

Efficient Centroid Computation of General Type-2 Fuzzy Sets with Linear Secondary Membership Function

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Abstract—This paper provides root finding methods for the centroid computation of interval type-2 fuzzy sets and general type-2 fuzzy sets with linear secondary membership function. When the FOU of the type-2 fuzzy sets are provided, the centroid computation methods in this paper is accurate and computational efficient, which provide new methods for the type-2 fuzzy systems computation.

Keywords—Type-2 fuzzy set (IT2 FS), Karnik-Mendel (KM) algorithms, centroid computation, root finding.

I. INTRODUCTION

Type-2 fuzzy logic system (FLS) allows for better modeling of uncertainty as type-2 fuzzy set (T2 FS) encompass a Footprint of Uncertainty (FOU) which, associated with its third dimension, gives more degrees of freedom to T2 FS in comparison to type-1 fuzzy set (T1 FS) [1], [14], [20], [30]. Interval type-2 fuzzy set (IT2 FS) is a simplified version of general T2 FS where the membership grade is crisp interval rather than functions. Most applications on type-2 fuzzy FLS involve only interval type-2 systems applications, and it has been shown that interval type-2 FLSs can outperform their type-1 FLSs counterparts in a variety of fields including information processing, fuzzy control and decision making [10], [28], [21], [6], [16]. General type-2 FLSs has only recently been investigated in more detail [19], [25], [11], [3], [24], [4], [23]. It is expected that the general T2 FSs will have the ability to model uncertainty more accurately than IT2 FSs which in turn will result in the potential for a superior performance in comparison to type-1 and interval type-2 FLSs. But the high complexity associated with their design and their computational requirements made them appear unsuitable for real-world use. One of the major obstacles to the use of generalized type-2 FLSs on real-world problems is the type reduction of general T2 FSs [19], [25], [3], [11], which is consisted of the steps of type reduction from general T2 FS into T1 FS, and T1 FS defuzzification to the final crisp output. Calculating the centroid of a general T2 FS typically requires a large amount of computation.

To date, type-reduction methods have been developed by Karnik and Mendel [9] and are elaborated upon in [14]. The efficient algorithms, called the Karnik-Mendel (KM) algorithms, have been developed for centroid type reduction for

interval type-2 FLSs. The KM algorithms have been applied to many applications of T2 FSs and play an important role in type-2 FLSs [20], [14], [7], [15], [16]. Mendel and Liu [18] proved monotonicity and super-exponential convergence of the algorithms. The iteration number of KM algorithm is usually less than seven for accuracies of 10^{-2} . Because of the iterative natures of the KM algorithms, there may be a computational bottleneck in real-time applications. So, Wu and Mendel [29] proposed an uncertainty bound method to approximate type reduction of ITS without computation burden. Greenfield, Chiclana, Coupland and Robert John [5] proposed a collapsing method of defuzzification for discretised IT2 FSs, which claimed can be outperformed the KM algorithms both in accuracy and speed. Finally, Wu and Mendel [27] proposed Enhanced KM (EKM) algorithms to reduce the computational cost of the standard KM algorithm.

However, no efficient strategies exist to date for centroid type-reduction of general T2 FLSs [11], [19], [25]. John and Czarnecki [8] and Lucas et al. [13] proposed the general centroid computation as the centroids of all vertical slices (each of which is a T1 FS). Liu [11] developed an efficient strategy to implement centroid type-reduction for general type-2 FLSs by decomposing a general T2 FS into IT2 FS α -planes with $\alpha \in [0, 1]$. Mendel, Liu and Zhai [19] use this method for the computation of general type-2 sets, which seems to be the first practical choices for extension of IT2 FSs to general T2 FSs. The centroid type reduction is performed using KM algorithms individually for each α -plane. A quasi-T2 fuzzy logic system with two α -plane sets are designed. Starczewski [24] also designed triangular type-2 fuzzy logic systems. A new efficient approximate iterative procedure based on the KM type-reduction is proposed. Wagner and Hagsras [25] introduced the zSlices-based representation, and proposed a similar type reduction method with the repetitive KM algorithm for the centroid of general T2 FSs. Except the α -plane (zslice) based centroid computation method mentioned [19], [25], which takes samples from the two dimensional directions of the membership function, and obtains approximate solution with repetitive computation and iterations, Coupland and John [2], [3], [4] presented a geometric representation of general type-2 sets. The geometric method [4], [3], [2] uses more simplified

and rough approximation method than the α -plane method with KM-algorithm, and it is claimed faster than the KM algorithm in computation. More recently, Zhai and Mendel [31] proposed introduced a centroid-flow (CF) algorithm, which only utilizes the KM/EKM algorithms to compute the centroid of the general type-2 fuzzy sets, which only need to compute the α -plane of the beginning. It should be noted that in all of the general T2 FSs publications for the authors [19], [11], [10], [3], [25], [4], [24], the second membership function are often assumed linear (triangular or trapezoidal), there seems no much justification to adopt other forms of second membership functions forms in general type-2 FLSs, especially in control and decision applications.

Up till now, in contrast to the centroid defuzzification of type-1 fuzzy sets, that is usually computed integral of the membership function of the output fuzzy set, almost all the current type reduction methods of T2 FS are discrete based, that is we take sampling data from the T2 FS, and compute the general centroid values approximately. This not only make the computation accuracy becomes very low but also make the computation complexity increases very fast, e. g. for 100 samples on the α -plane and x -domain respectively, we should compute 100 times iterate computation with 10,000 data; if we want a one more digit accuracy of the general centroid values, we have to take 1000 samples on two dimensions, and compute 1000 times iterate computation with 1,000,000 data. Some computational analysis of these methods can be found in [19], [11], [25]. In the present paper, we will extend the KM algorithm to the general root finding method. The KM algorithm iteration process become a part of rooting finding with Newton-Raphson method. In which, the centroid values can be computed directly and the accurately, and can integrate other numerical integral and root finding techniques in practice. For general T2 FSs under linear second membership function, the centroid value computation is only two times of the case of IT2 FSs, the repetitive IT2 FS slices computation can be avoided, and the computation is accurate.

The organization of the paper is as follows. Section II gives preliminaries about T2 FSs and the centroid computation of T2 FSs. Section III provides the root finding methods for the centroid of IT2 FS and the analytical method for the centroid computation of general T2 FS. Section IV illustrates the proposed methods with numerical examples. Section ?? summarizes the main results and draws conclusions.

II. PRELIMINARIES

A. The centroid of interval type-2 fuzzy set and general type-2 fuzzy set

Definition 1: [17] A T2 FS, denoted as $\tilde{A}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in J_x \subseteq [0, 1]$, i. e.

$$\tilde{A} = \{(x, u), \mu_{\tilde{A}}(x, u) | \forall x \in X, \forall u \in J_x \subseteq [0, 1]\} \quad (1)$$

where $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$, X is the domain of the fuzzy set, and J_x is the domain of the secondary membership function at x . $\tilde{A}$ can also be expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u) / (x, u) \quad \forall u \in J_x \subseteq [0, 1] \quad (2)$$

where $\int$ denotes union over all admissible x and u . J_x is called the primary membership of x in $\tilde{A}$. At each value of x say $x = x'$, the 2-D plane whose axes are u and $\mu_{\tilde{A}}(x', u)$ is called a vertical slice of $\tilde{A}$. Besides the vertical slice representation mentioned above, a general T2 FS can also be represented as a series of wavy slices where for discrete universes of discourse X and U [14].

The footprint of uncertainty, denoted $FOU(\tilde{A})$ is the union of all primary memberships, i.e., $FOU(\tilde{A}) = \bigcup_x \in X J_x$, which represents the uncertainty in the primary memberships of a T2 FS $\tilde{A}$.

Definition 2: [14] An IT2 FS $\tilde{A}$ is completely described by its lower and upper MFs, $\mu_{\tilde{A}}^-(x)$ and $\mu_{\tilde{A}}^+(x)$, respectively. The footprint of uncertainty (FOU) of an IT2 FS is described in terms of these MFs:

$$FOU(\tilde{A}) = \bigcup_{x \in X} [\mu_{\tilde{A}}^-(x), \mu_{\tilde{A}}^+(x)]$$

Definition 3: [19] An α -plane FOU , i.e., $FOU(\tilde{A}_\alpha)$ is

$$FOU(\tilde{A}_\alpha) = \alpha I_{\tilde{A}_\alpha}(x, u | \alpha). \quad (3)$$

where $I_{\tilde{A}_\alpha}$ is an indicator function

$$I_{\tilde{A}_\alpha}(x, u | \alpha) = \begin{cases} 1 & (x, u) \in \tilde{A}_\alpha \\ 0 & (x, u) \notin \tilde{A}_\alpha \end{cases}$$

The α -plane representation (theorem) for $\tilde{A}$ is

$$\tilde{A} = \bigcup_{\alpha \in [0, 1]} FOU(\tilde{A}_\alpha)$$

Definition 4: The centroid $c(A)$ of the T1 FS A is defined as

$$C(A) = \frac{\sum_{i=1}^N x_i \mu_A(x_i)}{\sum_{i=1}^N \mu_A(x_i)} \quad (4)$$

The centroid of a T2 FS, which was developed by Karnik and Mendel [9], has turned out to be a very important concept for IT2, as well as general T2 FSs and their associated FLSs.

Definition 5: [26] The centroid $C_{\tilde{A}}$ of an IT2 FS $\tilde{A}$ is the union of the centroids of all its embedded T1 FSs A_e , i.e.,

$$C(\tilde{A}) = \bigcup_{\forall A_e} c(A_e) = [c_l(\tilde{A}), c_r(\tilde{A})] \quad (5)$$

where $\bigcup$ is the union operation, and

$$c_l(\tilde{A}) = \min_{\forall A_e} c(A_e) \quad (6)$$

$$c_r(\tilde{A}) = \max_{\forall A_e} c(A_e) \quad (7)$$

Definition 6: [19] The centroid of general T2 FS $\tilde{A}$, is the union of the centroids of all the embedded T1 sets. With the α -plane representation method [19], [11], $C_{\tilde{A}}$ can be computed using the α -plane decomposition of $\tilde{A}$, the centroid of general

T2 FS $C_{\tilde{A}}(x)$ is the union of a series of IT2 FS at every α -plane, i.e.,

$$C_{\tilde{A}}(x) = \bigcup_{\alpha \in [0,1]} \alpha / C_{\tilde{A}_\alpha}(x)$$

where $C_{\tilde{A}_\alpha}(x)$ is the centroid of the α -plane $\tilde{A}$. Because each $\tilde{A}_\alpha$ is an interval-valued set, its centroid is an interval set, i.e.,

$$C_{\tilde{A}_\alpha}(x) = [l_{\tilde{A}_\alpha}, r_{\tilde{A}_\alpha}]$$

B. The centroid computation with KM algorithms

Let $x_i (i = 1, 2, \dots, N)$ represent the discretization of an IT2 FS $\tilde{A}$; $\underline{\mu}_{\tilde{A}}(x_i)$ and $\bar{\mu}_{\tilde{A}}(x_i)$ are the lower and upper membership functions that are associated with $\tilde{A}$ respectively. Using the representation theorem for a T2 FS [17], the centroid of IT2 FS $\tilde{A}$, $c_{\tilde{A}} = [c_l, c_r]$, can be computed as the optimal solutions of the following interval weighted average problems [9], [18]:

$$c_l = \min_{\forall \theta_i \in [\underline{\mu}_{\tilde{A}}(x_i), \bar{\mu}_{\tilde{A}}(x_i)]} \frac{\sum_{i=1}^N x_i \theta_i}{\sum_{i=1}^N x_i \theta_i} \quad (8)$$

$$c_r = \max_{\forall \theta_i \in [\underline{\mu}_{\tilde{A}}(x_i), \bar{\mu}_{\tilde{A}}(x_i)]} \frac{\sum_{i=1}^N x_i \theta_i}{\sum_{i=1}^N x_i \theta_i} \quad (9)$$

It is well known that c_l and c_r can be expressed, as:

$$c_l = \frac{\sum_{i=1}^L x_i \bar{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^N x_i \underline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^L \bar{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^N \underline{\mu}_{\tilde{A}}(x_i)} \quad (10)$$

$$c_r = \frac{\sum_{i=1}^R x_i \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^N x_i \bar{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^R \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^N \bar{\mu}_{\tilde{A}}(x_i)} \quad (11)$$

where L and R are called switch points with $x_L \leq c_l \leq x_{L+1}$ and $x_R \leq c_r \leq x_{R+1}$. The determination of L and R can be performed by using the KM algorithms [20].

Currently, the centroid of general T2 FS $\tilde{A}$ is computed with α -plane (or zslices) [19], [25] with $\alpha \in [0, 1]$, which decomposes the general T2 FS into a series of IT2 FS. The centroid of T2 FS is computed by the process in Table I.

III. THE ANALYTICAL DIRECT CENTROID COMPUTATION OF GENERAL TYPE-2 FUZZY SET WITH LINEAR SECONDARY MEMBERSHIP FUNCTION

A. Direct centroid computation of Interval type-2 sets

We assume all the x_i s are different, and they are bounded in $[a, b]$, where $a = \min_{1 \leq i \leq n} \{x_i\}$ and $b = \max_{1 \leq i \leq n} \{x_i\}$, which denote the smallest and the largest sampled values of x respectively*. Then, the continuous versions of (10) and (11) are

$$c_l = \frac{\int_a^{c_l} x \bar{\mu}_{\tilde{A}}(x) dx + \int_{c_l}^b x \underline{\mu}_{\tilde{A}}(x) dx}{\int_a^{c_l} \bar{\mu}_{\tilde{A}}(x) dx + \int_{c_l}^b \underline{\mu}_{\tilde{A}}(x) dx} \quad (12)$$

*As noted in [22, p. 363], if Gaussian MFs are used, one can extend the theoretical results to $a \rightarrow -\infty, b \rightarrow +\infty$; but, in practice, when truncations are used, a and b are again finite numbers.

$$c_r = \frac{\int_a^{c_r} x \underline{\mu}_{\tilde{A}}(x) dx + \int_{c_r}^b x \bar{\mu}_{\tilde{A}}(x) dx}{\int_a^{c_r} \underline{\mu}_{\tilde{A}}(x) dx + \int_{c_r}^b \bar{\mu}_{\tilde{A}}(x) dx} \quad (13)$$

From (12) and (13), we have

$$\int_a^{c_l} (c_l - x) \bar{\mu}_{\tilde{A}}(x) dx + \int_{c_l}^b (c_l - x) \underline{\mu}_{\tilde{A}}(x) dx = 0 \quad (14)$$

$$\int_a^{c_r} (c_r - x) \underline{\mu}_{\tilde{A}}(x) dx + \int_{c_r}^b (c_r - x) \bar{\mu}_{\tilde{A}}(x) dx = 0 \quad (15)$$

Let

$$\varphi(\xi) = \int_a^{\xi} (\xi - x) \bar{\mu}_{\tilde{A}}(x) dx + \int_{\xi}^b (\xi - x) \underline{\mu}_{\tilde{A}}(x) dx \quad (16)$$

$$\psi(\xi) = \int_a^{\xi} (\xi - x) \underline{\mu}_{\tilde{A}}(x) dx + \int_{\xi}^b (\xi - x) \bar{\mu}_{\tilde{A}}(x) dx \quad (17)$$

Consider (14), (15), then $\varphi(c_l) = 0$ and $\psi(c_r) = 0$. The computation of the centroid values of c_l and c_r can be transformed in the root finding problems of $\varphi(\xi) = 0$ and $\psi(\xi) = 0$. The equivalence of KM Algorithm and Newton-Raphson root-finding method and some more theoretical analysis were given in [12]. We can use the centroid finding equations (16) and (17) as the direct centroid computation methods.

B. Direct centroid computation of general type-2 sets

The centroid of general T2 FS is the combination of a series of α -plane IT2 FS with $\alpha \in [0, 1]$, which is in fact a normal type-1 fuzzy set with type reduction [19], [11]. It is natural that the centroid computation is the centroid of general T2 FS is more complicated, which is obtained by solving the centroids of a series of IT2 FS repeatedly [19], [25]. As the T2 FSs are obtained from T1 FS with an additional dimension, there is not common principle for the secondary membership determination. We often assume the secondary membership function of T2 FS to be assumed linear (triangular or trapezoidal membership function) [19], [11], [10], [3], [25], [4], [24]. It is very lucky that under such assumptions the centroid of a T2 FS can be obtained directly without solving every α -plane and we also get the type-1 centroid function analytically, which is convenient for final defuzzification.

If the second membership function of general T2 FS is linear, then α -plane decomposition of general T2 FS $\tilde{A}$ can be expressed as:

$$\underline{\mu}_{\tilde{A}_\alpha}(x) = (1 - \alpha) \underline{\mu}_{\tilde{A}_0}(x) + \alpha \underline{\mu}_{\tilde{A}_1}(x) \quad (18)$$

$$\bar{\mu}_{\tilde{A}_\alpha}(x) = (1 - \alpha) \bar{\mu}_{\tilde{A}_0}(x) + \alpha \bar{\mu}_{\tilde{A}_1}(x) \quad (19)$$

where $\underline{\mu}_{\tilde{A}_\alpha}(x)$ and $\bar{\mu}_{\tilde{A}_\alpha}(x)$ are the LMF and UMF of α -plane IT2 FS respectively.

We usually have $\forall x, \underline{\mu}_{\tilde{A}_0}(x) \leq \underline{\mu}_{\tilde{A}_1}(x)$ and $\bar{\mu}_{\tilde{A}_0}(x) \geq \bar{\mu}_{\tilde{A}_1}(x)$, so that $\underline{\mu}_{\tilde{A}_\alpha}(x)$ is increasing with α and $\bar{\mu}_{\tilde{A}_\alpha}(x)$ is decreasing with α . We can compute the centroid values of IT2 FS with FOU $\underline{\mu}_{\tilde{A}_\alpha}(x)$ and $\bar{\mu}_{\tilde{A}_\alpha}(x)$, in which α is a parameter of the IT2 FSs.

TABLE I
THE CENTROID COMPUTATION OF GENERAL TYPE-2 FUZZY SET

Step	The computation process
1	Sample $\alpha \in [0, 1]$ with $0 = \alpha_1 < \alpha_2 < \dots < \alpha_m = 1$, where m can be determined by the tolerance error bound of the problem. Then the general T2 FS is decomposed into a serial of α -planes in the form of IT2 FS $\tilde{A}_{\alpha_j}$, $j = 1, 2, \dots, m$.
2	For every $\tilde{A}_{\alpha_j}$, $j = 1, 2, \dots, m$, compute the general centroid values $c_l(\alpha_j)$ and $c_r(\alpha_j)$ ($j = 1, 2, \dots, m$) with the KM algorithms respectively.
3	Connect the points $c_l(\alpha_j)$ and $c_r(\alpha_j)$ for all $j = 1, 2, \dots, m$ together, thereby obtaining any approximate solution for $c_l(\alpha)$ and $c_r(\alpha)$ for $\alpha \in [0, 1]$.

With (16),

$$\begin{aligned}
\varphi(\alpha, l) &= \int_a^l (l-x) \underline{\mu}_{\tilde{A}_\alpha}(x) dx + \int_l^b (l-x) \underline{\mu}_{\tilde{A}_\alpha}(x) dx \\
&= \int_a^l (l-x) \left((1-\alpha) \underline{\mu}_{\tilde{A}_0}(x) + \alpha \underline{\mu}_{\tilde{A}_1}(x) \right) dx \\
&\quad + \int_l^b (l-x) \left((1-\alpha) \underline{\mu}_{\tilde{A}_0}(x) + \alpha \underline{\mu}_{\tilde{A}_1}(x) \right) dx \\
&= -\alpha \left(\int_a^l (l-x) \left(\underline{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_1}(x) \right) dx \right. \\
&\quad \left. + \int_l^b (l-x) \left(\underline{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_1}(x) \right) dx \right) \\
&\quad + \left(\int_a^l (l-x) \underline{\mu}_{\tilde{A}_0}(x) dx + \int_l^b (l-x) \underline{\mu}_{\tilde{A}_0}(x) dx \right)
\end{aligned}$$

Let $\varphi(\alpha, l) = 0$, we have

$$\begin{aligned}
\alpha_l &= \left(\int_a^l (l-x) \underline{\mu}_{\tilde{A}_0}(x) dx + \int_l^b (l-x) \underline{\mu}_{\tilde{A}_0}(x) dx \right) \\
&\quad / \left(\int_a^l (l-x) \left(\underline{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_1}(x) \right) dx \right. \\
&\quad \left. + \int_l^b (l-x) \left(\underline{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_1}(x) \right) dx \right) \quad (20)
\end{aligned}$$

This expression shows the relationship between l and α , if we have an additional restrict value of α with $\alpha \in [0, 1]$, then it can be regard the reverse function of $c_l(\alpha)$.

Similarly, with

$$\begin{aligned}
\psi(\alpha, r) &= \int_a^r (r-x) \underline{\mu}_{\tilde{A}_\alpha}(x) dx \\
&\quad + \int_r^b (l-x) \underline{\mu}_{\tilde{A}_\alpha}(x) dx \quad (21)
\end{aligned}$$

Let $\psi(\alpha, r) = 0$, we can also get that

$$\begin{aligned}
\alpha_r &= \left(\int_a^r (r-x) \underline{\mu}_{\tilde{A}_0}(x) dx + \int_l^b (r-x) \underline{\mu}_{\tilde{A}_0}(x) dx \right) \\
&\quad / \left(\int_a^r (r-x) \left(\underline{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_1}(x) \right) dx \right. \\
&\quad \left. + \int_r^b (r-x) \left(\underline{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_1}(x) \right) dx \right) \quad (22)
\end{aligned}$$

This provides a direct computation for the centroid of T2 FS with linear second membership function.

According to (20), solving $\alpha_l(l) = 0$ and $\alpha_l(l) = 1$ about l respectively. Suppose the solutions are l_0 and l_1 , then $\alpha_l(l)$ with $l \in [l_0, l_1]$ will be the left centroid. Similarly, according to (22), solving $\alpha_r(r) = 0$ and $\alpha_r(r) = 1$ about r respectively. Suppose the solutions are r_0 and r_1 , then $\alpha_r(r)$ with $r \in [r_1, r_0]$ will be the right centroid. The solution of $\alpha_l(l) = 0$ and $\alpha_l(l) = 1$, $\alpha_r(r) = 0$ and $\alpha_r(r) = 1$ can be obtained by the direct root finding method of Subsection III-A.

In summary, for general T2 FS with linear second membership function, the centroid function can be computed in a very simple way comparing to the α -plane method in Table I, which is given in Table II.

So, the type-1 centroid function of general T2 FS $\tilde{A}$ should be:

$$C_{\tilde{A}}(x) = \begin{cases} \alpha_l(x) & x \in [l_0, l_1] \\ 1 & x \in [l_1, r_1] \\ \alpha_r(x) & x \in [r_1, r_0] \end{cases} \quad (23)$$

With the centroid defuzzification method of type-1 set, the final defuzzification value can be computed:

$$c(\tilde{A}) = \frac{\int_{l_0}^{l_1} x \alpha_l(x) dx + \int_{l_1}^{r_1} x dx + \int_{r_1}^{r_0} x \alpha_r(x) dx}{\int_{l_0}^{l_1} \alpha_l(x) dx + \int_{l_1}^{r_1} dx + \int_{r_1}^{r_0} \alpha_r(x) dx} \quad (24)$$

IV. NUMERICAL EXAMPLES

Some numerical examples are provided to illustrate our proposed methods. If the membership functions is expressed with elementary function, the centroid function of T2 FS can even be expressed analytically. Here is an examples.

Example 1: Here is a T2 FS with non symmetrical FOU.

$$\underline{\mu}_{\tilde{A}_0}(x) = -\frac{4}{9}x^2 + \frac{4}{3}x \quad (25)$$

$$\underline{\mu}_{\tilde{A}_1}(x) = -\frac{4}{9}x^2 + \frac{8}{9}x \quad (26)$$

where $x \in [0, 2]$, which is shown in Figure 1.

The secondary MF of $\tilde{A}$ is chosen to be a triangular membership function whose base equals $\underline{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_1}(x)$ and whose apex is determined by a randomly generated value $w(x) \in [0, 1]$ as

$$Ape(x) = \underline{\mu}_{\tilde{A}_0}(x) + w(x) \left[\underline{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_1}(x) \right] \quad (27)$$

Here, we set $w = \frac{3}{5}$ and the domain is $x \in [0, 2]$.

TABLE II
THE CENTROID COMPUTATION OF GENERAL T2 FS WITH LINEAR SECOND MEMBERSHIP FUNCTION

Step	The computation process
1	Compute the centroid functions $\alpha_l(l)$ and $\alpha_r(r)$ without considering the domain of α -plane level value α with (20) and (22) respectively.
2	Find the corresponding bounds of l and r for $\alpha \in [0, 1]$ by solving equations $\alpha_l(l) = 0$ and $\alpha_l(l) = 1$ about l , and $\alpha_r(r) = 0$ and $\alpha_r(r) = 1$ about r respectively. Suppose the solutions are l_0, l_1 and r_0, r_1 respectively.
3	Replacing $\alpha_l(l), \alpha_r(r)$ with $\alpha_l(x), \alpha_r(x)$, then $\alpha_l(x)$ with $x \in [l_0, l_1]$ and $\alpha_r(x)$ with $x \in [r_0, r_1]$ will be the left and right centroid functions of general T2 FS respectively.

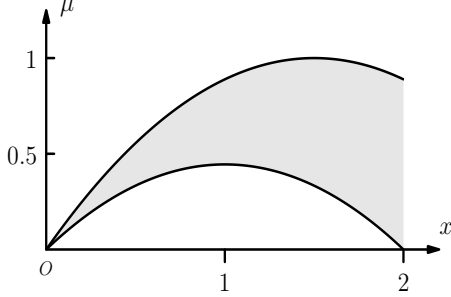


Fig. 1. The FOU of Example 1

The left centroid function and right centroid function can be computed respectively:

$$\alpha_l(l) = \frac{5(l^3 + 8l - 8)}{5l^3 - 36l + 48} \quad (28)$$

$$\alpha_r(r) = \frac{5(r^3 - 20r + 24)}{5r^3 - 24r + 32} \quad (29)$$

Find the solutions for $\alpha_l(l) = 0, \alpha_l(l) = 1$ for l and $\alpha_r(r) = 0, \alpha_r(r) = 1$ for r respectively. We can get that $l_0 = 0.907, l_1 = 1.158, r_1 = 1.158, r_0 = 1.313$.

Then the left centroid function of $\tilde{A}$ is the function α_l in the domain $l \in [l_0, l_1]$. The right centroid function of $\tilde{A}$ is the function α_r in the domain $r \in [r_1, r_0]$. The centroid of this T2 FS is in triangular shape, which can be expressed as (30) if the variables l and r are replaced with x .

$$\alpha(x) = \begin{cases} \frac{5(x^3 + 8x - 8)}{5x^3 - 36x + 48} & 0.907 \leq x \leq 1.158 \\ \frac{5(x^3 - 20x + 24)}{5x^3 - 24x + 32} & 1.158 \leq x \leq 1.313 \end{cases} \quad (30)$$

The results is plotted in Fig 2. It is shown that the centroid function is also in triangular shape. With (24), the final centroid defuzzification value is 1.1327.

Here are two more complicated examples.

Example 2: Piecewise linear LMF and UMF, and triangle secondary MFs [19], which is also similar to the example of [11]:

$$\underline{\mu}_{\tilde{A}_0}(x) = \max \left\{ \begin{cases} \frac{(x-1)}{2}, & 1 \leq x \leq 3 \\ \frac{(7-x)}{4}, & 4 \leq x \leq 7 \\ 0, & \text{otherwise} \end{cases}, \begin{cases} \frac{(x-2)}{5}, & 2 \leq x \leq 6 \\ \frac{(16-2x)}{5}, & 6 \leq x \leq 8 \\ 0, & \text{otherwise} \end{cases} \right\} \quad (31)$$

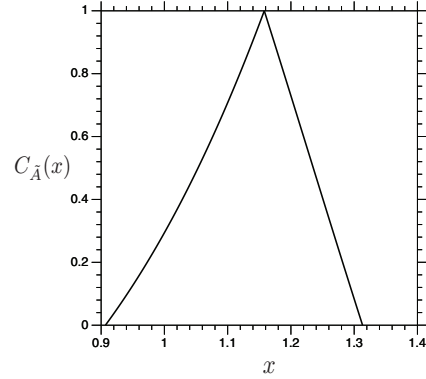


Fig. 2. The centroid function of Example 1

$$\underline{\mu}_{\tilde{A}_0}(x) = \max \left\{ \begin{cases} \frac{(x-1)}{6}, & 1 \leq x \leq 4 \\ \frac{(7-x)}{6}, & 4 \leq x \leq 7 \\ 0, & \text{otherwise} \end{cases}, \begin{cases} \frac{(x-3)}{6}, & 2 \leq x \leq 5 \\ \frac{(8-x)}{9}, & 5 \leq x \leq 8 \\ 0, & \text{otherwise} \end{cases} \right\} \quad (32)$$

which is shown in Figure 3. In this example, each secondary MF is chosen to be a triangle of height one whose base equals $\underline{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)$ and whose apex location Apex(x) is parameterized as

$$\text{Apex}(x) = \underline{\mu}(\tilde{A})(x) + w[\underline{\mu}(\tilde{A})(x) - \underline{\mu}(\tilde{A})(x)] \quad (33)$$

where $w = 0, 0.25, 0.5, 0.75, 1$.

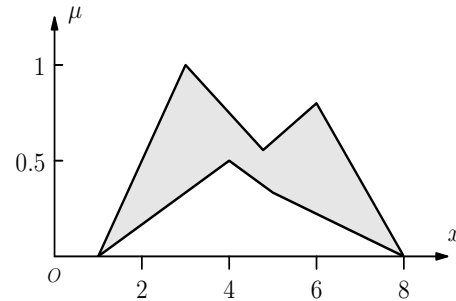


Fig. 3. The FOU of Example 2

Set $w = 0$, let $\alpha_l = 0, 1$ for (20), $\alpha_r = 0, 1$ for (22), then we can get $l_0 = 3.6605, l_1 = 4.2667$ and $r_0 = 4.9917, r_1 = 4.2667$, respectively. Then the left centroid function $\alpha_l(l)$ with $l \in [l_0, l_1]$ and right centroid function $\alpha_r(r)$ with $r \in [r_1, r_0]$

can be obtained. Replacing l, r with the same variable x , $\alpha_l(x)$ and $\alpha_r(x)$ can be plotted together.

The other cases for $w = 0.25, 0.5, 0.75, 1$ can also be obtained in a similar way. The two pairs of end points for the left centroid and right centroid functions for the different cases of w are listed in Table III. The left and right centroid functions are plotted in Fig. 4. These are almost the same as the results of [19] which are computed with $N = 10000$, $m = 100$ respectively.

TABLE III
THE CENTROID FUNCTIONS BOUNDS OF EXAMPLE 2

w	$[l_0, l_1]$	$[r_1, r_0]$
0	[3.6605, 4.2667]	[4.2667, 4.9917]
0.25	[3.6605, 4.3008]	[4.3008, 4.9917]
0.5	[3.6605, 4.3208]	[4.3208, 4.9917]
0.75	[3.6605, 4.3340]	[4.3340, 4.9917]
1	[3.6605, 4.3433]	[4.3433, 4.9917]

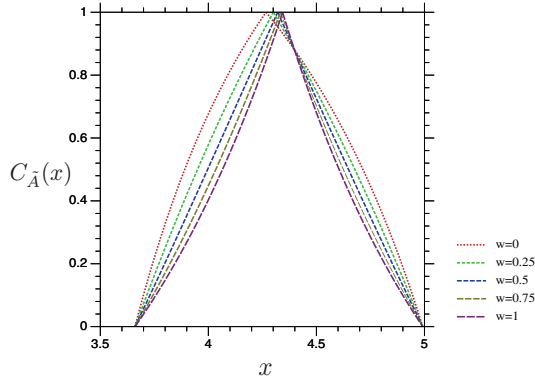


Fig. 4. The centroid functions of Example 2

All the centroid functions are kept in triangular shape with $l_1 = r_1$. With (24), the final centroid defuzzification values of these fuzzy sets are 4.4640, 4.4800, 4.4918, 4.5012, 4.5090 respectively.

Example 3: Gaussian LMF and UMF, and trapezoidal secondary MFs [19], which is also similar to the example of [11]:

$$\bar{\mu}_{\tilde{A}_0}(x) = \max \left\{ \exp \left[-\frac{(x-3)^2}{8} \right], 0.8 \exp \left[-\frac{(x-6)^2}{8} \right] \right\} \quad (34)$$

$$\underline{\mu}_{\tilde{A}_0}(x) = \max \left\{ 0.5 \exp \left[-\frac{(x-3)^2}{2} \right], 0.4 \exp \left[-\frac{(x-6)^2}{2} \right] \right\} \quad (35)$$

which is shown in Figure 5. The secondary MF of $\tilde{A}$ is chosen to be a trapezoid whose base equals $\bar{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_0}(x)$ and whose top is defined by left and right endpoints, which are $\underline{\mu}_{\tilde{A}_1}(x)$ and $\bar{\mu}_{\tilde{A}_1}(x)$, and both are parameterized as

$$\underline{\mu}_{\tilde{A}_1}(x) = \underline{\mu}_{\tilde{A}_0}(x) + 0.6w \left[\bar{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_0}(x) \right] \quad (36)$$

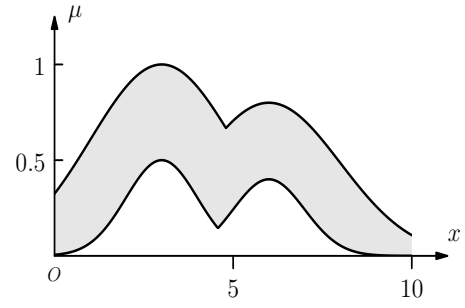


Fig. 5. The FOU of Example 3

$$\bar{\mu}_{\tilde{A}_1}(x) = \bar{\mu}_{\tilde{A}_0}(x) - 0.6(1-w) \left[\bar{\mu}_{\tilde{A}_0}(x) - \underline{\mu}_{\tilde{A}_0}(x) \right] \quad (37)$$

where $w = 0, 0.25, 0.5, 0.75, 1$, and the domain is $x \in [0, 10]$.

Set $w = 0$, let $\alpha_l = 0, 1$ for (20), $\alpha_r = 0, 1$ for (22), then we can get $l_0 = 3.156$, $l_1 = 3.680$, $r_1 = 5.059$, $r_0 = 5.675$, respectively. Then the left centroid function $\alpha_l(l)$ with $l \in [l_0, l_1]$ and right centroid function $\alpha_r(r)$ with $r \in [r_1, r_0]$ can be obtained. Replacing l, r with the same variable x , $\alpha_l(x)$ and $\alpha_r(x)$ can be plotted together.

The other cases for $w = 0.25, 0.5, 0.75, 1$ can also be obtained in a similar way. The two pairs of end points for the left centroid and right centroid functions for the different cases of w are listed in Table IV. The left and right centroid functions are plotted in Fig. 6. These are almost the same as the results of [19] which are computed with KM algorithms of $N = 10000$, $m = 100$ and Table I respectively.

TABLE IV
THE CENTROID FUNCTIONS BOUNDS OF EXAMPLE 3

w	$[l_0, l_1]$	$[r_1, r_0]$
0	[3.1557, 3.6800]	[5.0589, 5.6747]
0.25	[3.1557, 3.8281]	[4.9454, 5.6747]
0.5	[3.1557, 3.9286]	[4.8686, 5.6747]
0.75	[3.1557, 4.0013]	[4.8131, 5.6747]
1	[3.1557, 4.0565]	[4.7711, 5.6747]

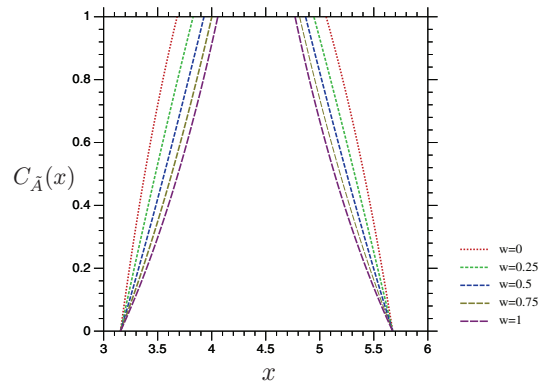


Fig. 6. The centroid functions of Example 3

All the centroid functions are kept in trapezoidal shape. With (24), the final centroid defuzzification values of these fuzzy sets are 4.9080, 4.8721, 4.8565, 4.8490, 4.8454 respectively.

This general type-2 defuzzification method not only utilizes the rooting finding method in centroid computation of the IT2 FSs, and also avoid in the data sampling in the second membership function, the repetitive iterate computation can be avoided, which can greatly reduce the amount of computation in application. In fact, except the possible root finding iteration at the two end points $\alpha = 0, 1$, there is no iteration in our method at all, which is helpful to loose the computation bottleneck for defuzzification applications, even for realtime implementation requirements.

Another advantage of our method is that, we use the integral operation of the membership function instead of the sum operation of the sampling data, the centroid value is the exact solution of the original problem, which is different from current approximate data sampling methods in essence. We can also use numerical integral method from the same sampling data of current sampling methods where computation error can be greatly reduced, and the error bound can be estimated in advance with the theory in numerical analysis.

Note that, both in the IT2 FS and general T2 FS centroid computation, we do not pay much attention on the lower level implementation of the algorithms. As we have mentioned, this can be done through the proper use of software packages or the programming of computer languages directly. The numerical integration and root finding methods in numerical analysis and the computation cost reduction techniques such as Enhanced KM algorithms [27] can be applied, but this is beyond the main topic of this paper.

V. CONCLUSION

With the continuous form of the centroid computation of interval type-2 (IT2 FS) fuzzy sets (IT2 FSs), this paper transforms the centroid computation of IT2 Fs into root finding problems. When the FOU's of the IT2 FS are given, the centroid of the IT2 Fs can be computed directly and accurately. Then efficient methods for the centroid computation of general type-2 fuzzy sets with linear membership function are provided. Numerical examples illustrate the proposed approaches.

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