

An Interval-Based Approach to Fuzzy Regression for Fuzzy Input-Output Data

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Abstract—A novel approach is introduced to construct a fuzzy regression model when the data available of independent and dependent variables are fuzzy numbers. The approach, consisting on the least-squares method, uses the α -level sets of fuzzy observations to estimate the crisp parameters of the model. A competitive study shows the performance and efficiency of the proposed approach with respect to some well-known methods.

Keywords— *capability index; fuzzy regression; hydrology; interval arithmetic; least squares method*

I. INTRODUCTION

Identification and analysis of the functional relationship between a dependent and some independent variables have made great interest in statistical analysis. Based on such functional relationship, which is called regression model, one can describe and predict the values of the dependent variable using the observations of independent variables.

However, when the data are reported as imprecise quantities we need to develop some soft methods to construct and analyze regression models. Fuzzy set theory provides appropriate tools for modeling regression when imprecise data have been observed. Over the last decades, different approaches to regression modeling have been widely proposed in the statistical literature, focussing on the situation of fuzzy environment. We refer the reader to [3], [4], [8], [12], [13], [15] for some recent works as well as some reviews of the main approaches to regression analysis in fuzzy environment.

In the present work, we investigate the situation in which the data available of independent and dependent variables are fuzzy numbers. We introduce a novel approach to model such a fuzzy data set. In the proposed approach, we apply a squared distance on the space of real intervals. Then, consisting on minimizing the total least-squares of errors, a linear regression model is expressed in terms of interval arithmetic between the α -level sets of fuzzy input-output data at each level of $\alpha \in [0, 1]$. The errors are considered based on the differences between the interval-valued data obtained from the α -level sets of the observations and the estimations of the dependent variable. Finally, by aggregating the coefficients obtained from the

interval regression model, we estimate the coefficients of the fuzzy regression model. The proposed method is investigated by two numerical competitive examples, which indicate the performance of the proposed model with respect to some well-known models.

The rest of this paper is organized as follows. In the next section, some preliminaries of fuzzy arithmetic that will be used in the sequel are recalled. Section III presents a novel approach to fuzzy regression for fuzzy input output data. In this section, the method of estimation of the parameters of the model is explained. In Section IV, two numerical examples are given to clarify the proposed approach. Finally, in Section V some concluding remarks are made.

II. PRELIMINARIES FROM FUZZY ARITHMETIC

A fuzzy set $\tilde{A}$ on the universal set $\mathbb{X}$ is defined by its membership function $\tilde{A}(x) : \mathbb{X} \rightarrow [0, 1]$. Through this paper we will consider that $\mathbb{X} = \mathbb{R}$ (the real line). The α -level set of a fuzzy set $\tilde{A}$ is defined as $A_\alpha = \{x \in \mathbb{R} : \tilde{A}(x) \geq \alpha\}$, $\alpha \in (0, 1]$, and $A_0 = \text{closure}\{x \in \mathbb{R} : \tilde{A}(x) > 0\}$. A fuzzy set $\tilde{A}$ of $\mathbb{R}$ is called a fuzzy number if any α -level set of $\tilde{A}$ is a non-empty compact interval, for any $\alpha \in [0, 1]$. Such intervals are represented by $A_\alpha = [A_\alpha^l, A_\alpha^u]$.

A specific type of fuzzy number, which is rich and flexible enough to cover most of the applications, is the so-called *LR-fuzzy number*, denoted by $\tilde{N} = (n, l, r)_{LR}$ with central value $n \in \mathbb{R}$, left and right spread values $l, r \in \mathbb{R}$, decreasing left and right shape functions $L, R : \mathbb{R}^+ \rightarrow [0, 1]$, with $L(0) = R(0) = 1$. Such *LR-fuzzy numbers* have the following membership function [16]

$$\tilde{N}(x) = \begin{cases} L(\frac{n-x}{l}) & \text{if } x \leq n, \\ R(\frac{x-n}{r}) & \text{if } x > n. \end{cases}$$

It is easy to see that each α -level set of the *LR-fuzzy number* $\tilde{N}$ is a closed interval as follows

$$N_\alpha = [n - L^{-1}(\alpha)l, n + R^{-1}(\alpha)r], \quad \alpha \in [0, 1].$$

In the special case, where $L(x) = R(x) = \max\{0, 1 - |x|\}$, $\tilde{N}$ is called a triangular fuzzy number and is denoted by $\tilde{N} = (n, l, r)_T$.

Algebraic operations on LR -fuzzy numbers are defined based on the interval arithmetic. For our proposes, we recall some definitions and results on this topic [16].

Proposition 2.1: Let $\tilde{M} = (m, l_m, r_m)_{LR}$ and $\tilde{N} = (n, l_n, r_n)_{LR}$ be two LR -fuzzy numbers, and λ be a real number. Then

$$\lambda \otimes \tilde{M} = \begin{cases} (\lambda m, \lambda l_m, \lambda r_m)_{LR} & \text{if } \lambda > 0, \\ \mathcal{I}_{\{0\}} & \text{if } \lambda = 0, \\ (\lambda m, |\lambda| r_m, |\lambda| l_m)_{RL} & \text{if } \lambda < 0, \end{cases}$$

$$\tilde{M} \oplus \tilde{N} = (m + n, l_m + l_n, r_m + r_n)_{LR}.$$

where $\mathcal{I}_A$ represents the indicator function of a crisp set A .

Definition 2.1: Let $I = [i_1, i_2]$ and $J = [j_1, j_2]$ be two intervals, and λ be a real number. Then scalar multiplication and addition of two intervals are defined as follows

$$\lambda \otimes I = \begin{cases} [\lambda i_1, \lambda i_2] & \lambda > 0, \\ \mathcal{I}_{\{0\}} & \lambda = 0, \\ [\lambda i_2, \lambda i_1] & \lambda < 0, \end{cases}$$

$$I \oplus J = [i_1 + j_1, i_2 + j_2].$$

Definition 2.2: The distance between two intervals $I = [i_1, i_2]$ and $J = [j_1, j_2]$ is defined as

$$\mathcal{D}(I, J) = (i_1 - j_1)^2 + (i_2 - j_2)^2.$$

The reader is referred to [10] for more on interval arithmetic.

III. FUZZY REGRESSION MODEL

In this section, we investigate the following fuzzy linear regression model

$$\tilde{y}_i = \beta_0 \oplus \beta_1 \otimes \tilde{x}_i, \quad i = 1, \dots, n, \quad (1)$$

in which the data $(\tilde{y}_i, \tilde{x}_i)$, $i = 1, \dots, n$, are fuzzy observations on n statistical units, and the coefficients β_0 and β_1 are crisp values.

In order to estimate the regression coefficients β_0 and β_1 , we first transform the fuzzy observations $(\tilde{y}, \tilde{x})$ into a set of interval-valued data obtained from the α -level sets of the observations, i.e. $x_\alpha = [x_\alpha^l, x_\alpha^u]$ and $y_\alpha = [y_\alpha^l, y_\alpha^u]$, for any $\alpha \in [0, 1]$. Then, we use the distance $\mathcal{D}$ in Definition 2.2 to minimize the following objective function

$$\begin{aligned} \mathcal{D}(b_0, b_1) &= \sum_{i=1}^n \mathcal{D}(y_{i\alpha}, b_0 \oplus b_1 \otimes x_{i\alpha}) \\ &= \sum_{i=1}^n \mathcal{D}([y_{i\alpha}^l, y_{i\alpha}^u], b_0 \oplus b_1 \otimes [x_{i\alpha}^l, x_{i\alpha}^u]). \end{aligned}$$

By such a minimization problem we can determine the best values for b_0 and b_1 in the following interval regression model

$$[y_{i\alpha}^l, y_{i\alpha}^u] = b_0 \oplus b_1 \otimes [x_{i\alpha}^l, x_{i\alpha}^u], \quad i = 1, \dots, n,$$

which models y_α on x_α at each α -level set, $\alpha \in [0, 1]$.

Note that, the objective function $\mathcal{D}(b_0, b_1)$ represents the sum of squared errors between the observed and estimated

values of the α -level sets of dependent variable. Now, concerning the sign of b_1 , we have the two following expressions for the objective function

$$\begin{aligned} \mathcal{D}^+(b_0, b_1) &= \sum_{i=1}^n [(y_{i\alpha}^l - b_0 - b_1 x_{i\alpha}^l)^2 \\ &\quad + (y_{i\alpha}^u - b_0 - b_1 x_{i\alpha}^u)^2] \quad b_1 \geq 0. \\ \mathcal{D}^-(b_0, b_1) &= \sum_{i=1}^n [(y_{i\alpha}^l - b_0 - b_1 x_{i\alpha}^u)^2 \\ &\quad + (y_{i\alpha}^u - b_0 - b_1 x_{i\alpha}^l)^2] \quad b_1 < 0. \end{aligned}$$

In this step, we can follow the procedure in two ways:

1) We may, first, pre-determine the sign of b_1 , for example, by fitting a classical regression model to the centers of the fuzzy observations, and then follow one of the above expressions with respect to the sign of the slope of the classical linear regression model.

2) We can also employ both expressions and estimate the related coefficients. In such a case, we consider the amounts b_0 or b_1 for which the related objective function is smaller than the other one. To do this, we have to solve the following equations

$$\frac{\partial}{\partial b_0} \mathcal{D}^+(b_0, b_1) = 0, \quad \frac{\partial}{\partial b_1} \mathcal{D}^+(b_0, b_1) = 0, \quad \text{for } b_1 \geq 0,$$

and

$$\frac{\partial}{\partial b_0} \mathcal{D}^-(b_0, b_1) = 0, \quad \frac{\partial}{\partial b_1} \mathcal{D}^-(b_0, b_1) = 0, \quad \text{for } b_1 < 0.$$

For the positive case, we obtain

$$\begin{aligned} b_0^+(\alpha) &= \bar{Y}(\alpha) - b_1^+(\alpha) \bar{X}(\alpha), \\ b_1^+(\alpha) &= \frac{\sum_{i=1}^n [x_{i\alpha}^l y_{i\alpha}^l + x_{i\alpha}^u y_{i\alpha}^u] - 2n \bar{X}(\alpha) \bar{Y}(\alpha)}{\sum_{i=1}^n [(x_{i\alpha}^l)^2 + (x_{i\alpha}^u)^2] - 2n \bar{X}^2(\alpha)}, \end{aligned}$$

where, $\bar{Y}(\alpha) = \frac{1}{n} \sum_{i=1}^n \frac{y_{i\alpha}^l + y_{i\alpha}^u}{2}$ and $\bar{X}(\alpha) = \frac{1}{n} \sum_{i=1}^n \frac{x_{i\alpha}^l + x_{i\alpha}^u}{2}$. For the negative case, we obtain

$$\begin{aligned} b_0^-(\alpha) &= \bar{Y}(\alpha) - b_1^-(\alpha) \bar{X}(\alpha), \\ b_1^-(\alpha) &= \frac{\sum_{i=1}^n [x_{i\alpha}^l y_{i\alpha}^u + x_{i\alpha}^u y_{i\alpha}^l] - 2n \bar{X}(\alpha) \bar{Y}(\alpha)}{\sum_{i=1}^n [(x_{i\alpha}^l)^2 + (x_{i\alpha}^u)^2] - 2n \bar{X}^2(\alpha)}. \end{aligned}$$

Finally, by integration, we aggregate the functions $b_0^+(\alpha)$ ($b_0^-(\alpha)$) and $b_1^+(\alpha)$ ($b_1^-(\alpha)$) to estimate the regression coefficients β_0 and β_1 , as follows

$$\begin{aligned} \hat{\beta}_0 &= \int_0^1 b_0^+(\alpha) d\alpha \quad (\text{or } \hat{\beta}_0 = \int_0^1 b_0^-(\alpha) d\alpha), \\ \hat{\beta}_1 &= \int_0^1 b_1^+(\alpha) d\alpha \quad (\text{or } \hat{\beta}_1 = \int_0^1 b_1^-(\alpha) d\alpha). \end{aligned}$$

Therefore, the estimate of the dependent variable, denoted by $\hat{\tilde{y}}$, can be obtained by the following fuzzy linear regression model

$$\hat{\tilde{y}}_i = \hat{\beta}_0 \oplus \hat{\beta}_1 \otimes \tilde{x}_i, \quad i = 1, \dots, n.$$

Goodness of fit criteria: To evaluate the goodness of fit of the fuzzy regression models we employ a well-known criteria due to Kim and Bishu [7] (see also [2], [6]).

TABLE I
FUZZY DATA IN EXAMPLE 4.1

i	$\tilde{x}_i = (x_i, l_i)_T$	$\tilde{y}_i = (y_i, s_i)_T$
1	(2.0, 0.5) _T	(4.0, 0.5) _T
2	(3.5, 0.5) _T	(5.5, 0.5) _T
3	(5.5, 1.0) _T	(7.5, 1.0) _T
4	(7.0, 0.5) _T	(6.5, 0.5) _T
5	(8.5, 0.5) _T	(8.5, 0.5) _T
6	(10.5, 1.0) _T	(8.0, 1.0) _T
7	(11.0, 0.5) _T	(10.5, 0.5) _T
8	(12.5, 0.5) _T	(9.5, 0.5) _T

Definition 3.1 ([7]): The capability index ($\mathcal{CI}$) of the fuzzy regression model (1) is defined by

$$\mathcal{CI} = \frac{1}{n} \sum_{i=1}^n \mathcal{CI}(i),$$

where

$$\mathcal{CI}(i) = \frac{\int |\tilde{y}_i(x) - \hat{y}_i(x)| dx}{\int \tilde{y}_i(x) dx}.$$

The capacity index $\mathcal{CI}$ represents the average of errors in estimation and, therefore, smaller values of $\mathcal{CI}$ indicates that the fuzzy linear regression model fits better to the data.

Remark 3.1: It should be noted that Bargiela et al. [1] considered a more or less similar approach to regression modelling with fuzzy data. They used the distance $\mathcal{D}$ for the α -level sets of fuzzy observations and then aggregated such distances to obtain the distance between fuzzy estimate and fuzzy observed values of the dependent variable. But, in our proposed approach, we first used the distances between the α -level sets of fuzzy observations and fuzzy estimations of $\tilde{y}$ to provide an interval-valued regression model. Then, we integrated the coefficients of the interval-valued regression model to provide the coefficients of the fuzzy regression model.

Remark 3.2: While, most common methods in fuzzy regression modeling require all the observations to have the same type of membership functions for the fuzzy observations, in the proposed approach there is no limitation in the types of the membership functions of the fuzzy observations. The proposed model, therefore, is applicable when the observations have different types of membership functions.

Remark 3.3: Note that, when fuzzy observations reduce to crisp values, the proposed method reduces to the conventional least-squares regression method.

IV. COMPETITIVE STUDY

In this section, we provide two numerical examples to explain how the proposed method is applicable to derive the regression model for fuzzy observations. We also compare the proposed model with some other models.

Example 4.1: In this example, we compare the proposed method in this paper with some other models, using the triangular fuzzy input-output data set given by Sakawa and Yano [14] (Table I).

By applying the introduced procedure in Section III, we obtain the following functions for each $\alpha \in [0, 1]$

$$\begin{aligned} \hat{b}_0^+(\alpha) &= \frac{-7\alpha^2 + 14\alpha + 11164}{112\alpha^2 - 224\alpha + 3239}, \\ \hat{b}_1^+(\alpha) &= \frac{112\alpha^2 - 224\alpha + 1736}{112\alpha^2 - 224\alpha + 3239}. \end{aligned}$$

Therefore, the estimated coefficients are derived as $\hat{\beta}_0 = 3.53$ and $\hat{\beta}_1 = 0.525$. So, the fuzzy linear regression model is as follows

$$\hat{y} = 3.53 \oplus 0.525 \otimes \tilde{x}.$$

To compare the proposed method with some other methods in fuzzy regression modeling, the index $\mathcal{CI}$ is employed to evaluate the goodness of fit of the models. Different models and their $\mathcal{CI}$ indices are given in Table II. The results indicate that the proposed method outperforms all the models.

Example 4.2: One of the classical problem in hydrology studies is the measurement of suspended load ($\frac{\text{Ton}}{\text{Day}}$) and discharge ($\frac{m^3}{s}$) in watersheds [9]. Based on a study in a part of Darband, situated in the province of Khorasan (North west of Iran) between the cities Bojnord and Esferayen, some water characteristics were measured using standard procedures. The daily discharge and suspended load of the watershed are measured for 51 days. But, due to some limitations in experimental environments, the observed data were reported as fuzzy numbers shown in Table IV. Since, in practical studies, it is important to estimate the amount of suspended load in term of discharge, we wish to model the relationship between suspended load as the dependent variable and discharge as independent variable.

According to the proposed method, the estimated coefficients are obtained as $\hat{\beta}_0 = 5.0845$ and $\hat{\beta}_1 = 0.3040$, and the fuzzy linear regression model is, therefore

$$\hat{y} = 5.0845 \oplus 0.3040 \otimes \tilde{x}.$$

For a competitive study we employ some well-known methods to model such a fuzzy data set, as well. Different models and their goodness of fit indices are given in Table III. The results show that the proposed method provides a better performance equal to that of Bargiela et al. and better performance with respect to three other models.

V. CONCLUSIONS

A fuzzy regression model was investigated for fuzzy input-output data set. By utilizing the method of least-squares as a fitting criterion and applying it to each interval-valued data obtained from α -level sets of imprecise observations, a regression model is provided at each level of α . The estimates of the parameters were obtained as functions of α . Then, by integrating the coefficients of the provided regression models, the coefficients of the fuzzy regression model were derived.

The proposed model is applicable when the observations have different types of membership functions. When all fuzzy observations reduce to crisp values, the proposed method

TABLE II
FUZZY REGRESSION MODELS AND THEIR CAPABILITY INDICES IN EXAMPLE 4.1

Proposed by	Model	$\mathcal{CI}$
Bargiela et al. [1]	$\hat{y}_i = (3.446 + 0.536x_i, 0.536l_i)_T$	1.3428
Kao and Chyu [6]	$\hat{y}_i = (3.554 + 0.522x_i, 0.951 + 0.522l_i, 0.949 + 0.522l_i)_T$	1.9951
Nasrabadi and Nasrabadi [11]	$\hat{y}_i = (3.5767 + 0.5467x_i, l_i)_T$	1.4879
The proposed model	$\hat{y}_i = (3.53 + 0.525x_i, 0.525l_i)_T$	1.3285

[†] For the values of $\tilde{E}_i$ see [2]

TABLE III
FUZZY REGRESSION MODELS AND THEIR CAPABILITY INDICES IN EXAMPLE 4.2

Proposed by	Model	$\mathcal{CI}$
Bargiela et al. [1]	$\hat{y}_i = (5.0674 + 0.3072x_i, 0.3072l_i)_T$	1.0107
Kao and Chyu [6]	$\hat{y}_i = (5.6371 + 0.1993x_i, 3.1887 + 0.1993l_i)_T$	2.1217
Nasrabadi and Nasrabadi [11]	$\hat{y}_i = (5.4464 + 0.1465x_i, 2.0058 + 1.2718l_i)_T$	1.6158
The proposed model	$\hat{y}_i = (5.0845 + 0.3040x_i, 0.3040l_i)_T$	1.0106

[†] For the values of $\tilde{E}_i$ see [2]

TABLE IV
FUZZY DATA IN EXAMPLE 4.2

i	$\tilde{x}_i = (x_i, l_i)_T$	$\tilde{y}_i = (y_i, s_i)_T$	i	$\tilde{x}_i = (x_i, l_i)_T$	$\tilde{y}_i = (y_i, s_i)_T$	i	$\tilde{x}_i = (x_i, l_i)_T$	$\tilde{y}_i = (y_i, s_i)_T$
1	(2.98, 0.30) _T	(4.90, 0.30) _T	18	(2.01, 0.20) _T	(5.36, 0.76) _T	35	(2.68, 0.30) _T	(5.30, 0.69) _T
2	(5.56, 0.60) _T	(7.23, 4.01) _T	19	(2.53, 0.25) _T	(8.47, 3.86) _T	36	(8.53, 1.00) _T	(7.99, 4.99) _T
3	(3.59, 0.40) _T	(5.53, 1.62) _T	20	(10.2, 2.50) _T	(7.96, 5.30) _T	37	(4.16, 0.60) _T	(6.72, 3.50) _T
4	(1.91, 0.20) _T	(4.63, 0.41) _T	21	(10.2, 2.50) _T	(5.73, 3.07) _T	38	(2.73, 0.30) _T	(7.84, 3.64) _T
5	(6.67, 0.70) _T	(8.03, 5.03) _T	22	(10.2, 2.50) _T	(5.37, 2.71) _T	39	(9.00, 1.25) _T	(9.62, 6.81) _T
6	(10.25, 2.50) _T	(7.41, 4.75) _T	23	(10.53, 2.50) _T	(7.96, 5.30) _T	40	(2.43, 0.25) _T	(5.81, 1.20) _T
7	(11.12, 2.70) _T	(7.90, 5.37) _T	24	(10.53, 2.50) _T	(5.14, 2.48) _T	41	(4.97, 0.65) _T	(6.80, 3.58) _T
8	(4.76, 0.50) _T	(6.57, 3.35) _T	25	(4.28, 0.60) _T	(7.15, 3.65) _T	42	(4.46, 0.60) _T	(6.26, 2.35) _T
9	(1.36, 0.20) _T	(6.31, 1.02) _T	26	(5.03, 0.60) _T	(8.10, 4.88) _T	43	(9.68, 1.25) _T	(7.79, 4.98) _T
10	(2.36, 0.30) _T	(4.98, 0.37) _T	27	(7.20, 0.80) _T	(8.04, 5.04) _T	44	(3.05, 0.40) _T	(4.83, 0.22) _T
11	(3.11, 0.40) _T	(6.34, 2.43) _T	28	(7.20, 0.80) _T	(7.56, 4.56) _T	45	(2.32, 0.30) _T	(5.37, 0.77) _T
12	(1.54, 0.20) _T	(6.26, 0.96) _T	29	(7.20, 0.80) _T	(8.29, 5.29) _T	46	(6.53, 0.75) _T	(7.83, 4.61) _T
13	(1.98, 0.25) _T	(5.14, 0.54) _T	30	(8.75, 1.00) _T	(8.96, 6.14) _T	47	(2.45, 0.35) _T	(6.90, 2.30) _T
14	(3.97, 0.45) _T	(5.51, 1.83) _T	31	(6.72, 0.75) _T	(5.50, 2.40) _T	48	(1.58, 0.20) _T	(7.66, 2.36) _T
15	(5.20, 0.60) _T	(7.86, 4.64) _T	32	(2.04, 0.20) _T	(4.69, 0.20) _T	49	(2.25, 0.25) _T	(4.90, 0.30) _T
16	(6.94, 0.75) _T	(7.24, 4.24) _T	33	(2.92, 0.30) _T	(5.95, 2.03) _T	50	(4.24, 0.55) _T	(6.91, 2.99) _T
17	(4.16, 0.50) _T	(5.98, 2.48) _T	34	(3.16, 0.40) _T	(6.68, 2.77) _T	51	(9.97, 1.75) _T	(7.88, 5.22) _T

becomes the conventional least-squares method in crisp environment, while some of the existing methods still generate fuzzy regression coefficients in such a case.

Although the discussion of this paper is confined to simple regression with one explanatory variable, the approach studied in this paper, could be extended to more general situation including several independent variables.

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