

Deriving the Input-Output Mathematical Relationship for a Class of Interval Type-2 Mamdani Fuzzy Controllers

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Abstract—Most fuzzy controllers, type-1 (T1) or type-2 (T2), have been used and treated as black boxes in that their explicit mathematical input-output mappings (i.e., analytical structures) are unknown. Revealing and analyzing the analytical structure is important as it will lay a solid foundation for better understanding, more insightful analysis, and more effective design of fuzzy control systems. We previously developed a general technique to derive the analytical structures of the type-1 fuzzy controllers that employed Zadeh AND operator. We now extend our study to a class of typical interval type-2 fuzzy controller that adopt Zadeh AND operator and the popular Karnik-Mendel iterative center-of-sets type reducer. A novel analytical structure deriving technique is developed. And the resulting input-output mathematical relationship for the controller is received.

Keywords- Type-2 fuzzy control; analytical structure; input-output mathematical relationship

I. INTRODUCTION

In the past several years, research results on type-2 (T2) fuzzy control have started to emerge in the literature [1-5]. Many results claim that T2 fuzzy control outperforms its counterpart type-1 (T1) fuzzy control (e.g., traditional fuzzy control). Logically, many questions arise. Why is this the case? Do the T2 fuzzy controllers work well because their input-output structures are more peculiar than T1 fuzzy controllers and other conventional nonlinear controllers? What are their structures then? All these questions are of considerable importance. And moreover, once the mathematical relationship information becomes available, analyzing and designing T2 fuzzy control systems can be studied in the framework of the well-developed conventional (nonlinear) control theory as the fuzzy control problems will be transformed to nonlinear control problems.

Actually, deriving the input-output mathematical relationship is no easy in most cases. The reality is that many fuzzy controllers are constructed via the “knowledge engineering approach” as opposed to the mathematical approaches exclusively adopted in conventional control. All the fuzzy controllers have been treated and used as black-box controllers in the sense that their analytical structures are unknown. And for conventional control, this is never an issue

because the analytical structure of the conventional control, linear or nonlinear, is always readily available for analysis and design. For T1 fuzzy controller, we previously did a lot of research on its analytical structure [6, 7]. Our research methods and results on the analytical structure of T1 fuzzy controllers create great help for better analysis and design of fuzzy controller, which have been widely used in literature [8]. Few of the present literatures about T2 fuzzy control, however, are concerned with mathematical input-output relationship of a T2 fuzzy controller [9, 10].

In the present paper, the input-output relationship of a class of interval T2 Mamdani fuzzy controller is derived. Their configurations are typical - any types of input fuzzy sets and output fuzzy sets, arbitrary fuzzy rules, Zadeh AND operator, the Karnik-Mendel center-of-sets type reducer, and the centroid defuzzifier.

II. CONFIGURATION OF A CLASS OF INTERVAL T2 MAMDANI FUZZY CONTROLLERS

The T2 fuzzy controllers in this study have two input variables, $x_1(n)$ and $x_2(n)$, and one output variable, $\Delta u(n)$, which is the change of controller output, where n represents the sampling instance. For simplicity, x_1 , x_2 and Δu will be used instead of $x_1(n)$, $x_2(n)$ and $\Delta u(n)$. We suppose that x_1 is defined on $[L_1, R_1]$ that is divided into $N_1 - 1$ subintervals: $[S_1, S_2], \dots, [S_i, S_{i+1}], \dots, [S_{N_1-1}, S_{N_1}]$, $1 \leq i \leq N_1$. There are N_1 interval T2 fuzzy sets, $A_1, \dots, A_i, \dots, A_{N_1}$. A_i is defined over $[S_{i-1}, S_{i+1}]$ and its membership value is zero everywhere else. The upper and lower primary membership functions of A_i are designated as $\bar{\mu}_{A_i}(x_1)$ and $\underline{\mu}_{A_i}(x_1)$, respectively. We assume that: (1) $\bar{\mu}_{A_i}(x_1) \geq \underline{\mu}_{A_i}(x_1)$, and (2) $\bar{\mu}_{A_i}(x_1)$ (and $\underline{\mu}_{A_i}(x_1)$) increases from 0, reaches its maximum (can be a range of the same maximum), and then decreases to 0. Most of the widely-used fuzzy sets in the literature (e.g., the trapezoidal ones) meet the assumptions. To illustrate the assumptions, Fig. 1 provides some example T2 fuzzy sets. Likewise, we suppose that x_2 is

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defined in $[L_2, R_2]$ that is divided into $N_2 - 1$ subintervals, namely $[M_1, M_2], \dots, [M_j, M_{j+1}], \dots, [M_{N_2-1}, M_{N_2}]$, over which N_2 interval T2 fuzzy sets meeting the same assumptions above are defined. Each of them is denoted B_j ($1 \leq j \leq N_2$) whose upper and lower primary membership functions are $\bar{\mu}_{B_j}(x_2)$ and $\underline{\mu}_{B_j}(x_2)$, respectively.

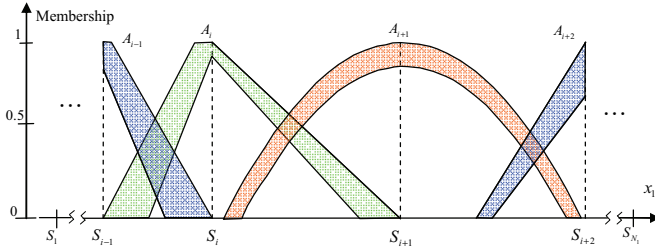


Fig. 1. Example interval T2 fuzzy sets for input variable x_1 .

We use a total of $N_1 \times N_2$ Mamdani fuzzy rules to cover all the possible combinations of the input fuzzy sets. At any sampling instance, only two adjacent A_i and two adjacent B_j will be involved in the fuzzification of x_1 and x_2 , respectively, due to the manner that the fuzzy sets are defined. Without losing generality, we assume that $A_i, A_{i+1}, B_j, B_{j+1}$ are the ones that are involved. Accordingly, the following four fuzzy rules are executed:

IF x_1 is A_i AND x_2 is B_j THEN Δu is $G_{h(i,j)}$. (Rule 1)

IF x_1 is A_i AND x_2 is B_{j+1} THEN Δu is $G_{h(i,j+1)}$. (Rule 2)

IF x_1 is A_{i+1} AND x_2 is B_j THEN Δu is $G_{h(i+1,j)}$. (Rule 3)

IF x_1 is A_{i+1} AND x_2 is B_{j+1} THEN Δu is $G_{h(i+1,j+1)}$ (Rule 4)

where Zadeh fuzzy AND operator (i.e., $\min()$) is used and $G_{h(i,j)}, G_{h(i,j+1)}, G_{h(i+1,j)}$ and $G_{h(i+1,j+1)}$ are interval T2 fuzzy sets for Δu , which can be any continuous T2 fuzzy sets, interval type or not. These output fuzzy sets are indexed by integer subscripts whose values are computed using function $h(\cdot)$ so that the input and output fuzzy sets are mathematically linked. Because $h(\cdot)$, chosen by the controller designer, can be any function, the fuzzy rules subsequently can be arbitrary in terms of input and output relation.

The firing interval for the input fuzzy sets in Rule k ($k=1,2,3,4$), denoted $[\underline{f}_k(x_1, x_2), \bar{f}_k(x_1, x_2)]$, is first calculated [2]:

$$\begin{aligned} f_1(x_1, x_2) &= [\underline{f}_1(x_1, x_2), \bar{f}_1(x_1, x_2)] \\ &= [\min(\underline{\mu}_{A_i}(x_1), \underline{\mu}_{B_j}(x_2)), \min(\bar{\mu}_{A_i}(x_1), \bar{\mu}_{B_j}(x_2))] \end{aligned} \quad (1)$$

$$\begin{aligned} f_2(x_1, x_2) &= [\underline{f}_2(x_1, x_2), \bar{f}_2(x_1, x_2)] \\ &= [\min(\underline{\mu}_{A_i}(x_1), \underline{\mu}_{B_{j+1}}(x_2)), \min(\bar{\mu}_{A_i}(x_1), \bar{\mu}_{B_{j+1}}(x_2))] \end{aligned} \quad (2)$$

$$\begin{aligned} f_3(x_1, x_2) &= [\underline{f}_3(x_1, x_2), \bar{f}_3(x_1, x_2)] \\ &= [\min(\underline{\mu}_{A_{i+1}}(x_1), \underline{\mu}_{B_j}(x_2)), \min(\bar{\mu}_{A_{i+1}}(x_1), \bar{\mu}_{B_j}(x_2))] \end{aligned} \quad (3)$$

$$\begin{aligned} f_4(x_1, x_2) &= [\underline{f}_4(x_1, x_2), \bar{f}_4(x_1, x_2)] \\ &= [\min(\underline{\mu}_{A_{i+1}}(x_1), \underline{\mu}_{B_{j+1}}(x_2)), \min(\bar{\mu}_{A_{i+1}}(x_1), \bar{\mu}_{B_{j+1}}(x_2))] \end{aligned} \quad (4)$$

In this paper, the Karnik-Mendel center-of-sets type reducer is used to reduce the T2 output sets to $\Delta U = [\Delta u_L, \Delta u_R]$, a T1 fuzzy set, which is then defuzzified to produce a crisp number Δu . In the Karnik-Mendel type reducer, the centroid for Rule k 's output fuzzy set, denoted $[\alpha_k, \gamma_k]$, is computed first, on-line or off-line [1]. The rather complex calculation procedure is described in [1]. Because we will only need the outcome of the reducer prior to the structure derivation and do not care how it is produced, the procedure is omitted here for better presentation. γ_k 's and α_k 's are then respectively arranged in the ascending orders and the results are represented by: $\gamma_1^* \leq \gamma_2^* \leq \gamma_3^* \leq \gamma_4^*$ and $\alpha_1^* \leq \alpha_2^* \leq \alpha_3^* \leq \alpha_4^*$. We then arrange $\bar{f}_k(x_1, x_2)$ and $\underline{f}_k(x_1, x_2)$ to correspond to $\gamma_1^* \leq \gamma_2^* \leq \gamma_3^* \leq \gamma_4^*$ and $\alpha_1^* \leq \alpha_2^* \leq \alpha_3^* \leq \alpha_4^*$, respectively, resulting in $\bar{f}_1^*(x_1, x_2) \leq \bar{f}_2^*(x_1, x_2) \leq \bar{f}_3^*(x_1, x_2) \leq \bar{f}_4^*(x_1, x_2)$ and $\underline{f}_1^*(x_1, x_2) \leq \underline{f}_2^*(x_1, x_2) \leq \underline{f}_3^*(x_1, x_2) \leq \underline{f}_4^*(x_1, x_2)$. The left-most point Δu_L and right-most point of the interval ΔU_R can then be represented as [2]:

$$\Delta u_L = \frac{\sum_{i=1}^{P_L} \bar{f}_i^*(x_1, x_2) \alpha_i + \sum_{j=P_L+1}^4 \bar{f}_j^*(x_1, x_2) \alpha_j}{\sum_{i=1}^{P_L} \bar{f}_i^*(x_1, x_2) + \sum_{j=P_L+1}^4 \bar{f}_j^*(x_1, x_2)} \quad (5)$$

$$\Delta u_R = \frac{\sum_{i=1}^{P_R} \underline{f}_i^*(x_1, x_2) \gamma_i + \sum_{j=P_R+1}^4 \underline{f}_j^*(x_1, x_2) \gamma_j}{\sum_{i=1}^{P_R} \underline{f}_i^*(x_1, x_2) + \sum_{j=P_R+1}^4 \underline{f}_j^*(x_1, x_2)} \quad (6)$$

where integers P_L ($1 \leq P_L \leq 3$) and P_R ($1 \leq P_R \leq 3$) are switching points that depend on the input and output fuzzy sets and the values of x_1 and x_2 , and hence vary with n . Their calculation procedure is fully described in [2]. Since ΔU_R is an interval set, the centroid defuzzifier is used to produce the crisp output Δu [4]:

$$\Delta u = \frac{1}{2}(\Delta u_L + \Delta u_R) \quad (7)$$

III. NEW TECHNIQUE FOR DERIVING ANALYTICAL STRUCTURE OF THE FUZZY CONTROLLERS

Without loss of generality, suppose arbitrarily that A_i, A_{i+1}, B_j , and B_{j+1} are as shown in Fig. 2 with the mathematical

definitions given in Table 1 (outside the intervals in the right column of Table 1, the membership values are 0).

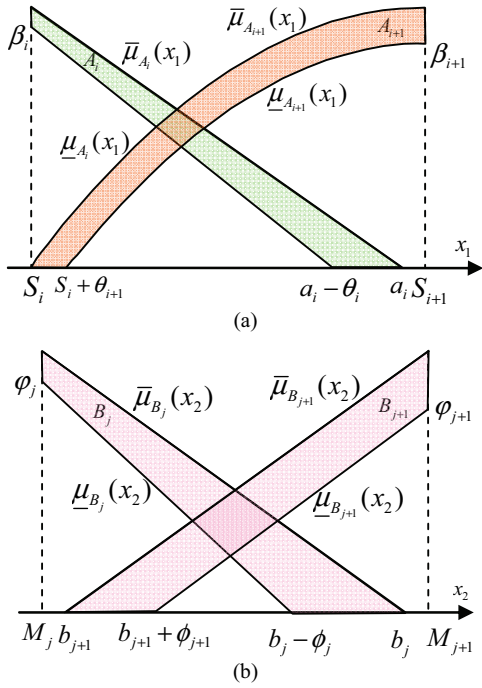


Fig. 2. Example interval T2 fuzzy sets used for (a) x_1 , and (b) x_2 at sampling time n .

TABLE I. MATHEMATICAL DEFINITIONS FOR THE PARTS OF THE PRIMARY MEMBERSHIP FUNCTIONS OF A_i , A_{i+1} , B_j , AND B_{j+1} .

Mathematical definition	Interval
$\bar{\mu}_{A_i}(x_1) = \frac{-1}{a_i - S_i} x_1 + \frac{S_i}{a_i - S_i} + 1$	$x_1 \in [S_i, a_i]$
$\underline{\mu}_{A_i}(x_1) = \frac{-\beta_i}{a_i - \theta_i - S_i} x_1 + \frac{S_i \beta_i}{a_i - \theta_i - S_i} + \beta_i$	$x_1 \in [S_i, a_i - \theta_i]$
$\bar{\mu}_{A_{i+1}}(x_1) = -\left(\frac{x_1 - S_{i+1}}{S_i - S_{i+1}}\right)^2 + 1$	$x_1 \in [S_i, S_{i+1}]$
$\underline{\mu}_{A_{i+1}}(x_1) = -\beta_{i+1} \left(\frac{x_1 - S_{i+1}}{S_i + \theta_{i+1} - S_{i+1}}\right)^2 + \beta_{i+1}$	$x_1 \in [S_i + \theta_{i+1}, S_{i+1}]$
$\bar{\mu}_{B_j}(x_2) = \frac{-1}{b_j - M_j} x_2 + \frac{M_j}{b_j - M_j} + 1$	$x_2 \in [M_j, b_j]$
$\underline{\mu}_{B_j}(x_2) = \frac{-\phi_j}{b_j - \phi_j - M_j} x_2 + \frac{M_j \phi_j}{b_j - \phi_j - M_j} + \phi_j$	$x_2 \in [M_j, b_j - \phi_j]$
$\bar{\mu}_{B_{j+1}}(x_2) = \frac{1}{M_{j+1} - b_{j+1}} x_2 - \frac{M_{j+1}}{M_{j+1} - b_{j+1}} + 1$	$x_2 \in [b_{j+1}, M_{j+1}]$
$\underline{\mu}_{B_{j+1}}(x_2) = \frac{\phi_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}} x_2 - \frac{M_{j+1} \phi_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}} + \phi_{j+1}$	$x_2 \in [b_{j+1} + \phi_{j+1}, M_{j+1}]$

To derive the explicit Δu expressions from (7), one must first determine the resulting membership value for each of the four rules involving the $\min()$ operation in (1)-(4) as a mathematical expression, as opposed to a numerical value. The only way to achieve this goal is to divide the area

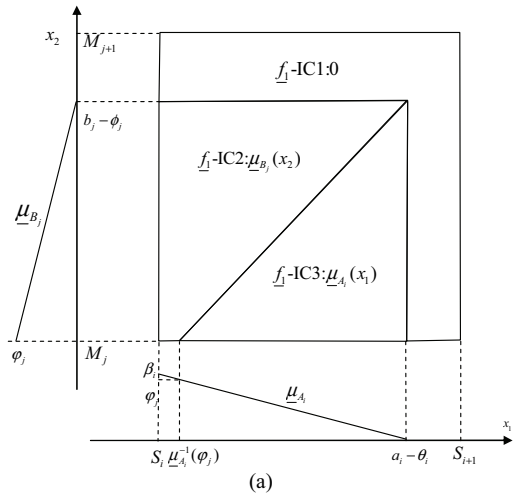
$[S_i, S_{i+1}] \times [M_j, M_{j+1}]$ into a number of regions, each of which is called IC (input combination) so that in each region, the inequality relationship between x_1 and x_2 is exclusive [6]. Without loss of generality, we assumed that $\phi_{j+1} \leq \beta_{i+1} \leq \phi_j \leq \beta_i$ in Table 1. Fig. 3 and 4 then show the input space division results for (1)-(4), where $\underline{\mu}_{A_i}^{-1}(\phi_j)$, $\underline{\mu}_{A_i}^{-1}(\phi_{j+1})$, $\underline{\mu}_{B_j}^{-1}(\beta_{i+1})$ and $\underline{\mu}_{A_{i+1}}^{-1}(\phi_{j+1})$ are the inverse functions of the linear parts of these four lower membership functions. For instance, Fig. 3(a) shows the divisions for $\underline{f}_1(x_1, x_2) = \min(\underline{\mu}_{A_i}(x_1), \underline{\mu}_{B_j}(x_2))$ from Rule 1. There are three regions and they are labeled as $\underline{f}_1$ -IC1, $\underline{f}_1$ -IC2 and $\underline{f}_1$ -IC3. In $\underline{f}_1$ -IC3, $\underline{\mu}_{A_i}(x_1) \leq \underline{\mu}_{B_j}(x_2)$, thus $\underline{f}_1(x_1, x_2) = \underline{\mu}_{A_i}(x_1)$. For $\underline{f}_1$ -IC1, because either $\underline{\mu}_{A_i}(x_1) = 0$ or $\underline{\mu}_{B_j}(x_2) = 0$, the result of $\min()$ is always 0. Accordingly, there should be eight different divisions for whole input space to determine the firing intervals in (1)-(4): division for $\underline{f}_1$ in (1), for $\bar{f}_1$ in (1), for $\underline{f}_2$ in (2), for $\bar{f}_2$ in (2), for $\underline{f}_3$ in (3), for $\bar{f}_3$ in (3), for $\underline{f}_4$ in (4), for $\bar{f}_4$ in (4). For space limitation, just two of them list in Fig. 3. And take Fig. 3(b) as another example. Boundary A, on which $\underline{\mu}_{A_{i+1}}(x_1) = \underline{\mu}_{B_j}(x_2)$, is obtained by solving the following equation:

$$\phi_j - \frac{\phi_j}{b_j - \phi_j - M_j} x_2 + \frac{M_j \phi_j}{b_j - \phi_j - M_j} = \beta_{i+1} - \beta_{i+1} \left(\frac{x_1 - S_{i+1}}{S_i + \theta_{i+1} - S_{i+1}} \right)^2 \quad (8)$$

Yielding

$$x_2 = \frac{\beta_{i+1}(b_j - \phi_j - M_j)}{\phi_j(S_i + \theta_{i+1} - S_{i+1})^2} (x_1 - S_{i+1})^2 + \frac{(\phi_j - \beta_{i+1})(b_j - \phi_j - M_j)}{\phi_j} + M_j \quad (9)$$

Once the boundary is determined, just compute the values of $\underline{\mu}_{A_{i+1}}(x_1)$ and $\underline{\mu}_{B_j}(x_2)$ using any pair of x_1 and x_2 in either IC. The function producing the smaller value should be the result of $\min()$ for that IC.



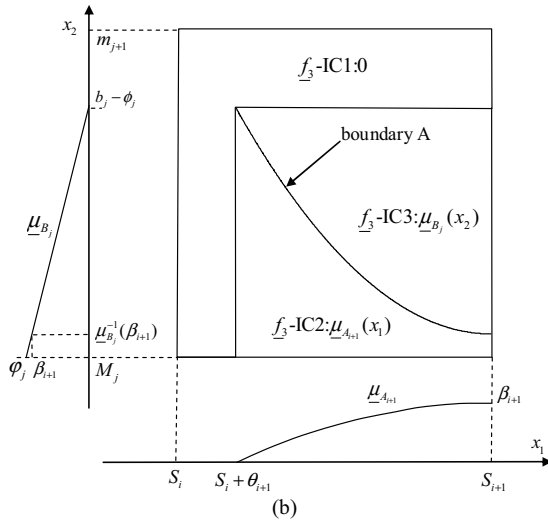


Fig. 3. Part of eight divisions of the input space to determine the firing intervals in (1)-(4): (a) for $\underline{f}_1$ in (1), (b) for $\underline{f}_3$ in (3).

These divisions are for each individual fuzzy rule only. Because all the rules are involved in calculating controller output Δu , all the four rules must be simultaneously considered to derive the controller structure. This simultaneous consideration is achieved if we superimpose the eight individual input space divisions shown in Fig. 3 to form an overall input space division (Fig. 4). It turns out that there are a total of 78 regions, labeled from IC1 to IC78.

Now, for each region in Fig. 4, we can decide $f_k(x_1, x_2)$ (i.e., $[f_k(x_1, x_2), \bar{f}_k(x_1, x_2)]$). For example, for IC40 (i.e., the region labeled 40 in Fig. 4), $\underline{f}_1(x_1, x_2) = \underline{\mu}_{B_j}(x_2)$, $\underline{f}_2(x_1, x_2) = \underline{\mu}_{A_i}(x_1)$, $\underline{f}_3(x_1, x_2) = \underline{\mu}_{B_j}(x_2)$, $\underline{f}_4(x_1, x_2) = \underline{\mu}_{B_{j+1}}(x_2)$, $\bar{f}_1(x_1, x_2) = \bar{\mu}_{B_j}(x_2)$, $\bar{f}_2(x_1, x_2) = \bar{\mu}_{A_i}(x_1)$, $\bar{f}_3(x_1, x_2) = \bar{\mu}_{B_j}(x_2)$ and

$\bar{f}_4(x_1, x_2) = \bar{\mu}_{B_{j+1}}(x_2)$. This process is carried out for all the 78 ICs. We do not list all the results here for space limitation. We emphasize that the result can be different if the input fuzzy sets are different.

For better notations, assume $\gamma_k^* = \gamma_k$ and $\alpha_k^* = \alpha_k$, and accordingly, $\bar{f}_k^*(x_1, x_2) = \bar{f}_k(x_1, x_2)$ and $\underline{f}_k^*(x_1, x_2) = \underline{f}_k(x_1, x_2)$. Putting the eight membership functions resulted from the min() operations in the four rules and the resulting centroids for an IC into the type reducer represented by (5) and (6), we will obtain Δu_R and Δu_L . Putting them into the defuzzifier (7) and making some algebraic simplifications, we will attain the analytical structure of the fuzzy controller for that particular IC if P_L and P_R in (5) and (6) are known.

According to the Karnik-Mendel type reducer, the exact values of P_L and P_R can always be computed as long as all the parameters of the input fuzzy sets, controller inputs, and the centroids of all the output fuzzy sets are numerically available. Hence, in principle for any specific fuzzy controller, the structure can always be derived. Nevertheless, our interest is on general controller configuration where these parameter values are not specified. As a result, P_L and P_R are unknown, making the analytical structure derivation significantly more complicated. Given that $P_L=1\sim3$ and $P_R=1\sim3$, there are a total of $3 \times 3 = 9$ possible outcomes, each of which is called a case, that (5) and (6) can produce for an IC. When $P_L=1$ and $P_R=1$, we define the case number as 1, the corresponding analytical structure of the T2 fuzzy controller with the general configuration is (10), the other eight cases can also be easily computed by Karnik-Mendel type reducer, which are omitted here.

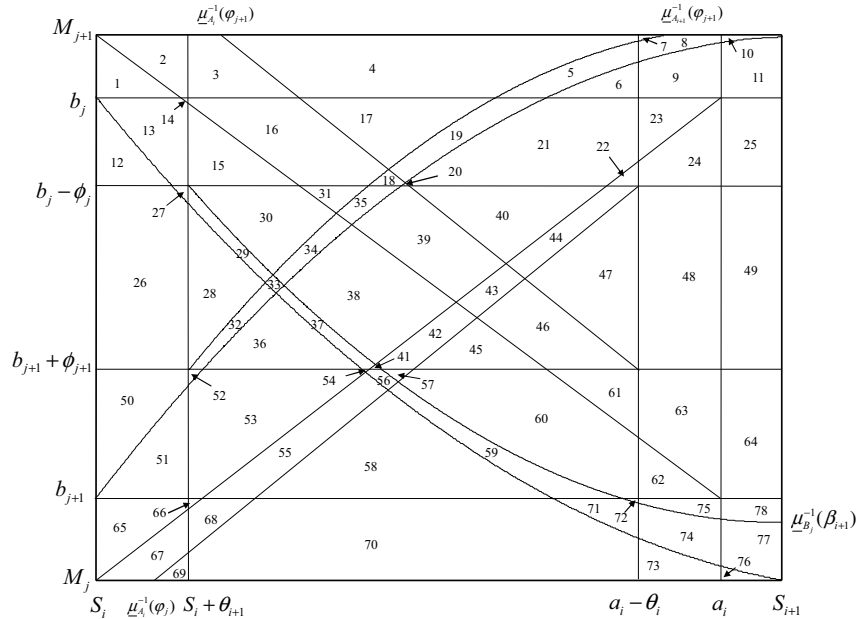


Fig. 4. Superimposing the ICs in Figs. 3(a)-3(h) to create an overall input space division (each region is labeled by an IC number)

$$\Delta u = \frac{1}{2} \left(\frac{\bar{f}_1^*(\cdot)\alpha_1 + \bar{f}_2^*(\cdot)\alpha_2 + \bar{f}_3^*(\cdot)\alpha_3 + \bar{f}_4^*(\cdot)\alpha_4}{\bar{f}_1^*(\cdot) + \bar{f}_2^*(\cdot) + \bar{f}_3^*(\cdot) + \bar{f}_4^*(\cdot)} + \frac{\bar{f}_1^*(\cdot)\gamma_1 + \bar{f}_2^*(\cdot)\gamma_2 + \bar{f}_3^*(\cdot)\gamma_3 + \bar{f}_4^*(\cdot)\gamma_4}{\bar{f}_1^*(\cdot) + \bar{f}_2^*(\cdot) + \bar{f}_3^*(\cdot) + \bar{f}_4^*(\cdot)} \right) \quad (10)$$

Continuing the IC40 example above, with different inputs, one can derive nine different analytical structures from the nine cases. We provide case No. 1 below: Eq. (11). As another instance, the analytical controller structure for another case number ($P_L=1$ and $P_R=2$) for IC28 is Eq.(12). For the 78 ICs shown in Fig. 4, there exist a total of $9 \times 78 = 702$ analytical structure expressions. We have derived all of them and found that their mathematical structures to be all similar to either (11) or (12).

The new structure-deriving method presented above is not restricted to the T2 fuzzy controllers with two input variables. Its principle is applicable to the fuzzy controllers involving more than two input variables. Nevertheless, the derivation task will be more challenging. The biggest difficulty is to divide the n-dimension input space ($n \geq 3$) into a number of n-dimensional ICs. Take $n=3$ as example - one must properly divide the 3D input space into many 3D ICs so that Zadeh AND operation can be carried out for each of the fuzzy rules (at least $2^3 = 8$ rules). This step is the same, in spirit, as what is shown in Fig. 3 except each IC is no longer two-dimensional, but 3D. Also, for all the $2^3 = 8$ rules, there will be $2 \times 2^3 = 16$ individual divisions that will be superimposed to generate overall 3D ICs, as suppose to $2 \times 2^2 = 8$ individual divisions in Fig. 3. The number of switching points will be more - $P_L=1 \sim 7$ and $P_R=1 \sim 7$, resulting in a total of $7 \times 7 = 49$ different cases.

IV. CONCLUSION

A novel general analytical structure-deriving technique for a class of interval T2 Mamdani fuzzy controllers that employ Zadeh fuzzy AND operator is developed. The

mathematical input-output mapping of this class of T2 fuzzy controllers is derived then. From the results we concluded that the universe of inputs can be divided into several regions, the mathematical input-output structure is not the same in different region. It depends on controller and controller inputs. The mathematical input-output relationships are complicated nonlinear function of inputs.

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$$\begin{aligned} \Delta u = & \frac{\left(\frac{-\gamma_2}{a_i - S_i} \right) x_1 + \left(\frac{-\gamma_2}{b_j - M_j} - \frac{\gamma_4}{b_{j+1} - M_{j+1}} + \frac{\gamma_1 \varphi_j}{M_j + \phi_j - b_j} \right) x_2 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_1 \varphi_j - \frac{\gamma_1 M_j \varphi_j}{M_j + \phi_j - b_j} + \frac{\gamma_2 S_i}{a_i - S_i} + \frac{\gamma_3 M_j}{b_j - M_j} + \gamma_4 \frac{M_{j+1}}{b_{j+1} - M_{j+1}}}{\left(\frac{-2}{a_i - S_i} \right) x_1 + \left(\frac{-2}{b_j - M_j} - \frac{2}{b_{j+1} - M_{j+1}} + \frac{2\varphi_j}{M_j + \phi_j - b_j} \right) x_2 + 6 + 2\varphi_j + \frac{2S_i}{a_i - S_i} + \frac{2M_j}{b_j - M_j} + \frac{2M_{j+1}}{b_{j+1} - M_{j+1}} - \frac{2M_j \varphi_j}{M_j + \phi_j - b_j}} \\ & + \frac{\left(\frac{-\alpha_2 \beta_j}{a_i - S_i - \theta_i} \right) x_1 + \left(\frac{-\alpha_1}{b_j - M_j} + \frac{\alpha_4 \varphi_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}} + \frac{\alpha_3 \varphi_j}{M_j + \phi_j - b_j} \right) x_2 + \alpha_1 + \frac{\alpha_1 M_j}{b_j - M_j} + \alpha_2 \beta_j + \frac{\alpha_2 S_i \beta_j}{a_i - S_i - \theta_i} + \alpha_3 \varphi_j - \frac{\alpha_3 \varphi_j M_j}{M_j + \phi_j - b_j} + \alpha_4 \varphi_{j+1} - \frac{\alpha_4 \varphi_{j+1} M_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}}}{\left(\frac{-2\beta_j}{a_i - S_i - \theta_i} \right) x_1 + \left(\frac{-2}{b_j - M_j} + \frac{2\varphi_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}} + \frac{2\varphi_j}{M_j + \phi_j - b_j} \right) x_2 + 2\beta_j + \frac{2S_i \beta_j}{a_i - S_i - \theta_i} + 2 + \frac{2M_j}{b_j - M_j} + 2\varphi_j - \frac{2\varphi_j M_j}{M_j + \phi_j - b_j} + 2\varphi_{j+1} - \frac{2\varphi_{j+1} M_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}}} \\ \Delta u = & \frac{\frac{(\alpha_3 + \alpha_4) \beta_{j+1}}{(S_i - S_{i+1} + \theta_{i+1})^2} x_1^2 + \frac{-2(\alpha_3 + \alpha_4) \beta_{j+1} S_{i+1}}{(S_i - S_{i+1} + \theta_{i+1})^2} x_1 + \left(\frac{\alpha_2 \varphi_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}} - \frac{\alpha_1}{b_j - M_j} \right) x_2 + (\alpha_3 + \alpha_4) \beta_{j+1} + (\alpha_3 + \alpha_4) \beta_{j+1} \frac{S_{i+1}^2}{(S_i - S_{i+1} + \theta_{i+1})^2} + \alpha_1 + \frac{\alpha_1 M_j}{b_j - M_j} + \alpha_2 \varphi_{j+1} - \frac{\alpha_2 \varphi_{j+1} M_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}}}{\frac{4\beta_{j+1}}{(S_i - S_{i+1} + \theta_{i+1})^2} x_1^2 + \frac{-8\beta_{j+1} S_{i+1}}{(S_i - S_{i+1} + \theta_{i+1})^2} x_1 + \left(\frac{2\varphi_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}} - \frac{2}{b_j - M_j} \right) x_2 + 4\beta_{j+1} + \frac{4\beta_{j+1} S_{i+1}^2}{(S_i - S_{i+1} + \theta_{i+1})^2} + 2 + \frac{2M_j}{b_j - M_j} + 2\varphi_{j+1} - \frac{2\varphi_{j+1} M_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}}} \\ & + \frac{\frac{-(\gamma_3 + \gamma_4)}{(S_i - S_{i+1})^2} x_1^2 + \frac{2(\gamma_3 + \gamma_4) S_{i+1}}{(S_i - S_{i+1})^2} x_1 + \left(\frac{\gamma_1 \varphi_j}{M_j + \phi_j - b_j} + \frac{\gamma_2 \varphi_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}} \right) x_2 + \gamma_3 + \gamma_4 - \frac{(\gamma_3 + \gamma_4) S_{i+1}^2}{(S_i - S_{i+1})^2} + \gamma_1 \varphi_j + \frac{\gamma_1 \varphi_j M_j}{M_j + \phi_j - b_j} + \gamma_2 \varphi_{j+1} - \frac{\gamma_2 \varphi_{j+1} M_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}}}{\frac{-4}{(S_i - S_{i+1})^2} x_1^2 + \frac{8S_{i+1}}{(S_i - S_{i+1})^2} x_1 + \left(\frac{2\varphi_j}{M_j + \phi_j - b_j} + \frac{2\varphi_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}} \right) x_2 + 4 - \frac{4S_{i+1}^2}{(S_i - S_{i+1})^2} + 2\varphi_j + \frac{2\varphi_j M_j}{M_j + \phi_j - b_j} + 2\varphi_{j+1} - \frac{2\varphi_{j+1} M_{j+1}}{M_{j+1} - b_{j+1} - \phi_{j+1}}} \end{aligned} \quad (12)$$