

A Fuzzy Representation for Non-additive Weights of AHP

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Abstract - AHP (Analytic Hierarchy Process) has been widely used in a domain of decision making, and there are a lot of extensions using fuzzy measure and integrals. In the previous research, we propose a kind of sensitivity analyses for AHP or fuzzy measure AHP, because it often occurs that data of AHP loses its reliability. In other words a comparison matrix there in does not always have enough consistency. Also in these cases, the authors propose one representation for weights using results from the sensitivity analysis. In this paper we refine fuzzy weights representation for fuzzy measure AHP. It can show how results of (fuzzy) AHP have fuzziness when the comparison matrix does not have relatively good consistency.

Keywords-decision making; AHP; fuzzy measure; fuzzy sets; sensitivity analysis

I. INTRODUCTION

The AHP was proposed by Saaty, T.L. in 1977 [2], [3], and it has been widely used in a field of decision making since it includes the natural feelings of human. Normalized eigenvector that corresponds to a Frobenius root of comparison matrix is usually used as weight.

On the other hand, fuzzy measure AHP [4], an extension of the normal AHP, allows us to obtain two types of decisions by use of fuzzy measure. Substitutive and complementary properties are considered in the method.

However, it often occurs that data of fuzzy measure AHP loses its reliability, because a comparison matrix does not always have consistency. The consistency of a comparison matrix A is usually measured by the following consistency index

$$C.I. = \frac{\lambda_A - n}{n - 1} \quad (1)$$

where n is the order of matrix A , and λ_A is its Frobenius root. In these cases, it is very important to investigate how components of a pairwise comparison matrix influence on its consistency or on weights. The sensitivity analysis[5] is useful to investigate them. In this study, the authors apply a sensitivity analysis for fuzzy measure AHP that some of the present authors have proposed before [6]. It evaluates a fluctuation of the consistency index and weights, when a comparison matrix is perturbed. Moreover, we suppose that answers from a decision-maker have ambiguous or fuzziness

in these cases. Therefore, it is natural for the weights to be represented by fuzzy sets. Then, we refine the representation by use of results from the sensitivity analysis and fuzzy sets operations.

II. NON-ADDITIVE WEIGHTS OF AHP

Since the weights of criteria in AHP are normalized as their sum becomes 1, it can be regarded as additive measure similar to the probability measure. Because of the additive property of the weights, substitutive and complementary evaluations are not considered in the normal AHP. The substitutive evaluation prefers an alternative that is particularly excellent, and the complementary evaluation prefers one with as few defects as possible. Further, reversals of priority orders can occur due to the addition of a similar criterion.

Meanwhile, with fuzzy measure AHP[4], we can obtain two types of decisions using the possibility measure and the necessity measure that are non-additive fuzzy measure. Substitutive and complementary decisions are considered in it. Results from the normal AHP can be regarded as the midway evaluation between substitutive and complementary. Moreover, the reversals of the priority orders due to the addition of a similar criterion are mitigated.

Let x_i and $w=(w_i)$ ($i=1, \dots, n$) denote criteria and their weights, y_p ($p=1, \dots, m$) be the alternatives, and $f_p(x_i)$ be a weight of x_i with respect to y_p . In the normal AHP the weight w is normalized as their sum of components becomes 1. But in fuzzy measure AHP, w is considered to be possibility measure (or necessity measure). Therefore, w_i is regarded as the degree of importance expressed by possibility (or necessity) measure of the i -th criterion. Weight w is eigenvector as in normal AHP, however the conditions are $0 \leq w_i \leq 1$, and $\exists i, w_i=1$. Weights $f_p(x_i)$ only satisfy $0 \leq f_p(x_i) \leq 1$. When w_i takes q values $r_1 < r_2 < \dots < r_q=1$, we can make a set A_l such as

$$A_l = \{x_i \mid w_i \geq r_l\}, \quad (l=1, \dots, q) \quad (2)$$

Thus, the basic probability assignment of A_l becomes

$$m(A_l) = r_l - r_{l-1} \quad (3)$$

where $r_0=0$. The upper limit expectation E^* based on possibility measure and the lower limit expectation E_* based on necessity measure are found as follows

$$E_p^* = \sum_l^q m(A_l) \max_{x \in A_l} f_p(x), \quad (4)$$

$$E_{*p} = \sum_l^q m(A_l) \min_{x \in A_l} f_p(x). \quad (5)$$

Accordingly, the most preferable alternatives are

$$y^* = \{y_p \mid \max_p E_p^*\},$$

$$y_* = \{y_p \mid \max_p E_{*p}\}.$$

If a decision-maker desires a substitutive decision, then y^* is acceptable. And if he or she prefers a complementary decision, y_* is acceptable.

It should be noted that weights of alternatives y^* and y_* in fuzzy measure AHP are not fuzzy value but crisp, although fuzzy measure is used.

III. SENSITIVITY ANALYSES

When we actually use AHP, it often occurs that a comparison matrix is not consistent or that there is not great difference among the overall weights of the alternatives. In these cases, it is very important to investigate how components of the pairwise comparison matrix influence on its consistency or on the weights. To analyse how results are influenced when a certain variable has changed, we can use sensitivity analysis.

In this study, we use a method that some of the present authors have proposed before. It evaluates a fluctuation of the consistency index and the weights when the comparison matrix is perturbed. It is useful because it does not change a structure of the data.

Since the pairwise comparison matrix is a positive square matrix, Perron-Frobenius theorem holds [1]. From Perron-Frobenius theorem, following theorem about a perturbed comparison matrix holds.

Theorem 1 Let $A = (a_{ij})$, $(i, j = 1, \dots, n)$ denote a comparison matrix and let $A(\varepsilon) = A + \varepsilon D_A$, $D_A = (a_{ij}d_{ij})$ denote a matrix that has been perturbed. Let λ_A be the Frobenius root of A , w be the eigenvector corresponding to λ_A , and v be the eigenvector corresponding to the Frobenius root of A' . Then, a Frobenius root $\lambda(\varepsilon)$ of $A(\varepsilon)$ and a corresponding eigenvector $w(\varepsilon)$ can be expressed as follows

$$\lambda(\varepsilon) = \lambda_A + \varepsilon \lambda^{(1)} + o(\varepsilon), \quad (6)$$

$$w(\varepsilon) = w + \varepsilon w^{(1)} + o(\varepsilon), \quad (7)$$

where

$$\lambda^{(1)} = \frac{v' D_A w}{v' w}, \quad (8)$$

$w^{(1)}$ is an n -dimension vector that satisfies

$$(A - \lambda_A I) w^{(1)} = -(D_A - \lambda^{(1)} I) w, \quad (9)$$

where $o(\varepsilon)$ denotes an n -dimension vector in which all components are $o(\varepsilon)$.

About a fluctuation of the consistency index, following corollary can be obtained from Theorem 1.

Corollary 1 Using appropriate g_{ij} , we can represent the consistency index $C.I.(\varepsilon)$ of the perturbed comparison matrix $A(\varepsilon)$ as follows

$$C.I.(\varepsilon) = C.I. + \varepsilon \sum_i^n \sum_j^n g_{ij} d_{ij} + o(\varepsilon). \quad (10)$$

To see g_{ij} in the equation (10) in Corollary 1, how the components of a comparison matrix impart influence on its consistency can be found.

A. Analysis for weights of normal AHP

About the fluctuation of the weights, following corollary also can be obtained from Theorem 1.

Corollary 2 Using appropriate $h_{ij}^{(k)}$, we can represent the fluctuation $w_k^{(1)} = (w_k^{(1)})$ of the weight (i.e. the eigenvector corresponding to the Frobenius root) as follows

$$w_k^{(1)} = \sum_i^n \sum_j^n h_{ij}^{(k)} d_{ij}. \quad (11)$$

From the equation (7) in Theorem 1, the component that has a great influence on weight $w(\varepsilon)$ is the component which has the greatest influence on $w^{(1)}$. Accordingly, from Corollary 2, how components of a comparison matrix impart influence on the weights, can be found, to see $h_{ij}^{(k)}$ in the equation (11).

Calculations or proofs of these theorem and corollaries are shown in [8].

B. Analysis for fuzzy measure AHP

About the fluctuation of the basic probability assignment, following corollary also can be obtained from corollary 2..

Corollary 3 Using appropriate $e_{ij}^{(k)}$, we can represent the fluctuation $m(A_l)^{(1)}$ of the basic probability assignment as follows

$$m(A_l)^{(1)} = \sum_i^n \sum_j^n e_{ij} d_{ij} + o(\varepsilon) \quad (12)$$

(Proof)

Let $w(\varepsilon) = (w(\varepsilon)_i)$, $(\max w(\varepsilon)_i = 1)$ be the eigenvector that corresponds to the Frobenius root of the perturbed comparison matrix $A(\varepsilon)$. It is possible to replace the elements as arranged the order from the smallest. When $w(\varepsilon)_i$ takes q values $r(\varepsilon)_1 < r(\varepsilon)_2 < \dots < r(\varepsilon)_q = 1$, from formula (2) the basic probability assignment is

$$m(A_l)(\varepsilon) = r(\varepsilon)_l - r(\varepsilon)_{l-1} \quad (13)$$

$$r(\varepsilon)_0 = 0, \quad (l = 1, \dots, q)$$

and from Corollary 2, it can be expressed as

$$m(A_l)(\varepsilon) = m(A_l) + \varepsilon \sum_{i=1}^n \sum_{j=1}^n (h_{ij}^{(l)} - h_{ij}^{(l-1)}) d_{ij} + o(\varepsilon) \quad (14)$$

However, different from the proof of corollary 2 of normal AHP, we should be careful that the standard of the calculation of $h_{ij}^{(l)}$ is $w_i = 0$, where t is satisfied $\max w_i = w_i$.

(Q.E.D)

Following theorem can be obtained from the process of fuzzy measure AHP and Corollary 3.

Theorem 2 Let $A = (a_{ij})$, $(i, j = 1, \dots, n)$ be a comparison matrix and E_p^* ($p = 1, \dots, m$) be the overall weight of the p -th alternative be. Let E_p^* (ε) denote weight of the p -th alternative based on a perturbed comparison matrix $A(\varepsilon) = A + \varepsilon D_A$, $D_A = (a_{ij} d_{ij})$. Then using appropriate $z_{ij}^{(p)}$, weights $E_p^*(\varepsilon)$ is represented as

$$E_p^*(\varepsilon) = E_p^* + \varepsilon \sum_{i=1}^n \sum_{j=1}^n z_{ij}^{(p)} d_{ij} + o(\varepsilon). \quad (15)$$

(Proof)

Let $f_p(x_i)$ denote weight of the criterion x_i of alternative p . From equation (4) and Corollary 3, the upper limit expectation (substitutive evaluation value) becomes as follows.

$$E_p^*(\varepsilon) = E_p^* + \varepsilon \sum_{i=1}^n \sum_{j=1}^n \left\{ \sum_{l=1}^q (h_{ij}^{(l)} - h_{ij}^{(l-1)}) \max_{x \in A_l} f_p(x) \right\} d_{ij} + o(\varepsilon) \quad (16)$$

Accordingly, the fluctuation part of the weight of the alternatives is also expressed by the linear combination of d_{ij} . (Q.E.D)

From Theorem 2, the component of the comparison matrix that has the greatest influence on weight $E_p^*(\varepsilon)$ can be found. In a complementary case, evaluation also becomes possible with a similar way.

IV. FUZZY WEIGHTS REPRESENTATIONS

It often occurs that data of fuzzy measure AHP loses its reliability, when a comparison matrix has not so good consistency (i.e. $0.1 < C.I. < 0.2$). In these cases, we consider that components of a comparison matrix of criteria have fuzziness, because they are results from fuzzy judgement of a decision-maker. Therefore weights could be treated as fuzzy sets. Here, to represent fuzziness of weights of alternatives, we use L-R fuzzy number.

The multiple of coefficients $g_{ij} z_{ij}^{(p)}$ found in Corollary 1 and Theorem 2 is considered as the influence concerned with a_{ij} from the fluctuation of the consistency index.

Since g_{ij} is always positive, if the coefficient $z_{ij}^{(p)}$ is positive, the real weight of alternative p is considered as bigger than

E_p^* , and if $h_{ij}^{(k)}$ is negative, the real weight of criterion k is considered as smaller. Therefore, a sign of $z_{ij}^{(p)}$ represents a direction of a spread of the fuzzy number. Of course the absolute value $g_{ij} |z_{ij}^{(p)}|$ represents the amount of the influence.

On the other hand, C.I. becomes bigger then the judgement becomes fuzzier.

Consequently, multiple C.I. $g_{ij} |z_{ij}^{(p)}|$ can be regarded as a spread of a fuzzy weight $\tilde{E}_p^*$ concerned with a_{ij} .

Definition (fuzzy weights for fuzzy measure AHP) Let E_p^* be the crisp weight of alternative p in fuzzy measure AHP using possibility measure. g_{ij} , $z_{ij}^{(p)}$ denote the coefficients found in Corollary 1 and Theorem 2 and C.I. denotes consistency index of the comparison matrix of the criteria. When C.I. < 0.2 holds, a fuzzy weight $\tilde{E}_p^*$ is defined by

$$\tilde{E}_p^* = (E_p^*, l_p, r_p)_{LR} \quad (17)$$

where

$$l_p = C.I. \sum_{i=1}^n \sum_{j=1}^n s(-, z_{ij}^{(p)}) g_{ij} |z_{ij}^{(p)}|, \quad (18)$$

$$r_p = C.I. \sum_{i=1}^n \sum_{j=1}^n s(+, z_{ij}^{(p)}) g_{ij} |z_{ij}^{(p)}|, \quad (19)$$

$$s(+, z) = \begin{cases} 1 & (z > 0), \\ 0 & (z \leq 0), \end{cases}$$

$$s(-, z) = \begin{cases} 0 & (z > 0), \\ 1 & (z \leq 0). \end{cases}$$

We can calculate the fuzzy weights of alternatives in fuzzy measure AHP using Definition 1. They show how the result from fuzzy measure AHP has fuzziness. In a complementary case, it is also possible to represent with a similar way.

And this definition can be used when weights of the alternatives with respect to criterion is also fuzzy number weights, even if comparison matrices of alternatives do not have enough consistency.

V. CONCLUSIONS

We propose one representation of the weights of fuzzy measure (non-additive weight) AHP by use of fuzzy sets and the result of the sensitivity analysis. It will be useful for case for the cases that consistency of a comparison matrix is not so good. It was refined from our old representation. It can show not only how to represent weights, but also how the result from the fuzzy measure AHP has fuzziness, especially when some bad consistency exists.

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