phi^T \mathbf{A}_i \bar{\mathbf{B}} \\ (\bar{\mathbf{B}}^T \bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T \mathbf{A}_i \Phi & (\bar{\mathbf{B}}^T \bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T \mathbf{A}_i \bar{\mathbf{B}} \end{bmatrix} \mathbf{v}(t) \\ &\quad + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} (\mathbf{u}(t) + \mathbf{h}(t, \mathbf{u}, \mathbf{x})). \end{aligned} \quad (15)$$

Substituting (12) into (15) and consider (11), we obtain the dynamics $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ as follows:

$$\begin{bmatrix} \dot{\mathbf{v}}_1(t) \\ \dot{\mathbf{v}}_2(t) \end{bmatrix} = \sum_{i=1}^r h_i(\phi_j) \begin{bmatrix} \Phi^T \mathbf{A}_i \Phi \mathbf{v}_1(t) - \Phi^T \mathbf{A}_i \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S} \Phi \mathbf{v}_1(t) \\ ((\bar{\mathbf{B}}^T \bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T \mathbf{A}_i - \mathbf{B}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S} \mathbf{A}_i) \mathbf{M}^{-1} \mathbf{v}(t) \end{bmatrix} \quad (16)$$

From (16), we have the reduced $(n-m)$ order sliding mode dynamics restricted to the switching surface $\sigma(\mathbf{x}) = \mathbf{S}\mathbf{x}(t) = 0$, i.e., the reduced $(n-m)$ order equivalent system dynamics is given by

$$\dot{\mathbf{v}}_1(t) = \sum_{i=1}^r h_i(\phi_j) [\Phi^T \mathbf{A}_i \Theta \Phi \mathbf{v}_1(t)]. \quad (17)$$

where $\Theta = \mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S}$.

Now, we consider the Lyapunov function candidate $\bar{V} = \mathbf{v}_1^T(t) \mathbf{P}_0^{-1} \mathbf{v}_1(t)$, then, the equivalent system dynamics (17) is asymptotically stable if there exists a positive definite matrix $\mathbf{P}_0^{-1} \in \Re^{(n-m) \times (n-m)}$ such that

$$\dot{\bar{V}}(t) = \mathbf{v}_1^T(t) \mathbf{P}_0^{-1} \dot{\mathbf{v}}_1(t) + \dot{\mathbf{v}}_1^T(t) \mathbf{P}_0^{-1} \mathbf{v}_1(t) < 0. \quad (18)$$

Substituting (17) into (18), we have

$$\begin{aligned}\dot{\bar{V}}(t) &= \sum_{i=1}^r h_i(\phi_j) \{ \mathbf{v}_1(t) \mathbf{P}_0^{-1} \Phi^T \mathbf{A}_i \Theta \mathbf{P}_0^{-1} \mathbf{v}_1(t) \\ &\quad + \mathbf{v}_1(t) \mathbf{P}_0^{-1} \mathbf{P}_0 \Phi^T \Theta^T \mathbf{A}_i^T \Phi \mathbf{P}_0^{-1} \mathbf{v}_1(t) \} \\ &= \sum_{i=1}^r h_i(\phi_j) \{ \mathbf{v}_1(t) \mathbf{P}_0^{-1} [\Phi^T \mathbf{A}_i \Theta \mathbf{P}_0 \\ &\quad + \mathbf{P}_0 \Phi^T \Theta^T \mathbf{A}_i^T \Phi] \mathbf{P}_0^{-1} \mathbf{v}_1(t) \} \\ &= \sum_{i=1}^r h_i(\phi_j) \{ \mathbf{v}_1(t) \mathbf{P}_0^{-1} \Phi^T [\Phi^T \mathbf{A}_i \Theta \mathbf{P}_0 \Phi^T \end{aligned}$$

$$+ \Phi \mathbf{P}_0 \Phi^T \Theta^T \mathbf{A}_i^T \Phi] \mathbf{P}_0^{-1} \mathbf{v}_1(t) \}$$

$$= \sum_{i=1}^r h_i(\phi_j) \{ \mathbf{v}_1(t) \mathbf{P}_0^{-1} \Phi^T [\mathbf{A}_i \mathbf{P}^T + \mathbf{P} \mathbf{A}_i^T] \mathbf{P}_0^{-1} \mathbf{v}_1(t) \} \quad (19)$$

where

$$\mathbf{P} = \Phi \mathbf{P}_0 \Phi^T \Theta^T = \Phi \mathbf{P}_0 \Phi^T (\mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S})^T. \quad (20)$$

From (19) and (20), we get the dynamics (17) is asymptotically stable if $\Phi^T (\mathbf{A}_i \mathbf{P}^T + \mathbf{P} \mathbf{A}_i^T) \Phi < 0$ and $\mathbf{P} > 0$ where the matrix $\mathbf{P}$ satisfy the condition (14). The dynamics (17) is asymptotically stable imply that the $\mathbf{v}_2(t) = 0$ by (11).

Lemma: $\mathbf{P} = \Phi \mathbf{P}_0 \Phi^T (\mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S})^T$ is a positive definite matrix in (20).

Proof:

Form $\mathbf{P} = \Phi \mathbf{P}_0 \Phi^T (\mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S})^T$ we obtain

$$(\mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S}) \mathbf{P} = (\mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S}) \Phi \mathbf{P}_0 \Phi^T (\mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S})^T \quad (21)$$

By (21) and must be satisfy (14), we have

$$\mathbf{P} = (\mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S}) \Phi \mathbf{P}_0 \Phi^T (\mathbf{I} - \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \mathbf{S})^T + \varepsilon \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T \quad (22)$$

where if there exists $\mathbf{P}_0$, ε in proposition 1 and $\mathbf{S}$ satisfying property 1, we have $\mathbf{P} > 0$.

Suppose that the proposition 1 has a solution for $\mathbf{P}$ and ε , then there exists a linear switching surface parameter matrix $\mathbf{S}$ satisfying property 1 and property 2, it can be parameterized as follows [2]:

$$\sigma(\mathbf{x}) = \mathbf{S}\mathbf{x}(t) = \bar{\mathbf{B}}^T \mathbf{P}^{-1} \mathbf{x}(t) \quad (23)$$

By the *Projection lemma* and *Finsler's lemma* [32], (13) is equivalent to the following linear matrix inequalities (LMIs) [31]:

$$\mathbf{P} > 0, \mathbf{A}_i \mathbf{P} + \mathbf{P} \mathbf{A}_i^T - \varepsilon \bar{\mathbf{B}}^T - \bar{\mathbf{B}} \mathbf{E}^T < 0, i=1,2,\dots,r. \quad (24)$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{E} \\ \mathbf{E}^T & \mathbf{I} \end{bmatrix} > 0 \quad (25)$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{I} \\ \mathbf{I} & \alpha \mathbf{I} \end{bmatrix} > 0 \quad (26)$$

where $\alpha > 0$. Therefore we have the following theorem.

Theorem 1: If there exists symmetric matrix $\mathbf{P}_0 \in \Re^{(n-m) \times (n-m)}$, matrix $\mathbf{E} \in \Re^{n \times m}$ and positive scalars $\varepsilon \in \Re$, $\alpha \in \Re$ satisfy the following LMIs

$$\Theta \Phi \mathbf{P}_0 \Phi^T \Theta^T + \varepsilon \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T > 0 \quad (27)$$

$$\begin{aligned} &\mathbf{A}_i \Theta \Phi \mathbf{P}_0 \Phi^T \Theta^T + \varepsilon \mathbf{A}_i \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T - \varepsilon \bar{\mathbf{B}}^T \\ &+ (\mathbf{A}_i \Theta \Phi \mathbf{P}_0 \Phi^T \Theta^T + \varepsilon \mathbf{A}_i \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T - \varepsilon \bar{\mathbf{B}} \mathbf{E}^T)^T < 0 \end{aligned} \quad (28)$$

$$\begin{bmatrix} \Theta \Phi \mathbf{P}_0 \Phi^T \Theta^T + \varepsilon \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T & \mathbf{E} \\ \mathbf{E}^T & \mathbf{I} \end{bmatrix} > 0 \quad (29)$$

$$\begin{bmatrix} \Theta \Phi \mathbf{P}_0 \Phi^T \Theta^T + \varepsilon \bar{\mathbf{B}}(\bar{\mathbf{S}}\bar{\mathbf{B}})^{-1} \bar{\mathbf{B}}^T & \mathbf{I} \\ \mathbf{I} & \alpha \mathbf{I} \end{bmatrix} > 0 \quad (30)$$

then the switching surface parameter matrix $\mathbf{S}$ satisfies *Property 1* and *Property 2*.

Proof:

By *Projection lemma* and *Finsler's lemma* we can show that any $\mathbf{S}$ satisfying *Property 1* and *Property 2* if exists a solution $\mathbf{P}_0$, $\mathbf{E}$ and ε , α satisfy (27)-(30) in which $\mathbf{P}$ can be obtained by (22). From *Lemma*, we know that $\mathbf{P}$ is a positive definite matrix and satisfy the condition (14) if there exists symmetric matrix $\mathbf{P}_0$ and positive scalar $\varepsilon \in \Re$, and from (13) we see that the sliding mode dynamics (17) is asymptotically stable.

Referring to [2] and [30], we consider the following switching feedback control law.

Control Rule i :

If ϕ_1 is F_1^i and $\dots$ and ϕ_n is F_n^i

$$\text{Then } \mathbf{u}(t) = -\eta_i \frac{\boldsymbol{\sigma}(\mathbf{x})}{\|\boldsymbol{\sigma}(\mathbf{x})\|} \quad (31)$$

where $\phi_j (j=1,2,\dots,n)$ are the state variables and $\eta_i (i=1,2,\dots,r) \in \Re$ are the controller parameters for each subsystems and $\eta_{\max} = \max\{\eta_1, \eta_2, \dots, \eta_r\}$ is assumed.

The overall controller inferred as the weighted average of the each local controller as follows:

$$\mathbf{u}(t) = -\sum_{i=1}^r h_i(\phi_j) \eta_i \frac{\boldsymbol{\sigma}(\mathbf{x})}{\|\boldsymbol{\sigma}(\mathbf{x})\|} \quad (32)$$

To support the following theorem, we make the assumptions as follows:

Assumption 4: There exist a known nonnegative constant

$$\rho = \frac{\rho_A + \rho_B}{1 - \varphi} \text{ such that } \mathbf{h}(t, \mathbf{u}, \mathbf{x}) \leq \rho \text{ for all } t.$$

Assumption 5: The difference $\eta_{\max} - \rho = \zeta$ is positive.

Theorem 2: Suppose that assumption 4 and assumption 5 hold and a switching surface $\boldsymbol{\sigma}(\mathbf{x})=0$ guaranteeing *Property 1* and *Property 2* are satisfied. Consider the uncertain T-S fuzzy system of (7) with the controller (32). The following region is an estimate of the ASR:

$$B_S = \left\{ \mathbf{x}(t) : \|\mathbf{x}(t)\| \leq \frac{\zeta}{\sqrt{\alpha}} \right\} \quad (33)$$

proof:

We define the Lyapunov function as

$$V = \mathbf{x}(t) \mathbf{P}^{-1} \mathbf{x}(t) \quad (34)$$

The time of derivative of the Lyapunov function V is as follows:

$$\begin{aligned} \dot{V} &= 2 \sum_{i=1}^r h_i(\phi_j) \mathbf{x}^T(t) \mathbf{P}^{-1} \mathbf{A}_i \mathbf{x}(t) + 2 \mathbf{x}^T(t) \mathbf{P}^{-1} \bar{\mathbf{B}} (\mathbf{u}(t) + \mathbf{h}(t, \mathbf{u}, \mathbf{x})) \\ &= \sum_{i=1}^r h_i(\phi_j) \mathbf{x}^T(t) \mathbf{P}^{-1} [\mathbf{A}_i \mathbf{P} + \mathbf{P} \mathbf{A}_i^T - \mathbf{E} \bar{\mathbf{B}}^T - \bar{\mathbf{B}} \mathbf{E}^T] \mathbf{P}^{-1} \mathbf{x}(t) \\ &\quad + 2 \mathbf{x}^T(t) \mathbf{P}^{-1} \bar{\mathbf{B}} (\mathbf{u}(t) + \mathbf{E}^T \mathbf{P}^{-1} \mathbf{x}(t) + \mathbf{h}(t, \mathbf{u}, \mathbf{x})) \end{aligned} \quad (35)$$

Substituting (32) into (35), we have

$$\begin{aligned} \dot{V} &= \sum_{i=1}^r h_i(\phi_j) \mathbf{x}^T(t) (-\mathbf{Q}_i) \mathbf{x}(t) \\ &\quad + 2 \mathbf{x}^T(t) \mathbf{P}^{-1} \bar{\mathbf{B}} \left(-\sum_{i=1}^r h_i(\phi_j) \eta_i \frac{\boldsymbol{\sigma}(\mathbf{x})}{\|\boldsymbol{\sigma}(\mathbf{x})\|} + \mathbf{E}^T \mathbf{P}^{-1} \mathbf{x}(t) \right. \\ &\quad \left. + \mathbf{h}(t, \mathbf{u}, \mathbf{x}) \right) \\ &\leq \sum_{i=1}^r h_i(\phi_j) \mathbf{x}^T(t) (-\mathbf{Q}_i) \mathbf{x}(t) \\ &\quad + \left\| 2 \mathbf{x}^T(t) \mathbf{P}^{-1} \bar{\mathbf{B}} \left(-\sum_{i=1}^r h_i(\phi_j) \eta_i \frac{\boldsymbol{\sigma}(\mathbf{x})}{\|\boldsymbol{\sigma}(\mathbf{x})\|} + \mathbf{E}^T \mathbf{P}^{-1} \mathbf{x}(t) \right. \right. \\ &\quad \left. \left. + \mathbf{h}(t, \mathbf{u}, \mathbf{x}) \right) \right\| \end{aligned} \quad (36)$$

where $\mathbf{P}^{-1} [\mathbf{A}_i \mathbf{P} + \mathbf{P} \mathbf{A}_i^T - \mathbf{E} \bar{\mathbf{B}}^T - \bar{\mathbf{B}} \mathbf{E}^T] \mathbf{P}^{-1} = -\mathbf{Q}_i$ and $\mathbf{Q}_i > 0, \forall i=1,2,\dots,r$.

From assumption 4 and assumption 5, (36) implies that

$$\begin{aligned} \dot{V} &\leq -\sum_{i=1}^r h_i(\phi_j) \lambda_{\min}(\mathbf{Q}_i) \|\mathbf{x}(t)\|^2 + 2 \left\| \mathbf{x}^T(t) \mathbf{P}^{-1} \bar{\mathbf{B}} \right\| \\ &\quad \cdot \left(\left\| \mathbf{E}^T \mathbf{P}^{-1} \mathbf{x}(t) \right\| - \zeta \right) \end{aligned} \quad (37)$$

Therefore, in the region

$$\Omega_0 = \{ \mathbf{x}(t) : \mathbf{x}^T(t) \mathbf{P}^{-1} \mathbf{E} \mathbf{E}^T \mathbf{P}^{-1} \mathbf{x}(t) \leq \zeta^2 \} \quad (38)$$

the following inequality holds:

$$\dot{V} \leq -\sum_{i=1}^r h_i(\phi_j) \lambda_{\min}(\mathbf{Q}_i) \|\mathbf{x}(t)\|^2 \leq 0 \quad (39)$$

and implies that the $\Omega_1 = \{ \mathbf{x}(t) : \mathbf{x}^T(t) \mathbf{P}^{-1} \mathbf{x}(t) \leq \zeta^2 \}$.

The constraint $\Omega_1 \subset \Omega_0$ is satisfied if $\mathbf{P}^{-1} > \mathbf{P}^{-1} \mathbf{E} \mathbf{E}^T \mathbf{P}^{-1}$ which can be rewritten as the LMI (25) by using the Schur complement formula [31]. From (26), we can find the inequality $\alpha \mathbf{I} > \mathbf{P}^{-1}$ satisfying the constraint $\Omega_{ASR} \subset \Omega_1$ where the region Ω_{ASR} is an estimate of the ASR as follows:

$$\Omega_{ASR} = \{ \mathbf{x}(t) : \mathbf{x}^T(t) \alpha \mathbf{I} \mathbf{x}(t) \leq \zeta^2 \} \quad (40)$$

From (40), we can obtain (33).

Remark: If the uncertain systems (1) satisfy assumption 1-assumption 5, then there exist a positive definite matrix $\mathbf{P}$ satisfy (24)-(26). The switching surface can be designed and some LMI problems of the (27)-(30) can be solved. The ASR can be estimated.

IV. SIMULATION EXAMPLE

We consider the inverted pendulum via gear train [27], described by the following sates equation:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ (g/l) \sin(x_1) + (NK_m/ml^2) x_3 \\ -(V_b/L_a) x_2 - (R_a/L_a) x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} u(t) \quad (41)$$

where $x_1(t)$ is the angle of the pendulum, $x_2(t)$ is the angle velocity, $x_3(t)$ is the current of the motor. K_m is the motor torque constant, V_b is the back emf constant, R_a is the resistance, L_a is the inductance of armature circuit, N is the gear ratio.

In this simulation, $g = 9.8m/s^2$, $l = 1m$, $m = 1kg$, $N = 10$, $K_m = 0.1Nm/A$, $K_b = 0.1Vs/rad$, $R_a = 1\Omega$, and $L_a = 100mH$. Using the values of the system parameter in (41), we have

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ 9.8\sin(x_1) + x_3 \\ -10x_2 - 10x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u(t) \quad (42)$$

The following dynamic fuzzy model is used to represent the nonlinear system (42):

Plant Rule 1 :

If x_1 is G_1

$$\text{Then } \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u(t) \quad (43)$$

Plant Rule 2:

If x_1 is G_2

$$\text{Then } \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u(t) \quad (44)$$

and we consider $\rho_A = 0.25$, $\varphi = 0.1$, $\rho_f = 0$, and $\eta_{\max} = 7$. The membership functions as follows [27]:

$$G_1 = \frac{1 - 1/[1 + \exp(-5(x_1 - 1))]}{1 + \exp(-5(x_1 - 1))} \text{ and } G_2 = 1 - G_1 \quad (45)$$

Now, we checking the $\bar{\mathbf{B}}^T \Phi = \mathbf{0}$ and $\Phi^T \Phi = \mathbf{I}$, the $\Phi^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ can be obtained from MATLAB code [33].

By solving the LMI problem we can obtain the following arguments:

$$\mathbf{P} = \begin{bmatrix} 2.6795 & -8.8122 & 16.3451 \\ -8.8122 & 32.6026 & -97.2679 \\ 16.3451 & -97.2679 & 632.8051 \end{bmatrix}, \mathbf{E} = \begin{bmatrix} -0.6149 \\ 2.1290 \\ -2.6274 \end{bmatrix}$$

$$\mathbf{P}_0 = \begin{bmatrix} 32.6026 & 8.8122 \\ 8.8122 & 2.6795 \end{bmatrix}, \alpha = 120.1430, \text{ and } \varepsilon = 1.8074.$$

And we can be obtained the estimate of ASR as follows:

$$ASR = \{\mathbf{x}(t) : \|\mathbf{x}(t)\| \leq 0.61\} \quad (46)$$

Fig. 1 gives the simulation results of $x_1(t)$. Fig. 2 gives the simulation results of $x_2(t)$. Fig. 3 gives the simulation results of $x_3(t)$. From Fig. 4, it can be clearly seen here that the system enters into a sliding mode and is confined to the

vicinity of the switching surface.

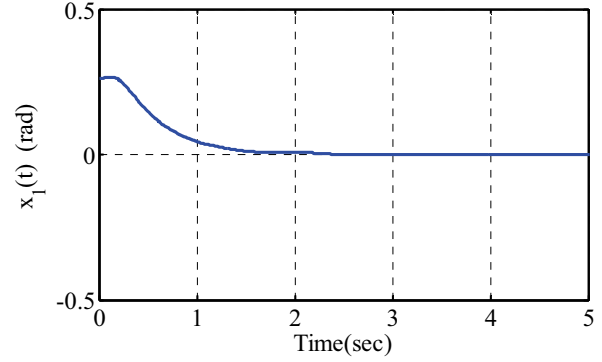


Fig. 1. The locus of $x_1(t)$ for initial conditions $\mathbf{x}(0) = [\pi/12 \ 0 \ 0]^T$.

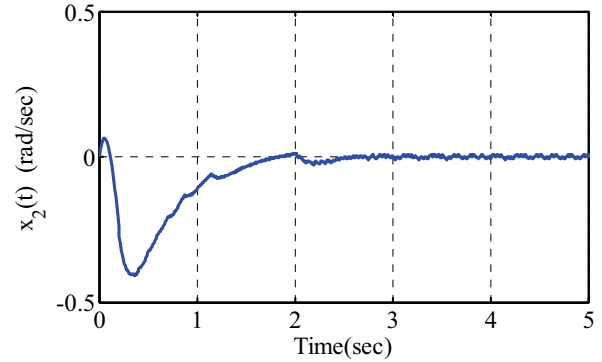


Fig. 2. The locus of $x_2(t)$ for initial conditions $\mathbf{x}(0) = [\pi/12 \ 0 \ 0]^T$.

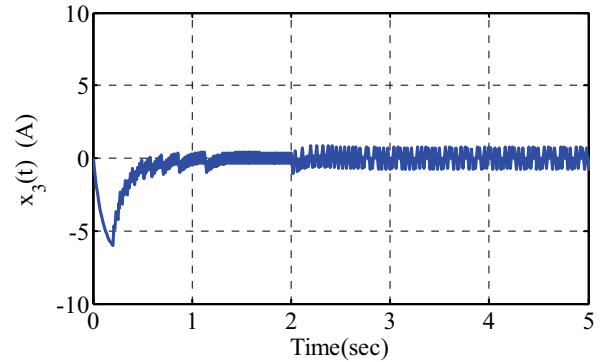


Fig. 3. The locus of $x_3(t)$ for initial conditions $\mathbf{x}(0) = [\pi/12 \ 0 \ 0]^T$.

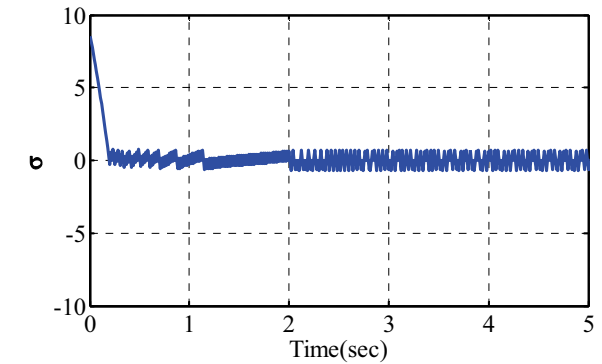


Fig. 4. The switching surface for initial conditions $\mathbf{x}(0) = [\pi/12 \ 0 \ 0]^T$.

V. CONCLUSIONS

The T-S fuzzy model is presented to approximate the uncertain nonlinear systems. By the Lyapunov stability theorem, the variable structure controllers can be designed with the linear matrix inequalities (LMIs). And the asymptotic stability region (ASR) can be obtained. The simulation example showed the effectiveness of the proposed scheme.

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