

T-S Fuzzy Systems Approach to Approximation and Robust Controller Design for General Nonlinear Systems

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Abstract—A novel approach to control of general nonlinear system based on T-S fuzzy model is presented in this paper. Firstly, it is shown that a general nonlinear system can be approximated by a generalized T-S fuzzy model to arbitrary degree of accuracy on any compact set. And the basic idea of the proposed approach is to stabilize the general nonlinear system by solving a robust stabilization problem of the developed T-S fuzzy system with the approximation errors as the uncertainty term. Then using a piecewise Lyapunov function, robust semi-global stabilization and generalized $\mathcal{H}_2$ control of the general nonlinear system are formulated in the form of linear matrix inequalities. Finally, an example is provided to demonstrate the effectiveness of the proposed approaches.

Index Terms—Approximation, General nonlinear systems, Piecewise Lyapunov functions, T-S fuzzy model, Robust Control.

I. INTRODUCTION

Takagi and Sugeno (T-S) type fuzzy models [1] have been widely studied recently and proved to be one of the most powerful approaches to nonlinear control [2]-[5]. This model describes a nonlinear system by a group of fuzzy IF-THEN rules in the form of local linear or affine models which are smoothly connected by fuzzy membership functions. And this structure provides a basis for systematic stability analysis and controller design of T-S fuzzy control systems in view of powerful conventional control theory.

In the past decades, a large volume of works on theoretical results of stability analysis and controller synthesis of T-S fuzzy systems have been published. Readers can refer to the survey paper [6] and the references therein for details. By using the direct Lyapunov method, it is shown that stability analysis and controller design of T-S fuzzy systems can be facilitated by searching for a common quadratic Lyapunov

function [7]. In order to reduce the conservatism of common quadratic Lyapunov function based approaches, some results based on piecewise quadratic and fuzzy Lyapunov functions have been proposed [8]-[16].

T-S fuzzy models have been shown to be universal function approximators in the sense that they are able to approximate any smooth nonlinear functions to arbitrary degree of accuracy in any convex compact region [17]-[19]. However, it is shown in [19] that the *commonly used* linear or affine T-S fuzzy models, where the control variables are not included in the premise variables are only able to approximate *affine nonlinear systems* to any degree of accuracy on any compact set. This implies that only the stabilization problem of *affine nonlinear systems* can be solved based on the *commonly used* T-S fuzzy models. In other words, to deal with more general nonlinear systems, a more general T-S fuzzy dynamic model should be employed.

Recently, some results on control design of more general nonlinear systems based on modified T-S fuzzy models have appeared in literature [20]-[22]. To describe conventional general nonlinear system approximately, the authors in [21] analyzed T-S fuzzy model with state variables and control variables contained in the conditional part of fuzzy rules, with each local model describes the dynamics of one state variable or control variable. The authors in [22] also analyzed the case that the control vector is a part of premise variables. However, only Mamdani-type fuzzy models were considered. Thus many fuzzy rules need to be used in these methods to describe global nonlinear systems and the construction of controllers is also very complex.

In this paper, we investigate the controller synthesis of general nonlinear systems based on a class of generalized T-S

fuzzy models, which are proved to be universal approximators. It is shown that the stabilization problem of a general nonlinear system can be solved as robust stabilization problems of a generalized T-S fuzzy system with the approximation errors as the uncertainty term. It is also noted that with the control variables included in the conditional part of fuzzy rules, the two mainly used control schemes, say, the normally parallel distributed compensation and the local compensation [2], [6], [8], can not be applied due to the logic conflicts between antecedences and consequences. Thus a dynamic state feedback controller is employed instead. Based on a piecewise Lyapunov function, it is shown that the robust semi-global stabilization and generalized $\mathcal{H}_2$ controller design results can be obtained based on the feasible solutions of a set of linear matrix inequalities that are numerically efficient with commercially available software.

The rest of this paper is structured as follows. The section II is devoted to the description of the generalized T-S fuzzy models and the proof of its universal approximation property. In section III and IV, the robust semi-global stabilization and generalized $\mathcal{H}_2$ control design results of the nonlinear system will be presented respectively. Simulation results will be provided in section V to demonstrate the effectiveness of the proposed approaches. Conclusions will be given in section VI.

Notations. The notations used in this paper are fairly standard. The notation $\star$ is used to indicate the terms that can be induced by symmetry. For example, $\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} A & \star \\ B & C \end{bmatrix}$. “ T ” represents vector or matrix transpose. $\mathbf{I}_n$ and $\mathbf{0}_{m \times n}$ are used to denote the $n \times n$ identity matrix and the $m \times n$ zero matrix, respectively. The subscripts n and $m \times n$ are omitted when the size is not relevant or can be determined from the context. The notation $l_2[0, N]$ will be used for discrete-time vector-valued functions, i.e., $z_d : [0, N] \rightarrow \mathbb{R}^k \in l_2[0, N]$ if $\sum_{t=1}^N z_d^T(t) z_d(t) < \infty$. The l_2 -norm is defined as $\|z_d(t)\|_2 = \sqrt{\sum_{t=1}^N z_d^T(t) z_d(t)}$ and the l_∞ -norm is defined as $\|z_d(t)\|_\infty = \sqrt{\sup_t \{z_d^T(t) z_d(t)\}}$.

II. GENERALIZED T-S FUZZY MODEL AND UNIVERSAL FUNCTION APPROXIMATION

Let us consider a general nonlinear system described by the following equation,

$$x(t+1) = G(x(t), u(t)), \quad (1)$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T \in X = \prod_{i=1}^n X_i \subset \mathbb{R}^n$ with $x_0 = [0, \dots, 0]^T \in X$ and $u(t) = [u_1(t), \dots, u_m(t)]^T \in U = \prod_{j=1}^m U_j \subset \mathbb{R}^m$ with $u_0 = [0, \dots, 0]^T \in U$. Throughout the paper, it is always assumed that the origin is the equilibrium of the system, that is, $G(0, 0) = 0$, and $G(x, u)$ is a continuously differentiable function on $X \times U$. Further it is assumed that $X \times U = \prod_{i=1}^n [\underline{x}_i, \bar{x}_i] \times \prod_{j=1}^m [\underline{u}_j, \bar{u}_j]$ is a compact set on $\mathbb{R}^n \times \mathbb{R}^m$.

Our objective is to develop an approach to control of such general nonlinear systems via a way of Takagi-Sugeno (T-S) fuzzy modelling. For this purpose, we consider to approximate the nonlinear systems by the following class of generalized T-S fuzzy models at first,

Plant rule $\mathcal{R}^l$:

IF $x_1(t)$ is μ_1^l AND ... AND $x_n(t)$ is μ_n^l ; $u_1(t)$ is ν_1^l AND ... AND $u_m(t)$ is ν_m^l

THEN $x(t+1) = A_l x(t) + B_l u(t); l \in \mathcal{L} := \{1, \dots, r\}$ (2)

where $\mathcal{R}^l$ denotes the l th rule, r the total number of rules, μ_i^l and ν_j^l the fuzzy sets, $x(t) \in \mathbb{R}^n$ the state vector, $u(t) \in \mathbb{R}^m$ the input vector, and $[A_l, B_l]$ the matrices of the l th local model.

Under the center-average defuzzifier, product inference and singleton fuzzifier, the T-S fuzzy system in (2) can be expressed as follows,

$$x(t+1) = F(x(t), u(t)), \quad (3)$$

with

$$\begin{cases} F(x, u) = \sum_{l=1}^r \mu_l(x, u) [A_l x(t) + B_l u(t)], \\ \mu_l(x, u) = \frac{\prod_{i=1}^n \mu_i^l(x_i) \prod_{j=1}^m \nu_j^l(u_j)}{\sum_{l=1}^r \prod_{i=1}^n \mu_i^l(x_i) \prod_{j=1}^m \nu_j^l(u_j)}, \end{cases} \quad (4)$$

where $\mu_l(x, u)$ are so-called normalized fuzzy membership functions satisfying $\mu_l(x, u) \geq 0$ and $\sum_{l=1}^r \mu_l(x, u) = 1$ [8].

Remark 2.1. In the commonly used T-S fuzzy model, which is referred as “the standard T-S fuzzy model” in [19], the control variable u is not included in the conditional part of the rule. It has been shown in [19] that the commonly used T-S fuzzy model, where the control variable u is not included in the premise part of the rule, are only able to approximate affine nonlinear systems. Thus, to represent the more general nonlinear systems given in (1), a more general T-S fuzzy model, such as those given in (2) or (3) is needed. These models are referred to be the generalized T-S fuzzy model in this paper.

Before giving the main results of this section, we first introduce a lemma.

Lemma 2.1. [19] If vector value function $f(z) = [f_1(z_1, \dots, z_N), \dots, f_n(z_1, \dots, z_N)]^T$ is continuously differentiable on $Z = \prod_{i=1}^N [\alpha_i, \beta_i]$ with $0 \in Z$ and $f(0) = 0$, then for $i = 1, \dots, N$ the vector value function

$$\begin{aligned} G_i(z) &= g_i(z_i, \dots, z_N) \\ &= \begin{cases} \frac{f(0, \dots, z_i, z_{i+1}, \dots, z_N) - f(0, \dots, 0, z_{i+1}, \dots, z_N)}{z_i}, & z_i \neq 0 \\ \frac{\partial f(0, \dots, 0, z_{i+1}, \dots, z_N)}{\partial z_i}, & z_i = 0 \end{cases} \end{aligned} \quad (5)$$

is continuous on Z and

$$f(z) = \sum_{i=1}^N G_i(z) z_i = \sum_{i=1}^N g_i(z_i, \dots, z_N) z_i. \quad (6)$$

Then one has the following result.

Theorem 2.1. Let a general nonlinear system be given in (1) where $G(x, u)$ on the compact set $X \times U$ is continuously

differentiable with $G(0,0) = 0$. Then, for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, there exists a T-S fuzzy model $F(x,u) = \sum_{l=1}^r \mu_l(x,u)[A_l x(t) + B_l u(t)]$ given in (3) such that, for any $(x,u) \in X \times U$

$$G(x,u) = F(x,u) + E(x,u), \quad (7)$$

and

$$\|E(x,u)\| \leq \epsilon_1 \|x\| + \epsilon_2 \|u\|. \quad (8)$$

Proof. See [23].

Remark 2.2. Noticing that $X \times U = \prod_{i=1}^n [\underline{x}_i, \bar{x}_i] \times \prod_{j=1}^m [\underline{u}_j, \bar{u}_j]$ are bounded, the norm bound of approximation error $\Delta E(x,u)$ will be as small as needed by choosing $\epsilon_1 > 0$ and $\epsilon_2 > 0$ accordingly. In other words, the fuzzy control system given in (3) can approximate the general nonlinear system (1) to any degree of accuracy.

Remark 2.3. It is noted the decomposition procedure in proof of Theorem 2.1 can be conducted both in continuous-time case and discrete-time case. Thus one can conclude that any given continuous-time general nonlinear systems satisfying the described conditions in Theorem 2.1 can also be approximated by a continuous-time generalized T-S fuzzy model to any degree of accuracy.

Remark 2.4. Based on Theorem 2.1, one can easily conclude that the stabilization of a general nonlinear system given in (1) can be solved as a robust stabilization problem of its corresponding T-S fuzzy model with the approximation errors as the uncertainty term.

III. ROBUST CONTROLLER DESIGN FOR NONLINEAR SYSTEMS

It has been shown in the previous section that a general nonlinear system described by (1) can be exactly expressed in a compact set by a T-S fuzzy model with some uncertainties as follows,

$$x(t+1) = F(x(t), u(t)) + \Delta A(x(t), u(t))x(t) + \Delta B(x(t), u(t))u(t), \quad (9)$$

where $F(x(t), u(t))$ is described by (4) and

$$\|\Delta A(x(t), u(t))\| \leq \epsilon_1 \quad \text{and} \quad \|\Delta B(x(t), u(t))\| \leq \epsilon_2.$$

In this section, we will develop a robust control design approach to the general nonlinear systems (1) via the way of T-S fuzzy modeling in terms of system (9) based on a piecewise Lyapunov function.

In order to facilitate the piecewise controller design, we partition the premise variable space S into crisp regions and fuzzy regions [3]. Denote those regions from the premise variable space partition as $\{S_i\}_{i \in \mathcal{I}}$ with $\mathcal{I}$ as the set of region indices.

Define $\mathcal{K}(i)$ the indices of fired rules in each region S_i , $i \in \mathcal{I}$, then the crisp regions and fuzzy regions can be defined respectively by

$$S_i := \{(x(t), u(t)) | \mu_k(x(t), u(t)) = 1, k \in \mathcal{K}(i)\}, \quad (10)$$

and

$$S_i := \{(x(t), u(t)) | 0 \leq \mu_k(x(t), u(t)) < 1, k \in \mathcal{K}(i)\}. \quad (11)$$

Based on such a partition method, the global fuzzy model (9) in each region can be rewritten by a blending of $k \in \mathcal{K}(i)$ subsystems,

$$x(t+1) = \sum_{k \in \mathcal{K}(i)} \mu_k(x,u) \left\{ (A_k + \Delta A(x,u))x(t) + (B_k + \Delta B(x,u))u(t) \right\}. \quad (12)$$

In order to stabilize the nonlinear system in (12), we employ the following piecewise dynamic state feedback controller,

$$u(t+1) = F_i x(t) + G_i u(t), (x,u) \in S_i, i \in \mathcal{I}. \quad (13)$$

Then the closed-loop control system can be described by

$$\bar{x}(t+1) = \left\{ \sum_{k \in \mathcal{K}(i)} \mu_k(\bar{x}(t)) \mathcal{A}_{ki} + R \Delta E \right\} \bar{x}(t), \bar{x}(t) \in S_i, \quad (14)$$

where $\mathcal{A}_{ki} = \bar{A}_k + \bar{B}_k \bar{K}_i$ and

$$\begin{cases} \bar{x}(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \bar{A}_k = \begin{bmatrix} A_k & B_k \\ \mathbf{0}_{m \times n} & \mathbf{0}_{m \times m} \end{bmatrix}, \\ \bar{B}_k = \begin{bmatrix} \mathbf{0}_{n \times m} \\ \mathbf{I}_m \end{bmatrix}, R = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0}_{m \times n} \end{bmatrix}, \\ \Delta E = \begin{bmatrix} \Delta A & \Delta B \end{bmatrix}, \bar{K}_i = \begin{bmatrix} F_i & G_i \end{bmatrix}. \end{cases}$$

In addition, we define a new set that represents all possible state transitions among regions of the closed-loop T-S fuzzy system in (14) as follows,

$$\Omega := \{(i,j) | \bar{x}(t) \in S_i, \bar{x}(t+1) \in S_j, i,j \in \mathcal{I}\}. \quad (15)$$

For a piecewise Lyapunov function candidate we consider functions of the following form:

$$V(\bar{x}(t)) = \bar{x}(t)^T P_i \bar{x}(t), \bar{x}(t) \in S_i, i \in \mathcal{I}. \quad (16)$$

Suppose the upper bound of the uncertainty ΔE is given by

$$\Delta E^T \Delta E = \begin{bmatrix} \Delta A^T \\ \Delta B^T \end{bmatrix} \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \leq \bar{\epsilon}^2 \mathbf{I}_{(m+n)}. \quad (17)$$

Then we have the following stabilization results of nonlinear system (1).

Theorem 3.1. The nonlinear system (1) can be semi-globally asymptotically stabilized by the fuzzy controller in (13), if there exist a set of positive-definite matrices $X_i, i \in \mathcal{I}$, a set of positive constants $\varepsilon_i, i \in \mathcal{I}$ and a set of matrices $Q_i, i \in \mathcal{I}$, such that the following LMIs are satisfied,

$$\begin{bmatrix} X_i & \star & \star & \star \\ A_k X_i + B_k Q_i & X_j & \star & \star \\ \bar{\epsilon} X_i & \mathbf{0} & \varepsilon_i \mathbf{I} & \star \\ \mathbf{0} & \varepsilon_i R^T & \mathbf{0} & \varepsilon_i \mathbf{I} \end{bmatrix} > 0, \quad k \in \mathcal{K}(i), (i,j) \in \Omega. \quad (18)$$

Moreover, the controller gains are given by

$$\bar{K}_i = Q_i X_i^{-1}, i \in \mathcal{I}. \quad (19)$$

Proof. See [23].

IV. ROBUST GENERALIZED $\mathcal{H}_2$ CONTROLLER DESIGN FOR NONLINEAR SYSTEMS

In this section, we will address the generalized $\mathcal{H}_2$ controller synthesis problem [11] for a general nonlinear system with disturbances, which is described by

$$x(t+1) = G(x(t), u(t)) + H(x(t), u(t))w(t), \quad (20)$$

where $w(t) \in \mathbb{R}^p$ is the external disturbance and other variables are the same as those defined in (1).

Based on the approximation results presented in section II, the nonlinear system (20) can be exactly expressed in a compact set by the following T-S fuzzy model with some uncertainties,

Plant rule $\mathcal{R}^l$: IF $x_1(t)$ is μ_1^l AND ... AND $x_n(t)$ is μ_n^l ; $u_1(t)$ is ν_1^l AND ... AND $u_m(t)$ is ν_m^l THEN

$$\begin{cases} x(t+1) = (A_l + \Delta A)x(t) + (B_l + \Delta B)u(t) \\ \quad + (H_l + \Delta H)w(t), \\ z(t) = C_l x(t) + D_l u(t), l \in \mathcal{L} := \{1, 2, \dots, r\}. \end{cases} \quad (21)$$

Then using the same controller described in (13), the same premise variable space partition method as in section III, and adopting the same definitions of symbols as in (15), the closed-loop control system consisting of (20) or (21) and (13) can be described by the following equations,

$$\begin{cases} \bar{x}(t+1) = \mathcal{A}_{ic}\bar{x}(t) + \mathcal{H}_{ic}w(t), \\ z(t) = C_{ic}\bar{x}(t), \quad \bar{x}(t) \in S_i \end{cases} \quad (22)$$

where

$$\begin{cases} \mathcal{A}_{ic} = \sum_{k \in \mathcal{K}(i)} \mu_k(\bar{x}) \mathcal{A}_{ki}^c = \sum_{k \in \mathcal{K}(i)} \mu_k(\bar{x}) (\mathcal{A}_{ki} + R\Delta E), \\ \mathcal{H}_{ic} = \sum_{k \in \mathcal{K}(i)} \mu_k(\bar{x}) \mathcal{H}_k^c = \sum_{k \in \mathcal{K}(i)} \mu_k(\bar{x}) R(H_k + \Delta H), \\ C_{ic} = \sum_{k \in \mathcal{K}(i)} \mu_k(\bar{x}) \bar{C}_k = \sum_{k \in \mathcal{K}(i)} \mu_k(\bar{x}) [C_k \quad D_k]. \end{cases}$$

Similar to the case of stabilization discussed in Section III, the global results on generalized $\mathcal{H}_2$ control of the model (21) would imply the semi-global results on generalized $\mathcal{H}_2$ control of the system (20) due to the restriction of a compact set for the description of (20) via (21).

We at first introduce the following definition of the generalized $\mathcal{H}_2$ control.

Definition 4.1. [12] Let a constant $\gamma > 0$ be given. The closed-loop control system (22) is said to be asymptotically stable with generalized $\mathcal{H}_2$ performance γ if both of the following conditions are satisfied,

1) The disturbance-free system is globally asymptotically stable,

2) The generalized $\mathcal{H}_2$ norm of the system, which is defined as

$$\mathcal{H}_2 := \sup_{N \in \{1, 2, \dots\}} \left\{ \|z_N\| : x_0 = 0, \sum_{k=0}^{N-1} (\omega_k^T(t) \omega_k(t)) \leq 1 \right\},$$

is less than γ .

Before present the generalized $\mathcal{H}_2$ controller design results, we first introduce two lemmas,

Lemma 4.1. [12] The discrete-time fuzzy system (22) is stable with generalized $\mathcal{H}_2$ performance, if there exist a set of positive-definite matrices $P_i, i \in \mathcal{I}$ such that the following matrix inequalities are satisfied,

$$\begin{cases} P_i - \frac{1}{\gamma^2} \bar{C}_k^T \bar{C}_k > 0, i \in \mathcal{I}, \end{cases} \quad (23a)$$

$$\begin{cases} (\mathcal{A}_{ki}^c)^T P_j \mathcal{H}_k^c (\mathcal{H}_k^c P_j \mathcal{H}_k^c - \mathbf{I})^{-1} (\mathcal{H}_k^c)^T P_j \mathcal{A}_{ki}^c \\ + (\mathcal{A}_{ki}^c)^T P_j \mathcal{A}_{ki}^c - P_i < 0. \end{cases} \quad (23b)$$

Lemma 4.2. [24] Given matrices $\Delta E(\mu)$ satisfying

$$\begin{cases} \Delta E^T(\mu) \Delta E(\mu) \leq \bar{\epsilon}^2 \mathbf{I} \quad \forall \mu \in M, \end{cases} \quad (24a)$$

$$\begin{cases} \Delta E^T(\mu^*) \Delta E(\mu^*) = \bar{\epsilon}^2 \mathbf{I} \quad \mu^* \in M, \end{cases} \quad (24b)$$

there exists a set of positive-definite symmetric matrix $P_i, i \in \mathcal{I}$ such that,

$$\begin{aligned} & (A + R\Delta E)^T P_j H (\mathbf{I} - H^T P_j H)^{-1} H^T P_j (A + R\Delta E) \\ & + (A + R\Delta E)^T P_j (A + R\Delta E) - P_i < 0, \end{aligned} \quad (25)$$

if and only if there exists a set of positive constants $\varepsilon_i, i \in \mathcal{I}$ such that the following conditions hold,

$$\begin{cases} \bar{P}_j = P_j^{-1} - H H^T - \varepsilon_i R R^T > 0, \end{cases} \quad (26a)$$

$$\begin{cases} A^T \bar{P}_j^{-1} A - P_i + \frac{1}{\varepsilon_i} \bar{\epsilon}^2 \mathbf{I} < 0. \end{cases} \quad (26b)$$

Suppose the upper bound of uncertainty ΔH is given by

$$\Delta H \Delta H^T \leq \bar{\epsilon}_h^2 \mathbf{I}_n. \quad (27)$$

then based on Lemmas 4.1, 4.2 and the upper bounds defined in (17), one has the following result.

Theorem 4.1. Given a constant $\gamma > 0$, the nonlinear system (20) can be semi-globally asymptotically stabilized with generalized $\mathcal{H}_2$ performance γ via the controller in (13), if there exist a set of positive definite matrices $X_i, i \in \mathcal{I}$, a set of matrices $Q_i, i \in \mathcal{I}$ and a set of positive constants $\varepsilon_i, i \in \mathcal{I}$ such that the following LMIs are satisfied,

$$\begin{bmatrix} X_i & \star \\ \bar{C}_k X_i & \gamma^2 \mathbf{I} \end{bmatrix} > 0, \quad (28)$$

$$\begin{bmatrix} X_i & \star & \star \\ \bar{\epsilon} X_i & \varepsilon_i \mathbf{I}_n & \star \\ \bar{A}_k X_i + \bar{B}_k Q_i & \mathbf{0} & \Upsilon_{kij} \end{bmatrix} > 0, \quad (29)$$

where $\Upsilon_{kij} = X_j - R(2(H_k H_k^T + \bar{\epsilon}_h^2 \mathbf{I}_n) - \varepsilon_i \mathbf{I}_n) R^T$.

Moreover, the controller gains can be determined by

$$\bar{K}_i = Q_i X_i^{-1}, i \in \mathcal{I}. \quad (30)$$

Proof. The proof is omitted here.

It follows from Theorem 4.1 that the minimum generalized $\mathcal{H}_2$ performance index γ can be obtained by the following convex optimization algorithm [25].

Algorithm 4.1: $\min_{X_i, Q_i, \varepsilon_i} \gamma^2$, subject to LMIs (28) and (29).

V. SIMULATION STUDIES

Control of the inverted pendulum is one of the benchmark examples to demonstrate novel nonlinear control approaches. It is also noted that actuator models in the inverted pendulum are often given by the hyperbolic tangent functions of control input u . To show the performance of the above controller design results, we consider the balancing problem of an inverted pendulum on a cart. The equation of motion for the pendulum are

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g \sin(x_1) - a m l x_2^2 \sin(2x_1)/2 - a \cos(x_1)v}{4l/3 - a m l \cos^2(x_1)} + w \\ v &= (\arctan(u) + 0.15u) * 10^3\end{aligned}$$

where x_1 denotes the angle of pendulum from the vertical, and x_2 is the angular velocity. $g = 9.8m/s^2$ is the gravity constant, m is the mass of pendulum, M is the mass of the cart, $2l$ is the length of the pendulum and w is the disturbance. Note the input force is given by $\arctan(u) + 0.15u$ with an amplifier 1000 connected. In this simulation, we choose $m = 2.0\text{kg}$, $M = 8.0\text{kg}$, $2l = 1.0\text{m}$.

We linearize the plant around the following operating points, $(x; u) = (0; 0; 0)$, $(0; 0; \pm 3)$, $(\pm 88^\circ; 0; 0)$, and $(\pm 88^\circ; 0; \pm 3)$ respectively, and consider the approximation errors between the linearized local model and the original nonlinear models as norm-bounded uncertainties. By the standard discretization procedure [14] with sampling time $T = 0.01\text{s}$, we can obtain the following uncertain discrete-time dynamic T-S fuzzy model,

Plant rule $\mathcal{R}^l$: IF $|x_1|$ is μ^l And $|u|$ is ν^l THEN

$$\begin{cases} x(t+1) = (A_l + \Delta A)x(t) + (B_l + \Delta B)u(t), \\ \quad \quad \quad + (H_l + \Delta H)w(t) \\ z(t) = C_l x(t) + D_l u(t), l \in \mathcal{L} := \{1, 2, 3, 4\}. \end{cases}$$

where

$$\begin{aligned}A_1 &= A_2 = \begin{bmatrix} 1 & 0.01 \\ 0.1729 & 1 \end{bmatrix}, \\ A_3 &= A_4 = \begin{bmatrix} 1 & 0.01 \\ 0.0036 & 1 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0 \\ -2.03 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -1.38 \end{bmatrix}, \\ B_3 &= \begin{bmatrix} 0 \\ -0.060 \end{bmatrix}, B_4 = \begin{bmatrix} 0 \\ -0.041 \end{bmatrix}, \\ C_1 &= C_2 = C_3 = C_4 = [0.5 \quad 1], \\ H_1 &= H_2 = H_3 = H_4 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \\ D_1 &= 0.05, D_2 = D_3 = D_4 = 0.01.\end{aligned}$$

Suppose the upper bounds of uncertainties ΔA and ΔB are given by (17) with $\bar{\varepsilon} = 0.005$. It is noted that the uncertainty $\Delta H = 0$.

By using the space partition method given in section III, the premise variable space is partitioned into six regions. The membership functions and the corresponding local regions are shown in Fig. 1.

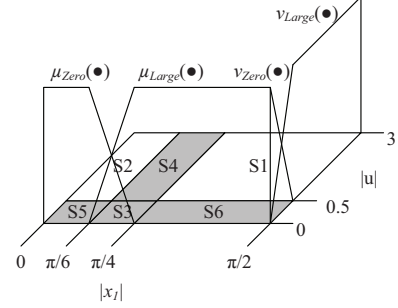


Fig. 1. MFs and the partitioned regions

By applying Algorithm 4.1, the minimum $\mathcal{H}_2$ performance $\gamma_{\min} = 0.0572$ is obtained and the corresponding controller gains are given by

$$\begin{aligned}\bar{K}_1 &= [0.5660 \quad 0.4997 \quad -0.0202], \\ \bar{K}_2 &= [0.6613 \quad 0.5111 \quad -0.6974], \\ \bar{K}_3 &= [1.1039 \quad 0.9744 \quad -1.0127], \\ \bar{K}_4 &= [0.8436 \quad 0.7292 \quad -0.5135], \\ \bar{K}_5 &= [0.7935 \quad 0.6286 \quad -1.0604], \\ \bar{K}_6 &= [0.5673 \quad 0.5019 \quad -0.0245].\end{aligned}$$

To illustrate the performance of the approaches proposed in this paper, we first show the state trajectories and control input of the closed-loop system under initial condition $x(0) = (89^\circ, 0)$ in Fig. 2. It can be found that the two states converge to zero when the external disturbance $w(t) = 0$. Next we illustrate the disturbance attenuation performance. Assume zero initial conditions, Fig. 3 shows the response of the ratio $\frac{\max |z(1:\kappa)|}{\sqrt{\sum_{i=0}^{\kappa} w^T(i)w(i)}}$ of the closed-loop control system. The disturbance is $w(t) = e^{-0.02t} \sin(\pi t)$. It shows that the performance is satisfactory and the ratio $\frac{\max |z(1:\kappa)|}{\sqrt{\sum_{i=0}^{\kappa} w^T(i)w(i)}}$ is less than 0.01, which is less than the minimum disturbance attenuation level $\gamma_{\min} = 0.0572$.

A number of simulations have been conducted and it is found that the pendulum can be balanced from $x_1 \in [-89^\circ, 89^\circ]$ with guaranteed performance. Note that the control input is also bounded.

VI. CONCLUSIONS

A class of generalized T-S fuzzy models have been proposed and shown to be universal approximators to general nonlinear

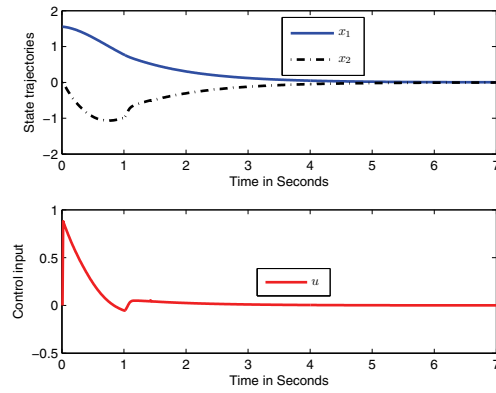


Fig. 2. State trajectories and control input

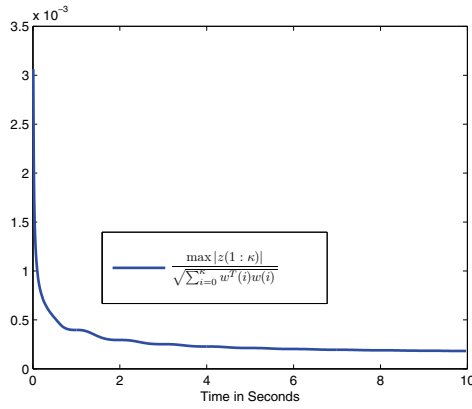


Fig. 3. Response of the ratio $\frac{\max|z(1:\kappa)|}{\sqrt{\sum_{i=0}^{\kappa} w^T(i)w(i)}}$

systems. Based on this kind of T-S fuzzy models and piecewise quadratic Lyapunov functions, approaches to stabilization and generalized $\mathcal{H}_2$ control of those general nonlinear systems have been developed. It is shown that the general nonlinear systems can be semi-globally stabilized with/without generalized $\mathcal{H}_2$ performance by solving a set of linear matrix inequalities. An example is presented to demonstrate the effectiveness of the proposed approaches. It is believed that the approaches proposed in this paper provide a new avenue to solve much more difficult control problem for general nonlinear systems.

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