

Fuzzy Approaches for Multiobjective Stochastic Linear Programming Problems Considering Both Probability Maximization and Fractile Optimization

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Abstract—In this paper, we propose two kinds of fuzzy approaches to obtain a satisfactory solution for multiobjective stochastic linear programming problems, in which the criteria of probability maximization and fractile optimization are considered simultaneously. In the first approach, a probability maximization model is applied to multiobjective stochastic linear programming problems, where the decision maker is required to specify not permissible objective levels but the corresponding membership functions which represent fuzzy goals for permissible objective levels. By adopting the fuzzy decision to integrate both the membership functions for permissible objective levels and the ones for the probability functions, a probability maximization model is transformed to the maxmin problem without permissible objective levels as parameters. Similarly, in the second approach, a fractile optimization model is applied to multiobjective stochastic linear programming problems, where the decision maker is required to specify not permissible probability levels but the corresponding membership functions which represent fuzzy goals for permissible probability levels. By adopting the fuzzy decision to integrate both the membership functions for permissible probability levels and the ones for the objective functions, a fractile optimization model is transformed to the maxmin problem without permissible probability levels as parameters. It is shown that such two fuzzy approaches are same.

Keywords- multiobjective stochastic linear programming; probability maximization; fractile optimization; fuzzy decision

I. INTRODUCTION

In the real world decision making situations, we often have to make a decision under uncertainty. In order to deal with decision problems involving uncertainty, stochastic programming approaches and fuzzy programming approaches have been developed. In stochastic programming approaches, two stage problems [3] and chance constrained programming models [1] have been investigated in various ways, and they were extended to multiobjective stochastic programming problems [7],[10]. In fuzzy programming approaches, various types of fuzzy programming problems have been formulated and investigated [5],[6],[12]. As a natural extension, multiobjective fuzzy programming technique first proposed by Zimmermann [11], and many methods have been proposed [8],[12].

Recently, in order to deal with probabilistic uncertainty and fuzzy one simultaneously, the hybrid approaches of stochastic programming and fuzzy programming have been proposed [4].

Especially, Sakawa et al. [9] proposed an interactive fuzzy satisficing method for multiobjective linear programming problem with random variable coefficients. In their method, a probability maximization model was adopted to transform stochastic programming problems into well-defined mathematical programming ones. When adopting a probability maximization model, it is required in advance that the decision maker specifies permissible levels for objective functions (called permissible objective levels) in his/her subjective manner. However, it seems to be very difficult to specify such values in advance, because there exist conflicts among permissible objective levels and the corresponding distribution function values. This means that the less values of permissible objective levels for minimization problems results in the less values of the corresponding distribution function. On the other hand, when a fractile optimization model is adopted, it is required in advance that the decision maker specifies permissible levels for the probability levels (called permissible probability levels) in his/her subjective manner. Similar to a probability maximization model, there exist conflicts among permissible probability levels and the corresponding objective function values. This means that the larger values of permissible probability levels results in the larger values of the corresponding objective function for minimization problems.

In this paper, we propose two kinds of fuzzy approaches to obtain a satisfactory solution for multiobjective stochastic linear programming problems, in which the criteria of probability maximization and fractile optimization are considered simultaneously. In section II, in order to deal with a probability maximization model and a fractile optimization model for multiobjective stochastic linear programming problems, P-Pareto optimal solution concept and F-Pareto optimal solution concept are defined respectively. The relationships between them are investigated. In section III, a probability maximization model is applied to multiobjective stochastic linear programming problems, where the decision maker is required to specify the membership functions which represent fuzzy goals for permissible objective levels. By adopting the fuzzy decision [8],[12] to integrate both the membership functions for permissible objective levels and the ones for the probability functions, a probability maximization model is transformed

to the maxmin problem without permissible objective levels as parameters. In section IV, a fractile optimization model is applied to multiobjective stochastic linear programming problems, where the decision maker is required to specify the membership functions which represent fuzzy goals for permissible probability levels. By adopting the fuzzy decision [8],[12] to integrate both the membership functions for permissible probability levels and the ones for the objective functions, a fractile optimization model is transformed to the maxmin problem without permissible probability levels as parameters. It is shown that two types of maxmin problems based on a probability maximization model and a fractile optimization model are same. In section V, an illustrative numerical example, which is formulated by Sakawa et al. [9], for multiobjective stochastic linear programming problems demonstrates the feasibility of the proposed approach.

II. TWO FORMULATIONS FOR MULTIOBJECTIVE STOCHASTIC LINEAR PROGRAMMING PROBLEMS THROUGH A PROBABILITY MAXIMIZATION MODEL AND A FRACTILE OPTIMIZATION MODEL

In this section, we focus on multiobjective linear programming problem involving random variable coefficients in objective functions and the right-hand side of constraints, which is called multiobjective stochastic linear programming problem (MOSLP). MOSLP can be formally formulated as follows.

[MOSLP]

$$\min_{\mathbf{x} \geq \mathbf{0}} \mathbf{z}(\mathbf{x}, \omega) = (z_1(\mathbf{x}, \omega), \dots, z_k(\mathbf{x}, \omega)) \quad (1)$$

subject to

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}(\omega) \quad (2)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ is an n dimensional decision variable column vector,

$$z_i(\mathbf{x}, \omega) = \mathbf{c}_i(\omega)\mathbf{x} + \alpha_i(\omega), i = 1, \dots, k, \quad (3)$$

are multiobjective functions involving random variable coefficients, $\mathbf{c}_i(\omega)$ is a n dimensional random variable row vector expressed by $\mathbf{c}_i(\omega) = \mathbf{c}_i^1 + t_i(\omega)\mathbf{c}_i^2$ where $t_i(\omega)$ is a random variable and $\alpha_i(\omega)$ is a random variable row vector expressed by $\alpha_i(\omega) = \alpha_i^1 + t_i(\omega)\alpha_i^2$. $\mathbf{A}$ is an $(m \times n)$ coefficient matrix, $\mathbf{b}(\omega) = (b_1(\omega), \dots, b_m(\omega))^T$ is an m dimensional column vector, whose elements $b_j(\omega), j = 1, \dots, m$ are random variables independent each other. Let us denote distribution functions of random variables $t_i(\omega)$ and $b_j(\omega)$ as follows.

$$T_i(r) \stackrel{\text{def}}{=} \Pr(t_i(\omega) < r), i = 1, \dots, k \quad (4)$$

$$F_j(r) \stackrel{\text{def}}{=} \Pr(b_j(\omega) < r), j = 1, \dots, m \quad (5)$$

It is assumed that distribution functions $T_i(\cdot), i = 1, \dots, k$ and $F_j(\cdot), j = 1, \dots, m$ are strictly increasing and continuous.

Since MOSLP contains random variable coefficients in objective functions and the right-hand side of constraints, mathematical programming techniques can not be directly applied. In order to deal with MOSLP, we interpret the constraints

(2) as chance constrained conditions [1], which mean that the j -th constraint of (2) is required to be satisfied with a certain probability β_j (called a constraint probability level) and over.

$$X(\beta) = \{\mathbf{x} \geq \mathbf{0} \mid \Pr(\mathbf{a}_j\mathbf{x} \leq b_j(\omega)) \geq \beta_j, j = 1, \dots, m\} \quad (6)$$

where $\mathbf{a}_j$ is the j -th row vector of $\mathbf{A}$. Using a distribution function $F_j(r)$, the j -th constraint in (6) can be rewritten as follows.

$$\begin{aligned} & \Pr(\mathbf{a}_j\mathbf{x} \leq b_j(\omega)) \geq \beta_j, \\ \Leftrightarrow & 1 - F_j(\mathbf{a}_j\mathbf{x}) \geq \beta_j, \\ \Leftrightarrow & F_j(\mathbf{a}_j\mathbf{x}) \leq 1 - \beta_j, \\ \Leftrightarrow & \mathbf{a}_j\mathbf{x} \leq F_j^{-1}(1 - \beta_j) \end{aligned}$$

where $F_j^{-1}(\cdot)$ means a pseudo-inverse function corresponding to $F_j(\cdot)$ defined by $F_j^{-1}(h) = \inf\{y \mid F_j(y) \geq h\}$.

Under the above chance constrained conditions (6), if we adopt a probability maximization model for the objective functions in MOSLP, we can substitute the minimization of the objective functions $z_i(\mathbf{x}, \omega) = \mathbf{c}_i(\omega)\mathbf{x} + \alpha_i(\omega), i = 1, \dots, k$ for the maximization of the probability that each objective functions $z_i(\mathbf{x}, \omega)$ is less than or equal to a certain permissible level $\bar{f}_i$ (called a permissible objective level). Such a probability function can be defined as follows.

$$p_i(\mathbf{x}, \bar{f}_i) \stackrel{\text{def}}{=} \Pr(z_i(\mathbf{x}, \omega) \leq \bar{f}_i), i = 1, \dots, k \quad (7)$$

where

$$\bar{\mathbf{f}} = (\bar{f}_1, \dots, \bar{f}_k). \quad (8)$$

is a k -dimensional vector of permissible objective levels which are specified by the decision maker in his/her subjective manner. Then, MOSLP can be converted as the following form. **[MOP1($\bar{\mathbf{f}}, \beta$)]**

$$\max_{\mathbf{x} \in X(\beta)} (p_1(\mathbf{x}, \bar{f}_1), \dots, p_k(\mathbf{x}, \bar{f}_k)) \quad (9)$$

In MOP1($\bar{\mathbf{f}}, \beta$), assuming that $\mathbf{c}_i^2\mathbf{x} + \alpha_i^2 > 0, i = 1, \dots, k$ for any $\mathbf{x} \in X(\beta)$, we can rewrite the objective function $p_i(\mathbf{x}, \bar{f}_i)$ as follows.

$$\begin{aligned} p_i(\mathbf{x}, \bar{f}_i) &= \Pr(z_i(\mathbf{x}, \omega) \leq \bar{f}_i) \\ &= \Pr(\mathbf{c}_i(\omega)\mathbf{x} + \alpha_i(\omega) \leq \bar{f}_i) \\ &= \Pr((\mathbf{c}_{r\ell}^1 + t_{r\ell}(\omega)\mathbf{c}_{r\ell}^2)\mathbf{x} \\ &\quad + (\alpha_{r\ell}^1 + t_{r\ell}(\omega)\alpha_{r\ell}^2) \leq \bar{f}_{r\ell}) \\ &= \Pr\left(t_i(\omega) \leq \frac{\bar{f}_i - (\mathbf{c}_i^1\mathbf{x} + \alpha_i^1)}{\mathbf{c}_i^2\mathbf{x} + \alpha_i^2}\right) \\ &= T_i\left(\frac{\bar{f}_i - (\mathbf{c}_i^1\mathbf{x} + \alpha_i^1)}{\mathbf{c}_i^2\mathbf{x} + \alpha_i^2}\right) \end{aligned}$$

In order to deal with MOP1($\bar{\mathbf{f}}, \beta$), we introduce a P-Pareto optimal solution concept as follows.

Definition 1.

$\mathbf{x}^* \in X(\beta)$ is said to be a P-Pareto optimal solution to MOP1($\bar{\mathbf{f}}, \beta$), if and only if there does not exist another $\mathbf{x} \in X(\beta)$ such that $p_i(\mathbf{x}, \bar{f}_i) \geq p_i(\mathbf{x}^*, \bar{f}_i), i = 1, \dots, k$, with strict inequality holding for at least one i .

On the other hand, if we adopt a fractile optimization model for the objective functions in MOSLP under the chance constrained conditions (6), we can convert MOSLP to the following multiobjective programming problem.

[MOP2($\bar{p}, \beta$)]

$$\min_{\mathbf{x} \in X(\beta), \bar{f}_i, i=1, \dots, k} (\bar{f}_1, \dots, \bar{f}_k) \quad (10)$$

subject to

$$p_i(\mathbf{x}, \bar{f}_i) \geq \bar{p}_i, i = 1, \dots, k \quad (11)$$

where

$$\bar{p} = (\bar{p}_1, \dots, \bar{p}_k) \quad (12)$$

is a k -dimensional vector of permissible probability levels which are specified by the decision maker in his/her subjective manner.

In MOP2($\bar{p}, \beta$), since a distribution function $T_i(\cdot)$ is continuous and strictly increasing and continuous, the constraints (11) can be transformed to the following form.

$$\begin{aligned} \bar{p}_i &\leq p_i(\mathbf{x}, \bar{f}_i) = T_i\left(\frac{\bar{f}_i - (c_i^1 \mathbf{x} + \alpha_i^1)}{c_i^2 \mathbf{x} + \alpha_i^2}\right) \\ \Leftrightarrow \bar{f}_i &\geq T_i^{-1}(\bar{p}_i) \cdot (c_i^2 \mathbf{x} + \alpha_i^2) + (c_i^1 \mathbf{x} + \alpha_i^1) \end{aligned} \quad (13)$$

Let us define the right-hand side of the inequality (13) as follows.

$$f_i(\mathbf{x}, \bar{p}_i) \stackrel{\text{def}}{=} T_i^{-1}(\bar{p}_i) \cdot (c_i^2 \mathbf{x} + \alpha_i^2) + (c_i^1 \mathbf{x} + \alpha_i^1) \quad (14)$$

Then, MOP2($\bar{p}, \beta$) can be equivalently reduced to the following simple form, where $\bar{f}_i, i = 1, \dots, k$ are disappeared.

[MOP3($\bar{p}, \beta$)]

$$\min_{\mathbf{x} \in X(\beta)} (f_1(\mathbf{x}, \bar{p}_1), \dots, f_k(\mathbf{x}, \bar{p}_k)) \quad (15)$$

In order to deal with MOP3($\bar{p}, \beta$), we introduce an F-Pareto optimal solution concept as follows.

Definition 2.

$\mathbf{x}^* \in X(\beta)$ is said to be an F-Pareto optimal solution to MOP3($\bar{p}, \beta$), if and only if there does not exist another $\mathbf{x} \in X(\beta)$ such that $f_i(\mathbf{x}, \bar{p}_i) \leq f_i(\mathbf{x}^*, \bar{p}_i), i = 1, \dots, k$, with strict inequality holding for at least one i .

The relationships between a P-Pareto optimal solution to MOP1($\bar{f}, \beta$) and an F-Pareto optimal solution to MOP3($\bar{p}, \beta$) can be characterized by the following theorem.

Theorem 1.

(1) If $\mathbf{x}^* \in X(\beta)$ is a P-Pareto optimal solution to MOP1($\bar{f}, \beta$) for some given permissible objective levels $\bar{f}$, then, there exist some permissible probability levels $\bar{p}$ such that $\mathbf{x}^* \in X(\beta)$ is an F-Pareto optimal solution to MOP3($\bar{p}, \beta$).

(2) If $\mathbf{x}^* \in X(\beta)$ is an F-Pareto optimal solution to MOP3($\bar{p}, \beta$) for some given permissible probability levels $\bar{p}$, then, there exist some permissible objective levels $\bar{f}$ such that $\mathbf{x}^* \in X(\beta)$ is a P-Pareto optimal solution to MOP1($\bar{f}, \beta$).

(Proof)

(1) Assume that $\mathbf{x}^* \in X(\beta)$ is a P-Pareto optimal solution to MOP1($\bar{f}, \beta$) for some given permissible objective levels $\bar{f}$.

Then, from Definition 1, there does not exist $\mathbf{x} \in X(\beta)$ such that

$$\begin{aligned} p_i(\mathbf{x}, \bar{f}_i) &\geq p_i(\mathbf{x}^*, \bar{f}_i) \\ \Leftrightarrow T_i\left(\frac{\bar{f}_i - (c_i^1 \mathbf{x} + \alpha_i^1)}{c_i^2 \mathbf{x} + \alpha_i^2}\right) &\geq T_i\left(\frac{\bar{f}_i - (c_i^1 \mathbf{x}^* + \alpha_i^1)}{c_i^2 \mathbf{x}^* + \alpha_i^2}\right) \\ i = 1, \dots, k \end{aligned} \quad (16)$$

with strict inequality holding for at least one i . For the left-hand side of the inequality (16), let us set as :

$$\bar{p}_i = T_i\left(\frac{\bar{f}_i - (c_i^1 \mathbf{x}^* + \alpha_i^1)}{c_i^2 \mathbf{x}^* + \alpha_i^2}\right). \quad (17)$$

Then, because of the definition (14) and $c_i^2 \mathbf{x} + \alpha_i^2 > 0$, the inequality (16) can be transformed as follows.

$$\begin{aligned} T_i\left(\frac{\bar{f}_i - (c_i^1 \mathbf{x} + \alpha_i^1)}{c_i^2 \mathbf{x} + \alpha_i^2}\right) &\geq \bar{p}_i \\ \Leftrightarrow \bar{f}_i - (c_i^1 \mathbf{x} + \alpha_i^1) &\geq T_i^{-1}(\bar{p}_i) \cdot (c_i^2 \mathbf{x} + \alpha_i^2) \\ \Leftrightarrow \bar{f}_i &\geq f_i(\mathbf{x}, \bar{p}_i) \end{aligned} \quad (18)$$

On the other hand, from the relationship (17), it immediately holds that

$$\begin{aligned} \bar{f}_i &= T_i^{-1}(\bar{p}_i) \cdot (c_i^2 \mathbf{x}^* + \alpha_i^2) + (c_i^1 \mathbf{x}^* + \alpha_i^1) \\ &= f_i(\mathbf{x}^*, \bar{p}_i). \end{aligned}$$

This means that there does not exist $\mathbf{x} \in X(\beta)$ such that $f_i(\mathbf{x}, \bar{p}_i) \leq f_i(\mathbf{x}^*, \bar{p}_i), i = 1, \dots, k$ with strict inequality holding for at least one i . Therefore, $\mathbf{x}^* \in X(\beta)$ is an F-Pareto optimal solution to MOP3($\bar{p}, \beta$).

(2) The proof is same to (1).

Sakawa et al.[9] formulated multiobjective linear programming problems with random variable coefficients in objective functions through a probability maximization model. In their approach, assuming that the decision maker has a fuzzy goal for each distribution function $p_i(\mathbf{x}, \bar{f}_i)$, the satisfactory solution is derived from among M-Pareto optimal solution set by updating the reference membership levels in membership space. However, in a probability maximization model, the permissible objective levels $\bar{f}_i, i = 1, \dots, k$ for the original objective functions $z_i(\mathbf{x}, \omega), i = 1, \dots, k$ must be specified in advance by the decision. The less values for the permissible objective levels $\bar{f}_i, i = 1, \dots, k$ in MOP1($\bar{f}, \beta$) will give the less values for the corresponding distribution functions $p_i(\mathbf{x}, \bar{f}_i), i = 1, \dots, k$. As a result, if the decision maker specifies the less values for the permissible objective levels $\bar{f}_i, i = 1, \dots, k$, he/she may not be able to accept any candidate of the satisfactory solution, because the corresponding values of the distribution functions $p_i(\mathbf{x}, \bar{f}_i), i = 1, \dots, k$ will become close to zero.

On the other hand, instead of a probability maximization model, let us consider the case where a fractile optimization model is adopted to formulate multiobjective linear programming problems with random variable coefficients in objective functions. In such a model, the permissible probability levels $\bar{p}_i, i = 1, \dots, k$ must be specified by the decision maker in advance. The larger values for the permissible probability

levels $\bar{p}_i, i = 1, \dots, k$ will give the larger values for the objective functions $f_i(\mathbf{x}, \bar{p}_i), i = 1, \dots, k$ for minimization problems. As a result, similar to the case where a probability maximization model is adopted, if the decision maker specifies the larger values for the permissible probability levels $\bar{p}_i, i = 1, \dots, k$, he/she may not be able to accept any candidate of the satisfactory solution, because the corresponding values of the objective functions $f_i(\mathbf{x}, \bar{p}_i), i = 1, \dots, k$ will become the larger values.

In the next section, we assume that, the decision maker has fuzzy goals for not only each distribution function $p_i(\mathbf{x}, \bar{f}_i)$ but also each permissible objective levels $\bar{f}_i$ for MOSLP through a probability maximization model. After eliciting the membership functions for such fuzzy goals, we propose a fuzzy approach for MOSLP through a probability maximization model. to obtain the satisfactory solution.

III. A FUZZY APPROACH FOR MULTIOBJECTIVE STOCHASTIC LINEAR PROGRAMMING PROBLEMS THROUGH A PROBABILITY MAXIMIZATION MODEL

In this section, we propose a fuzzy approach to MOP1($\bar{\mathbf{f}}, \beta$) where MOSLP is formulated through a probability maximization model. In order to deal with MOP1($\bar{\mathbf{f}}, \beta$), the decision maker must specify his/her permissible objective levels $\bar{\mathbf{f}}$ in advance. However, in general, the decision maker seems to prefer not only the larger value of the distribution function $p_i(\mathbf{x}, \bar{f}_i)$ but also the less value of permissible objective level $\bar{f}_i$. Since these values conflict with each other, the less values of permissible objective level $\bar{f}_i$ results in the less value of probability function $p_i(\mathbf{x}, \bar{f}_i)$. From such a point of view, we consider the following multiobjective programming problem which can be regarded as a natural extension of MOP1($\bar{\mathbf{f}}, \beta$).

$$\begin{aligned} & \max_{\mathbf{x} \in X(\beta), \bar{f}_i, i=1, \dots, k} (p_1(\mathbf{x}, \bar{f}_1), \dots, p_k(\mathbf{x}, \bar{f}_k), -\bar{f}_1, \dots, -\bar{f}_k) \\ & \quad (19) \end{aligned}$$

It should be noted in MOP4(β) that the permissible objective levels $\bar{f}_i, i = 1, \dots, k$ are not the fixed values but the decision variables.

Considering the imprecise nature of the decision maker's judgment, it is natural to assume that the decision maker have fuzzy goals for each objective function in MOP4(β). In this section, we assume that such fuzzy goals can be quantified by eliciting the corresponding membership functions. Let us denote a membership function of a probability function $p_i(\mathbf{x}, \bar{f}_i)$ as $\mu_{p_i}(p_i(\mathbf{x}, \bar{f}_i))$, and a membership function of a permissible objective level $\bar{f}_i$ as $\mu_{\bar{f}_i}(\bar{f}_i)$ respectively. Then, MOP4(β) can be transformed as the following multiobjective programming problem.

$$\begin{aligned} & \max_{\mathbf{x} \in X(\beta), \bar{f}_i, i=1, \dots, k} (\mu_{p_1}(p_1(\mathbf{x}, \bar{f}_1)), \dots, \mu_{p_k}(p_k(\mathbf{x}, \bar{f}_k)), \\ & \quad \mu_{\bar{f}_1}(\bar{f}_1), \dots, \mu_{\bar{f}_k}(\bar{f}_k)) \\ & \quad (20) \end{aligned}$$

Throughout this section, we make the following assumptions.

Assumption 1.

$\mu_{\bar{f}_i}(\bar{f}_i), i = 1, \dots, k$ are strictly decreasing and continuous with respect to $\bar{f}_i$.

Assumption 2.

$\mu_{p_i}(p_i(\mathbf{x}, \bar{f}_i)), i = 1, \dots, k$ are strictly increasing and continuous with respect to $p_i(\mathbf{x}, \bar{f}_i)$.

For example, we can define the membership functions $\mu_{\bar{f}_i}(\bar{f}_i), \mu_{p_i}(p_i(\mathbf{x}, \bar{f}_i)), i = 1, \dots, k$ as follows. Considering the individual minimum and maximum of $E(z_i(\mathbf{x}, \omega))$:

$$\min_{\mathbf{x} \in X(\beta)} / \max_{\mathbf{x} \in X(\beta)} E(z_i(\mathbf{x}, \omega)), i = 1, \dots, k,$$

the decision maker subjectively specifies a sufficiently satisfactory maximum value $f_{i\min}$ and an acceptable maximum value $f_{i\max}$. Then, the domain of $\mu_{\bar{f}_i}(\bar{f}_i)$ can be set as:

$$F_i = [f_{i\min}, f_{i\max}]. \quad (21)$$

According to Assumption 1, the decision maker can define his/her membership function $\mu_{\bar{f}_i}(\bar{f}_i)$ on the domain F_i . Corresponding to the domain F_i , let us denote the domain of $\mu_{p_i}(p_i(\mathbf{x}, \bar{f}_i))$ as:

$$P_i(F_i) = [p_{i\min}, p_{i\max}]. \quad (22)$$

where $p_{i\min}$ is an acceptable minimum value and $p_{i\max}$ is a sufficiently satisfactory minimum value. $p_{i\max}$ can be obtained by solving the following problem.

$$p_{i\max} = \max_{\mathbf{x} \in X(\beta)} p_i(\mathbf{x}, f_{i\max}) \quad (23)$$

It should be noted here that the above problem is equivalent to the following linear fractional programming problem [2] because a distribution function $T_i(\cdot)$ is strictly increasing and continuous.

$$\max_{\mathbf{x} \in X(\beta)} \left(\frac{f_{i\max} - (c_i^1 \mathbf{x} + \alpha_i^1)}{c_i^2 \mathbf{x} + \alpha_i^2} \right) \quad (24)$$

On the other hand, in order to obtain $p_{i\min}$, we first solve $\max_{\mathbf{x} \in X(\beta)} p_i(\mathbf{x}, f_{i\min})$, and denote the corresponding optimal solution as $\mathbf{x}_i$. Using the optimal solutions $\mathbf{x}_i, i = 1, \dots, k, p_{i\min}$ can be obtained as follows.

$$p_{i\min} = \min_{\ell=1, \dots, k, \ell \neq i} p_i(\mathbf{x}_\ell, f_{i\min}) \quad (25)$$

According to Assumption 2, the decision maker can define his/her membership function $\mu_{p_i}(p_i(\mathbf{x}, \bar{f}_i))$ on the domain $P_i(F_i)$.

Then, if the decision maker adopts the fuzzy decision [8],[12] as an aggregation operator for MOP5(β), the satisfactory solution is obtained by solving the following maxmin problem.

[MAXMIN1]

$$\max_{\mathbf{x} \in X(\beta), \bar{f}_i \in F_i, i=1, \dots, k, \lambda \in [0,1]} \lambda \quad (26)$$

subject to

$$\mu_{p_i}(p_i(\mathbf{x}, \bar{f}_i)) \geq \lambda, i = 1, \dots, k \quad (27)$$

$$\mu_{\bar{f}_i}(\bar{f}_i) \geq \lambda, i = 1, \dots, k \quad (28)$$

Because of Assumption 2 and $c_i^2 x + \alpha_i^2 > 0$, the constraints (27) can be transformed as:

$$\begin{aligned} & \mu_{p_i}(p_i(x, \bar{f}_i)) \geq \lambda, \\ \Leftrightarrow & p_i(x, \bar{f}_i) \geq \mu_{p_i}^{-1}(\lambda), \\ \Leftrightarrow & T_i\left(\frac{\bar{f}_i - (c_i^1 x + \alpha_i^1)}{c_i^2 x + \alpha_i^2}\right) \geq \mu_{p_i}^{-1}(\lambda), \\ \Leftrightarrow & \bar{f}_i - (c_i^1 x + \alpha_i^1) \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda)) \cdot (c_i^2 x + \alpha_i^2), \end{aligned} \quad (29)$$

where $\mu_{p_i}^{-1}(\cdot)$ is a pseudo-inverse function. From the constraints (28) and Assumption 1, it holds that $\bar{f}_i \leq \mu_{\bar{f}_i}^{-1}(\lambda)$. Therefore, the constraint (29) can be reduced to the following inequality where a permissible objective level $\bar{f}_i$ is disappeared.

$$\mu_{\bar{f}_i}^{-1}(\lambda) - (c_i^1 x + \alpha_i^1) \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda)) \cdot (c_i^2 x + \alpha_i^2) \quad (30)$$

Then, MAXMIN1 is equivalently transformed into the following problem.

[MAXMIN2]

$$\max_{x \in X(\beta), \lambda \in [0, 1]} \lambda \quad (31)$$

subject to

$$\mu_{\bar{f}_i}^{-1}(\lambda) - (c_i^1 x + \alpha_i^1) \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda)) \cdot (c_i^2 x + \alpha_i^2), \quad i = 1, \dots, k \quad (32)$$

It should be noted here that the constraints (32) can be reduced to a set of linear inequalities for some fixed value $\lambda \in [0, 1]$. This means that an optimal solution (x^*, λ^*) of MAXMIN2 is obtained by combined use of the bisection method with respect to λ and the first-phase of the two-phase simplex method of linear programming.

The relationships between the optimal solution (x^*, λ^*) of MAXMIN2, P-Pareto optimal solutions to MOP1($\bar{f}^*, \beta$), and F-Pareto optimal solutions to MOP3($\bar{p}^*, \beta$) can be characterized by the following theorem.

Theorem 2.

(1) If (x^*, λ^*) is a unique optimal solution of MAXMIN2, then $x^* \in X(\beta)$ is a P-Pareto optimal solution to MOP1($\bar{f}^*, \beta$), where $\bar{f}^* = (\bar{f}_1^*, \dots, \bar{f}_k^*) = (\mu_{\bar{f}_1}^{-1}(\lambda^*), \dots, \mu_{\bar{f}_k}^{-1}(\lambda^*))$.

(2) If (x^*, λ^*) is a unique optimal solution of MAXMIN2, then $x^* \in X(\beta)$ is an F-Pareto optimal solution to MOP3($\bar{p}^*, \beta$), where $\bar{p}^* = (\bar{p}_1^*, \dots, \bar{p}_k^*) = (\mu_{p_1}^{-1}(\lambda^*), \dots, \mu_{p_k}^{-1}(\lambda^*))$.

(Proof)

(1) Let (x^*, λ^*) be a unique optimal solution of MAXMIN2. Since an optimal solution (x^*, λ^*) satisfies the constraints (32), it holds that

$$\begin{aligned} & \mu_{\bar{f}_i}^{-1}(\lambda^*) - (c_i^1 x^* + \alpha_i^1) \\ & \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (c_i^2 x^* + \alpha_i^2), \\ \Leftrightarrow & T_i\left(\frac{\mu_{\bar{f}_i}^{-1}(\lambda^*) - (c_i^1 x^* + \alpha_i^1)}{c_i^2 x^* + \alpha_i^2}\right) = p_i(x^*, \mu_{\bar{f}_i}^{-1}(\lambda^*)) \\ & \geq \mu_{p_i}^{-1}(\lambda^*), i = 1, \dots, k. \end{aligned}$$

Assume that $x^* \in X(\beta)$ is not a P-Pareto optimal solution to MOP1($\bar{f}^*, \beta$), where $\bar{f}^* = (\mu_{\bar{f}_1}^{-1}(\lambda^*), \dots, \mu_{\bar{f}_k}^{-1}(\lambda^*))$, then there exists $x \in X(\beta)$ such that

$$\begin{aligned} & p_i(x, \mu_{\bar{f}_i}^{-1}(\lambda^*)) \geq p_i(x^*, \mu_{\bar{f}_i}^{-1}(\lambda^*)) \geq \mu_{p_i}^{-1}(\lambda^*), \\ & i = 1, \dots, k. \end{aligned}$$

This means that there exists $x \in X(\beta)$ such that

$$\begin{aligned} \mu_{\bar{f}_i}^{-1}(\lambda^*) - (c_i^1 x + \alpha_i^1) & \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (c_i^2 x + \alpha_i^2), \\ & i = 1, \dots, k, \end{aligned}$$

which contradicts the fact that (x^*, λ^*) is a unique optimal solution of MAXMIN2.

(2) Assume that $x^* \in X(\beta)$ is not an F-Pareto optimal solution to MOP3($\bar{p}^*, \beta$), where $\bar{p}^* = \mu_{p_i}^{-1}(\lambda^*), i = 1, \dots, k$. Then, because of (14), there exists $x \in X(\beta)$ such that

$$\begin{aligned} & f_i(x, \bar{p}_i^*) \leq f_i(x^*, \bar{p}_i^*) \\ \Leftrightarrow & T_i^{-1}(\bar{p}_i^*) \cdot (c_i^2 x + \alpha_i^2) + (c_i^1 x + \alpha_i^1) \\ & \leq T_i^{-1}(\bar{p}_i^*) \cdot (c_i^2 x^* + \alpha_i^2) + (c_i^1 x^* + \alpha_i^1) \stackrel{\text{def}}{=} \bar{f}_i^* \\ \Leftrightarrow & \bar{f}_i^* - (c_i^1 x + \alpha_i^1) \geq T_i^{-1}(\bar{p}_i^*) \cdot (c_i^2 x + \alpha_i^2) \\ \Leftrightarrow & \bar{f}_i^* - (c_i^1 x + \alpha_i^1) \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (c_i^2 x + \alpha_i^2) \end{aligned}$$

On the other hand, since (x^*, λ^*) is an optimal solution of MAXMIN2, from (32), it holds that

$$\mu_{\bar{f}_i}^{-1}(\lambda^*) - (c_i^1 x^* + \alpha_i^1) \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (c_i^2 x^* + \alpha_i^2).$$

This means that

$$\begin{aligned} \mu_{\bar{f}_i}^{-1}(\lambda^*) & \geq (c_i^1 x^* + \alpha_i^1) + T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (c_i^2 x^* + \alpha_i^2) \\ & = \bar{f}_i^*. \end{aligned}$$

Therefore, it holds that

$$\begin{aligned} \mu_{\bar{f}_i}^{-1}(\lambda^*) - (c_i^1 x + \alpha_i^1) & \geq T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (c_i^2 x + \alpha_i^2), \\ & i = 1, \dots, k, \end{aligned}$$

which contradicts the fact that (x^*, λ^*) is a unique optimal solution of MAXMIN2.

IV. A FUZZY APPROACH FOR MULTIOBJECTIVE STOCHASTIC LINEAR PROGRAMMING PROBLEMS THROUGH A FRACTILE OPTIMIZATION MODEL

In this section, we propose a fuzzy approach to MOP3($\bar{p}, \beta$) where MOSLP is formulated through a fractile optimization model, and show that the corresponding formulation is same to MAXMIN2. In order to deal with MOP3($\bar{p}, \beta$), the decision maker must specify permissible probability levels $\bar{p}$ in advance. However, in general, the decision maker seems to prefer not only the less value of the objective function $f_i(x, \bar{p}_i)$ of MOP3($\bar{p}, \beta$) but also the larger value of the permissible probability level $\bar{p}_i$. Since these values conflict with each other, the larger value of the permissible probability level $\bar{p}_i$ results in the larger value of the objective function $f_i(x, \bar{p}_i)$ of MOP3($\bar{p}, \beta$). From such a point of view, we consider the following multiobjective programming problem which can be regarded as a natural extension of MOP3($\bar{p}, \beta$).

[MOP6(β)]

$$\min_{\mathbf{x} \in X(\beta), \bar{p}_i, i=1, \dots, k} (f_1(\mathbf{x}, \bar{p}_1), \dots, f_k(\mathbf{x}, \bar{p}_k), -\bar{p}_1, \dots, -\bar{p}_k) \quad (33)$$

It should be noted in MOP6(β) that permissible probability levels $\bar{p}_i, i = 1, \dots, k$ are not the fixed values but the decision variables.

Considering the imprecise nature of the decision maker's judgment, we assume that the decision maker has fuzzy goals for each objective function in MOP6(β). Such fuzzy goals can be quantified by eliciting the corresponding membership functions. Let us denote a membership function of an objective function $f_i(\mathbf{x}, \bar{p}_i)$ as $\mu_{f_i}(f_i(\mathbf{x}, \bar{p}_i))$, and a membership function of a permissible probability level $\bar{p}_i$ as $\mu_{\bar{p}_i}(\bar{p}_i)$ respectively. Then, MOP6(β) can be transformed as the following multiobjective programming problem.

[MOP7(β)]

$$\max_{\mathbf{x} \in X(\beta), \bar{p}_i, i=1, \dots, k} (\mu_{f_1}(f_1(\mathbf{x}, \bar{p}_1)), \dots, \mu_{f_k}(f_k(\mathbf{x}, \bar{p}_k)), \mu_{\bar{p}_1}(\bar{p}_1), \dots, \mu_{\bar{p}_k}(\bar{p}_k)) \quad (34)$$

Throughout this section, we make the following assumptions.

Assumption 3.

$\mu_{\bar{p}_i}(\bar{p}_i), i = 1, \dots, k$ are strictly increasing and continuous with respect to $\bar{p}_i$.

Assumption 4.

$\mu_{f_i}(f_i(\mathbf{x}, \bar{p}_i)), i = 1, \dots, k$ are strictly decreasing and continuous with respect to $f_i(\mathbf{x}, \bar{p}_i)$.

For example, we can define the membership functions $\mu_{\bar{p}_i}(\bar{p}_i), \mu_{f_i}(f_i(\mathbf{x}, \bar{p}_i))$ as follows. At first, the decision maker specifies permissible probability intervals in his/her subjective manner,

$$P_i = [p_{i\min}, p_{i\max}], i = 1, \dots, k, \quad (35)$$

where $p_{i\min}$ is an acceptable minimum value and $p_{i\max}$ is a sufficiently satisfactory minimum value. According to Assumption 3, the decision maker can define his/her membership function $\mu_{\bar{p}_i}(\bar{p}_i)$ on the domain P_i . Corresponding to the domain P_i , Let us denote the domain of $\mu_{f_i}(f_i(\mathbf{x}, \bar{p}_i))$ as:

$$F_i(P_i) = [f_{i\min}, f_{i\max}], \quad (36)$$

where $f_{i\min}$ is a sufficiently satisfactory maximum value and $f_{i\max}$ is an acceptable maximum value. $f_{i\min}$ can be obtained by solving the following linear programming problem.

$$f_{i\min} = \min_{\mathbf{x} \in X(\beta)} f_i(\mathbf{x}, p_{i\min}) \quad (37)$$

In order to obtain $f_{i\max}$, we first solve the following k linear programming problems.

$$\min_{\mathbf{x} \in X(\beta)} f_i(\mathbf{x}, p_{i\max}), i = 1, \dots, k \quad (38)$$

Let $\mathbf{x}_i, i = 1, \dots, k$ be the above optimal solution. Using the optimal solutions $\mathbf{x}_i, i = 1, \dots, k$, $f_{i\max}$ can be obtained as follows.

$$f_{i\max} = \max_{\ell=1, \dots, k, \ell \neq i} f_i(\mathbf{x}_\ell, p_{i\max}) \quad (39)$$

According to Assumption 4, the decision maker can define his/her membership function $\mu_{f_i}(f_i(\mathbf{x}, \bar{p}_i))$ on the domain $F_i(P_i)$.

Then, if the decision maker adopts the fuzzy decision [8],[12] as an aggregation operator for MOP7(β), the satisfactory solution is obtained by solving the following maxmin problem.

[MAXMIN3]

$$\max_{\mathbf{x} \in X(\beta), \bar{p}_i \in P_i, i=1, \dots, k, \lambda \in [0, 1]} \lambda \quad (40)$$

subject to

$$\mu_{f_i}(f_i(\mathbf{x}, \bar{p}_i)) \geq \lambda, i = 1, \dots, k \quad (41)$$

$$\mu_{\bar{p}_i}(\bar{p}_i) \geq \lambda, i = 1, \dots, k \quad (42)$$

Because of Assumption 4 and $\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2 > 0$, the constraints (41) can be transformed as:

$$\begin{aligned} & \mu_{f_i}(f_i(\mathbf{x}, \bar{p}_i)) \geq \lambda, \\ \Leftrightarrow & f_i(\mathbf{x}, \bar{p}_i) \leq \mu_{f_i}^{-1}(\lambda), \\ \Leftrightarrow & T_i^{-1}(\bar{p}_i) \cdot (\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2) + (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1) \leq \mu_{f_i}^{-1}(\lambda), \\ \Leftrightarrow & \bar{p}_i \leq T_i \left(\frac{\mu_{f_i}^{-1}(\lambda) - (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1)}{\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2} \right) \end{aligned} \quad (43)$$

where $\mu_{f_i}^{-1}(\cdot)$ is a pseudo-inverse function. From the constraints (42) and Assumption 3, it holds that $\bar{p}_i \geq \mu_{\bar{p}_i}^{-1}(\lambda)$. Therefore, the constraint (43) can be reduced to the following inequality where a permissible probability level $\bar{p}_i$ is disappeared.

$$\begin{aligned} & \mu_{\bar{p}_i}^{-1}(\lambda) \leq T_i \left(\frac{\mu_{f_i}^{-1}(\lambda) - (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1)}{\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2} \right) \\ \Leftrightarrow & \mu_{f_i}^{-1}(\lambda) - (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1) \\ & \geq T_i^{-1}(\mu_{\bar{p}_i}^{-1}(\lambda)) \cdot (\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2) \end{aligned} \quad (44)$$

Then, MAXMIN3 is equivalently transformed into the following problem.

[MAXMIN4]

$$\max_{\mathbf{x} \in X(\beta), \lambda \in [0, 1]} \lambda \quad (45)$$

subject to

$$\begin{aligned} & \mu_{f_i}^{-1}(\lambda) - (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1) \geq T_i^{-1}(\mu_{\bar{p}_i}^{-1}(\lambda)) \cdot (\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2), \\ & i = 1, \dots, k \end{aligned} \quad (46)$$

It should be noted here that MAXMIN4 is equivalent to MAXMIN2, and the constraints (46) can be reduced to a set of linear inequalities for some fixed value $\lambda \in [0, 1]$. This means that an optimal solution $(\mathbf{x}^*, \lambda^*)$ of MAXMIN4 is obtained by combined use of the bisection method with respect to λ and the first-phase of the two-phase simplex method of linear programming.

The relationship between the optimal solution $(\mathbf{x}^*, \lambda^*)$ of MAXMIN4, P-Pareto optimal solutions to MOP1($\bar{\mathbf{f}}, \beta$), and F-Pareto optimal solutions to MOP3($\bar{\mathbf{p}}, \beta$) can be characterized by the following theorem.

Theorem 3.

(1) If $(\mathbf{x}^*, \lambda^*)$ is a unique optimal solution of MAXMIN4, then $\mathbf{x}^* \in X(\beta)$ is a P-Pareto optimal solution to MOP1($\bar{\mathbf{f}}^*, \beta$), where $\bar{\mathbf{f}}^* = (\bar{f}_1^*, \dots, \bar{f}_k^*) = (\mu_{f_1}^{-1}(\lambda^*), \dots, \mu_{f_k}^{-1}(\lambda^*))$.

(2) If $(\mathbf{x}^*, \lambda^*)$ is a unique optimal solution of MAXMIN4, then $\mathbf{x}^* \in X(\beta)$ is an F-Pareto optimal solution to MOP3($\bar{\mathbf{p}}^*, \beta$), where $\bar{\mathbf{p}}^* = (\bar{p}_1^*, \dots, \bar{p}_k^*) = (\mu_{\bar{p}_1}^{-1}(\lambda^*), \dots, \mu_{\bar{p}_k}^{-1}(\lambda^*))$.

(Proof)

(1) Let $(\mathbf{x}^*, \lambda^*)$ be a unique optimal solution of MAXMIN4. Since an optimal solution $(\mathbf{x}^*, \lambda^*)$ satisfies the constraints (46), it holds that

$$\begin{aligned} & \mu_{f_i}^{-1}(\lambda^*) - (\mathbf{c}_i^1 \mathbf{x}^* + \alpha_i^1) \\ & \geq T_i^{-1}(\mu_{\bar{p}_i}^{-1}(\lambda^*)) \cdot (\mathbf{c}_i^2 \mathbf{x}^* + \alpha_i^2), \\ \Leftrightarrow & T_i \left(\frac{\mu_{f_i}^{-1}(\lambda^*) - (\mathbf{c}_i^1 \mathbf{x}^* + \alpha_i^1)}{\mathbf{c}_i^2 \mathbf{x}^* + \alpha_i^2} \right) = p_i(\mathbf{x}^*, \mu_{f_i}^{-1}(\lambda^*)) \\ & \geq \mu_{\bar{p}_i}^{-1}(\lambda^*), i = 1, \dots, k. \end{aligned}$$

Assume that $\mathbf{x}^* \in X(\beta)$ is not a P-Pareto optimal solution to MOP1($\bar{\mathbf{f}}^*, \beta$), where $\bar{\mathbf{f}}^* = (\mu_{f_1}^{-1}(\lambda^*), \dots, \mu_{f_k}^{-1}(\lambda^*))$. Then there exists $\mathbf{x} \in X(\beta)$ such that

$$\begin{aligned} p_i(\mathbf{x}, \mu_{f_i}^{-1}(\lambda^*)) & \geq p_i(\mathbf{x}^*, \mu_{f_i}^{-1}(\lambda^*)) \geq \mu_{\bar{p}_i}^{-1}(\lambda^*), \\ i & = 1, \dots, k. \end{aligned}$$

This means that there exists $\mathbf{x} \in X(\beta)$ such that

$$\begin{aligned} \mu_{f_i}^{-1}(\lambda^*) - (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1) & \geq T_i^{-1}(\mu_{\bar{p}_i}^{-1}(\lambda^*)) \cdot (\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2), \\ i & = 1, \dots, k, \end{aligned}$$

which contradicts the fact that $(\mathbf{x}^*, \lambda^*)$ is a unique optimal solution of MAXMIN4.

(2) Assume that $\mathbf{x}^* \in X(\beta)$ is not an F-Pareto optimal solution to MOP3($\bar{\mathbf{p}}^*, \beta$), where $\bar{\mathbf{p}}^* = \mu_{\bar{p}_i}^{-1}(\lambda^*), i = 1, \dots, k$. Then, because of (14), there exists $\mathbf{x} \in X(\beta)$ such that

$$\begin{aligned} f_i(\mathbf{x}, \bar{\mathbf{p}}^*) & \leq f_i(\mathbf{x}^*, \bar{\mathbf{p}}^*) \\ \Leftrightarrow & T_i^{-1}(\bar{\mathbf{p}}^*) \cdot (\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2) + (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1) \\ & \leq T_i^{-1}(\bar{\mathbf{p}}^*) \cdot (\mathbf{c}_i^2 \mathbf{x}^* + \alpha_i^2) + (\mathbf{c}_i^1 \mathbf{x}^* + \alpha_i^1) \stackrel{\text{def}}{=} \bar{f}_i^* \\ \Leftrightarrow & \bar{f}_i^* - (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1) \geq T_i^{-1}(\bar{\mathbf{p}}^*) \cdot (\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2) \\ \Leftrightarrow & \bar{f}_i^* - (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1) \geq T_i^{-1}(\mu_{\bar{p}_i}^{-1}(\lambda^*)) \cdot (\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2) \end{aligned}$$

On the other hand, since $(\mathbf{x}^*, \lambda^*)$ is an optimal solution of MAXMIN4, from (46), it holds that

$$\mu_{f_i}^{-1}(\lambda^*) - (\mathbf{c}_i^1 \mathbf{x}^* + \alpha_i^1) \geq T_i^{-1}(\mu_{\bar{p}_i}^{-1}(\lambda^*)) \cdot (\mathbf{c}_i^2 \mathbf{x}^* + \alpha_i^2).$$

This means that

$$\begin{aligned} \mu_{f_i}^{-1}(\lambda^*) & \geq (\mathbf{c}_i^1 \mathbf{x}^* + \alpha_i^1) + T_i^{-1}(\mu_{\bar{p}_i}^{-1}(\lambda^*)) \cdot (\mathbf{c}_i^2 \mathbf{x}^* + \alpha_i^2) \\ & = \bar{f}_i^*. \end{aligned}$$

Therefore, it holds that

$$\begin{aligned} \mu_{f_i}^{-1}(\lambda^*) - (\mathbf{c}_i^1 \mathbf{x} + \alpha_i^1) & \geq T_i^{-1}(\mu_{\bar{p}_i}^{-1}(\lambda^*)) \cdot (\mathbf{c}_i^2 \mathbf{x} + \alpha_i^2), \\ i & = 1, \dots, k, \end{aligned}$$

which contradicts the fact that $(\mathbf{x}^*, \lambda^*)$ is a unique optimal solution of MAXMIN4.

TABLE I
CONSTANT COEFFICIENTS OF OBJECTIVE FUNCTIONS AND CONSTRAINTS

$\mathbf{x}$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
$\mathbf{c}_1^1$	19	48	21	10	18	35	46	11	24	33
$\mathbf{c}_2^1$	3	2	2	1	4	3	1	2	4	2
$\mathbf{c}_3^1$	12	-46	-23	-38	-33	-48	12	8	19	20
$\mathbf{c}_4^1$	1	2	4	2	2	1	2	1	2	1
$\mathbf{c}_5^1$	-18	-26	-22	-28	-15	-29	-10	-19	-17	-28
$\mathbf{c}_6^1$	2	1	3	2	1	2	3	3	2	1
$\mathbf{a}_1$	12	-2	4	-7	13	-1	-6	6	11	-8
$\mathbf{a}_2$	-2	5	3	16	6	-12	12	4	-7	-10
$\mathbf{a}_3$	3	-16	-4	-8	-8	2	-12	-12	4	-3
$\mathbf{a}_4$	-11	6	-5	9	-1	8	-4	6	-9	6
$\mathbf{a}_5$	-4	7	-6	-5	13	6	-2	-5	14	-6
$\mathbf{a}_6$	5	-3	14	-3	-9	-7	4	-4	-5	9
$\mathbf{a}_7$	-3	-4	-6	9	6	18	11	-9	-4	7

V. A NUMERICAL EXAMPLE

We consider the following three-objective linear programming problems involving random variable coefficients, which is formulated by Sakawa et al. [9], to demonstrate the feasibility of our proposed fuzzy approach.

[MOSLP]

$$\min (\mathbf{c}_1^1 + t_1(\omega)\mathbf{c}_1^2)\mathbf{x} + (\alpha_1^1 + t_1(\omega)\alpha_1^2)$$

$$\min (\mathbf{c}_2^1 + t_2(\omega)\mathbf{c}_2^2)\mathbf{x} + (\alpha_2^1 + t_2(\omega)\alpha_2^2)$$

$$\min (\mathbf{c}_3^1 + t_3(\omega)\mathbf{c}_3^2)\mathbf{x} + (\alpha_3^1 + t_3(\omega)\alpha_3^2)$$

subject to $\mathbf{a}_i \mathbf{x} \leq b_i(\omega), i = 1, \dots, 7, \mathbf{x} \geq \mathbf{0}$

where $\mathbf{x} = (x_1, x_2, \dots, x_{10})^T$ is a ten-dimensional decision column vector, $\mathbf{a}_i, i = 1, \dots, 7, \mathbf{c}_1^1, \mathbf{c}_1^2, \mathbf{c}_2^1, \mathbf{c}_2^2, \mathbf{c}_3^1, \mathbf{c}_3^2$ are ten-dimensional coefficient row vectors (Table 1), and $\alpha_1^1 = -18, \alpha_1^2 = 5, \alpha_2^1 = -27, \alpha_2^2 = 6, \alpha_3^1 = -10, \alpha_3^2 = 4$. $t_i(\omega), i = 1, \dots, 3, b_j(\omega), j = 1, \dots, 7$ are Gaussian random variables defined as follows.

$$\begin{aligned} t_1(\omega) & \sim N(4, 2^2), & t_2(\omega) & \sim N(3, 3^2), \\ t_3(\omega) & \sim N(3, 2^2), & b_1(\omega) & \sim N(164, 30^2), \\ b_2(\omega) & \sim N(-190, 20^2), & b_3(\omega) & \sim N(-184, 15^2), \\ b_4(\omega) & \sim N(99, 22^2), & b_5(\omega) & \sim N(-150, 17^2), \\ b_6(\omega) & \sim N(154, 35^2), & b_7(\omega) & \sim N(142, 42^2). \end{aligned}$$

Let us assume that the hypothetical decision maker sets constraint probability levels as $\beta = (\beta_1, \dots, \beta_7) = (0.85, 0.95, 0.8, 0.9, 0.85, 0.8, 0.9)$ for the constraints in MOSLP, which are same values to the numerical example [9]. Since, in the numerical example [9], a probability maximization model is adopted to formulate MOSLP, we adopt MAXMIN2 for MOSLP in order to compare our proposed fuzzy approach with the results of Sakawa et al.[9].

Let us set the membership functions $\mu_{p_i}(p_i(\mathbf{x}, \bar{f}_i))$ for distribution functions $p_i(\mathbf{x}, \bar{f}_i), i = 1, 2, 3$ as follows, which

are same to [9].

$$\begin{aligned}\mu_{p_1}(p_1(\mathbf{x}, \bar{f}_1)) &= \frac{p_1(\mathbf{x}, \bar{f}_1) - 0.502}{(0.88 - 0.502)} \\ \mu_{p_2}(p_2(\mathbf{x}, \bar{f}_2)) &= \frac{p_2(\mathbf{x}, \bar{f}_2) - 0.06}{(0.783 - 0.06)} \\ \mu_{p_3}(p_3(\mathbf{x}, \bar{f}_3)) &= \frac{p_3(\mathbf{x}, \bar{f}_3) - 0.446}{(0.664 - 0.446)}\end{aligned}$$

In the numerical example [9], as permissible objective levels for a probability maximization model, $\bar{\mathbf{f}} = (\bar{f}_1, \bar{f}_2, \bar{f}_3) = (2150, 450, -950)$ are adopted. Corresponding to these values, we assume that the hypothetical decision maker sets his/her membership functions $\mu_{\bar{f}_i}(\bar{f}_i)$ for permissible objective levels $\bar{f}_i$, $i = 1, 2, 3$ as follows.

$$\begin{aligned}\mu_{\bar{f}_1}(\bar{f}_1) &= \frac{3000 - \bar{f}_1}{(3000 - 2150)} \\ \mu_{\bar{f}_2}(\bar{f}_2) &= \frac{1000 - \bar{f}_2}{(1000 - 450)} \\ \mu_{\bar{f}_3}(\bar{f}_3) &= \frac{-900 - \bar{f}_3}{((-900) - (-950))}\end{aligned}$$

For these membership functions, MAXMIN2 is solved by combined use of the bisection method with respect to λ and the first-phase of the two-phase simplex method of linear programming. The optimal solution $(\mathbf{x}^*, \lambda^*)$ is obtained at $\lambda^* = 0.826714$. The corresponding membership function values are obtained as $\mu_{\bar{f}_i}(\bar{f}_i^*) = \mu_{p_i}(p_i(\mathbf{x}^*, \bar{f}_i^*)) = 0.826714$, $i = 1, 2, 3$, where

$$\begin{aligned}\bar{f}_i^* &= (c_i^1 \mathbf{x}^* + \alpha_i^1) + T_i^{-1}(\mu_{p_i}^{-1}(\lambda^*)) \cdot (c_i^2 \mathbf{x}^* + \alpha_i^2), \\ p_i(\mathbf{x}^*, \bar{f}_i^*) &= T_i \left(\frac{\bar{f}_i^* - (c_i^1 \mathbf{x}^* + \alpha_i^1)}{c_i^2 \mathbf{x}^* + \alpha_i^2} \right).\end{aligned}$$

In Table 2, not only the optimal values $\bar{f}_i^*, p_i(\mathbf{x}^*, \bar{f}_i^*)$, $i = 1, 2, 3$ of MAXMIN2 but also the results of Sakawa et al. (especially, $p_i(\mathbf{x}^*, \bar{f}_i)$ for the fixed values of permissible objective levels $(\bar{f}_1, \bar{f}_2, \bar{f}_3) = (2150, 450, -950)$ at the first interaction of Table 3 [9]) are shown. From Table 2, it can be understood that, at the optimal solution of MAXMIN2, the distribution function $p_i(\mathbf{x}^*, \bar{f}_i^*)$ is improved at the expense of the permissible objective level in comparison with the results of Sakawa et al. [9]. From this result, it is expected that the proper balance between the distribution function $p_i(\mathbf{x}, \bar{f}_i)$ and the permissible objective level $\bar{f}_i$ in a probability maximization model will be attained at the optimal solution of MAXMIN2. Similarly, it is also expected that the proper balance between the objective function $f_i(\mathbf{x}, \bar{p}_i)$ and the permissible probability level $\bar{p}_i$ in a fractile optimization model will be attained at the optimal solution of MAXMIN4.

VI. CONCLUSION

In this paper, two fuzzy approaches to multiobjective stochastic linear programming problems through a probability maximization model and a fractile optimization model are proposed. MAXMIN2 is formulated for multiobjective stochastic linear programming problems through a probability

TABLE II
THE OPTIMAL SOLUTIONS OF MAXMIN2 AND THE FIRST INTERACTION
IN TABLE 3 [9]

	MAXMIN2	The first interaction in Table 3 [9]
$\bar{f}_1^*$	2297.29	2150.0
$\bar{f}_2^*$	545.308	450.0
$\bar{f}_3^*$	-941.336	-950.0
$p_1(\mathbf{x}^*, \bar{f}_1^*)$	0.814498	0.719
$p_2(\mathbf{x}^*, \bar{f}_2^*)$	0.657714	0.474
$p_3(\mathbf{x}^*, \bar{f}_3^*)$	0.626224	0.571

maximization model to obtain a satisfactory solution of the decision maker. Similarly, MAXMIN4 is formulated for multiobjective stochastic linear programming problems through a fractile optimization model to obtain a satisfactory solution of the decision maker. Both problems can be easily solved by linear programming technique and, at the optimal solution, it is expected that a proper balance between permissible objective levels and the corresponding distribution function values or permissible probability levels and the corresponding objective function values is attained. It has been also shown that such two problems are same. Although a probability maximization model or a fractile optimization model is one of the most efficient tool to transform stochastic programming problems into well-defined mathematical programming ones, appropriate permissible levels are not known for the decision maker in advance. In order to overcome such a difficulty, we have proposed fuzzy approaches for multiobjective stochastic linear programming problems.

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