

Relaxed Fuzzy Lyapunov Approach for Dynamic Local Model Networks

Christian H. Mayr^{*}, Christoph Hametner[†], Martin Kozek[‡] and Stefan Jakubek[§]

Christian Doppler Laboratory for Model Based Calibration Methodologies

at the Institute of Mechanics and Mechatronics

Vienna University of Technology

Wiedner Hauptstrasse 8-10/325/A5, 1040 Vienna, Austria

^{*}christian.mayr@tuwien.ac.at

[†]christoph.hametner@tuwien.ac.at

[‡]martin.kozek@tuwien.ac.at

[§]stefan.jakubek@tuwien.ac.at

Abstract—This paper deals with the problem of stability analysis of dynamic local model networks. Established methods in this context are mainly based on Lyapunov stability theory and are targeted to be as little conservative as possible. In this respect it is essential to take into account the transitions between the different local models. For that purpose this paper presents and discusses a method to determine possible model transitions of such dynamic local model networks utilizing identification or simulation data. The effectiveness of the proposed method is shown by a simulation example in connection with the fuzzy Lyapunov approach as a stability criterion. The example demonstrates how identification data can be used to reduce the conservatism compared to standard approaches.

Keywords—local model networks, fuzzy Lyapunov function, stability, system identification

I. INTRODUCTION

Local model networks (LMN) are a versatile architecture for the representation of a large class of nonlinear systems. The approximation is based on a partitioning of the nonlinearities into subdomains which can be accurately represented by simpler submodels, for example linear models [1]. The architecture of LMN offers good interpretability which is an important advantage for the design of controllers or observers. LMN are usually generated from measured input and output data of the process to be approximated.

When the structure of LMN is applied to nonlinear dynamic systems a main challenge is to provide stability criteria. A major difficulty lies in the fact that a dynamic LMN can become unstable in the large even when all local linear models are stable, [2].

Stability proofs of local model networks are mainly based on Lyapunov's direct method which results in linear matrix inequalities (LMIs) as stability criterion. Lyapunov stability criteria provide *sufficient* rather than a necessary conditions for stability. In this context the issue of *conservatism* is inherently linked to these methods: Depending on the type and structure of the chosen Lyapunov function a stability proof may fail even

though the investigated system is stable.

The basic criterion is the so-called common quadratic Lyapunov approach, [2]. The main drawback of this approach is its conservatism since it aims to find one common Lyapunov function for all local linear models in an LMN. The reason lies in the fact that the LMI solver has to find a single positive definite matrix and must simultaneously fulfill as many LMIs as the number of local models. Thus, by increasing the number of local models, and increasing the accuracy of the LMN, it will be more difficult to find a common Lyapunov function.

In recent years advanced methods, based on the less conservative fuzzy Lyapunov approach have been developed, [3], [4], [5], [6], [7]. Basically this approach aims to find a Lyapunov function for each local model which are connected by so-called model transitions. A model transition occurs when the system state transits from one local model to another. The conservatism of the stability criterion is directly increased with the number of model transitions. Thus, in order to prove stability using the fuzzy Lyapunov approach without any additional structural information of the LMN it is mandatory to take into account how state trajectories can progress through different local models. This helps to reduce the conservatism because impossible transitions need not be considered. Due to the nature of local model networks only a limited number of model transitions actually take place or are theoretically feasible. Note, that the number of theoretically possible model transitions is increased with the number of local models.

For simple and low dimensional partitioning strategies (e.g. two dimensional trapezoid validity functions) it is possible to determine the model transition using the phase portrait of the considered local model network or the method presented in [4]. For high dimensional partition spaces and complex partitioning strategies such as [8], [9], [10], [11] and [12] the method of [4] is no longer feasible.

In this paper a systematic approach is introduced to determine all occurring model transitions in a dynamic LMN. The basic idea is to capture the model transitions by means of simulation data. Therefore an optimally designed input data sequence

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regarding design of experiment (DoE) is required which must be *extensive* and *tight*, see [13]. For local model networks generated from a real physical process the identification data sequence can be used because it must fulfill the same requirements to capture the dynamics of the considered system within the operating area. For artificial local model networks the input data sequence has to be generated. Thus, all occurring model transition can be determined using simulation data generated by an optimally designed input data sequence. It must be noted, however, that the characteristics of the simulation data significantly contribute to the effectiveness of the proposed method. This is not a severe disadvantage, because the range of the excitation signal is either given (identification data) or user defined and can be chosen to the area of interest (e.g. operating area) for the stability criterion.

With the proposed method the conservatism of the fuzzy Lyapunov approach is reduced significantly making it a feasible method to prove stability of LMN. The main advantages of the transition determination of dynamic local model networks by utilizing the identification data sequence or an optimally designed excitation data sequence are:

- The conservatism does not grow automatically with the number of local models.
- It is independent regarding the dimensions of the partition space and the type of the validity function.
- It is applicable to arbitrary partitioning strategies, such as: equidistant grid [4]; recursive, axis orthogonal [8], [9], [14]; axis oblique [10], [15]; arbitrary functions [12]; hinging hyperplanes [11].

This paper is organized as follows: The architecture of local model networks is briefly described in section II. Section III generally addresses stability, the common quadratic Lyapunov approach, and the fuzzy Lyapunov approach for discrete-time LMN. Section IV describes the proposed procedure to determine all occurring model transitions utilize a DoE optimal excitation signal. In section V on the one hand the transition determination is described and on the other hand the improvement of finding a fuzzy Lyapunov function to prove stability using the proposed concept is demonstrated by means of illustrative examples. The paper is concluded by some remarks in section VI.

II. LOCAL MODEL NETWORKS

The architecture of dynamic local model networks is illustrated in Fig. 1.

First, an ordered set for the indices of the local models is defined as:

$$\mathcal{I} = (i \in \mathbb{N} | 1 \leq i \leq I) \quad (1)$$

where I denotes the number of local models. Local model networks with external dynamics have an input vector $\mathbf{z}(k)$ with past inputs and outputs according to Fig. 1:

$$\mathbf{z}(k) = [\mathbf{u}_{\mathcal{J}}(k - \mathcal{M}) \ \hat{\mathbf{y}}(k - \mathcal{N})], \ \mathbf{z}(k) \in \mathbb{R}^O \quad (2)$$

where the ordered sets for the number of the input channels $\mathcal{J}$, the orders of the inputs $\mathcal{M}$, the orders of the feedback output

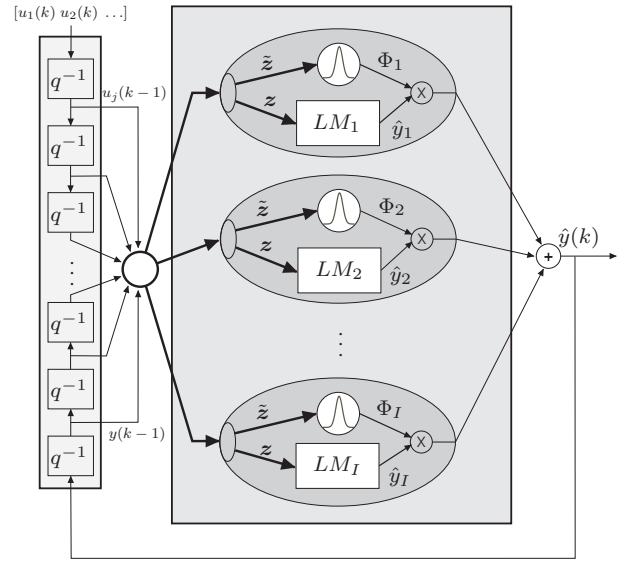


Fig. 1. Architecture of a local model network with external dynamics [15]

(=system order) $\mathcal{N}$ and the index $\mathcal{O}$ are as follows:

$$\mathcal{J}(j) = (j \in \mathbb{N} | 1 \leq j \leq J) \quad (3)$$

$$\mathcal{M}(m) = (m \in \mathbb{N} | 1 \leq m \leq M) \quad (4)$$

$$\mathcal{N}(n) = (n \in \mathbb{N} | 1 \leq n \leq N) \quad (5)$$

$$\mathcal{O}(o) = (o \in \mathbb{N} | 1 \leq o \leq O). \quad (6)$$

where J denotes the number of inputs, M denotes the order of the inputs, N denotes the system order and O denotes the dimension of the input vector. From Fig. 1 it becomes obvious that the input vector for the validity functions $\Phi_{\mathcal{I}}(\tilde{\mathbf{z}}(k))$ can be chosen differently to the input vector for the local model:

$$\tilde{\mathbf{z}}(k) = [\mathbf{u}_{\tilde{\mathcal{J}}}(k - \tilde{\mathcal{M}}) \ \hat{\mathbf{y}}(k - \tilde{\mathcal{N}})] \subseteq \mathbf{z}(k), \ \tilde{\mathbf{z}}(k) \in \mathbb{R}^{\tilde{\mathcal{O}}} \quad (7)$$

where the ordered sets $\tilde{\mathcal{J}}, \tilde{\mathcal{M}}, \tilde{\mathcal{N}}, \tilde{\mathcal{O}}$ are subsets of their corresponding sets as follows:

$$\tilde{\mathcal{J}}(\tilde{j}) \subseteq \mathcal{J}(j) \quad (8)$$

$$\tilde{\mathcal{M}}(\tilde{m}) \subseteq \mathcal{M}(m) \quad (9)$$

$$\tilde{\mathcal{N}}(\tilde{n}) \subseteq \mathcal{N}(n) \quad (10)$$

$$\tilde{\mathcal{O}}(\tilde{o}) \subseteq \mathcal{O}(o) \quad (11)$$

All local model outputs

$$\hat{y}_i(k) = \mathbf{z}(k)\boldsymbol{\theta}_i, \quad (12)$$

with the local parameter vector $\boldsymbol{\theta}_i$ are used subsequently to form the global model output $\hat{y}(k)$ by weighted aggregation, see Fig. 1:

$$\hat{y}(k) = \sum_{\mathcal{I}} \Phi_i(k) \hat{y}_i(k) \quad (13)$$

where the validity functions are constrained:

$$\sum_{\mathcal{I}} \Phi_i = 1 \quad (14)$$

$$0 \leq \Phi_i \leq 1, \ \forall i \in \mathcal{I}. \quad (15)$$

For stability analysis using Lyapunov approaches the local dynamics must be defined in state space notation. Thus, the discrete-time state space notation of local model networks can be generally defined using (1):

$$\begin{aligned} \mathbf{x}(k+1) &= \sum_{\mathcal{I}} \Phi_i(\tilde{\mathbf{z}}(k))(\mathbf{A}_i \mathbf{x}(k) + \mathbf{B}_i \mathbf{u}(k)) \\ \hat{\mathbf{y}}(k) &= \mathbf{c}^T \mathbf{x}(k). \end{aligned} \quad (16)$$

III. STABILITY CONCEPTS FOR LOCAL MODEL NETWORKS

In this section stability for discrete-time systems in general, the common quadratic Lyapunov approach and the fuzzy Lyapunov approach are described.

A. General

An autonomous dynamic system in discrete-time is represented by a set of difference equations in the form

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k)), \quad \mathbf{f} \in \mathbb{R}^{n \times 1} \quad (17)$$

where $\mathbf{f}$ is a nonlinear vector function, $\mathbf{x}$ is the $n \times 1$ state vector, and k denotes the sampling instant.

Nonlinear systems can have much more complex behavior than linear systems. A number of refined stability concepts, such as marginal stability and asymptotic stability, are needed, [16]:

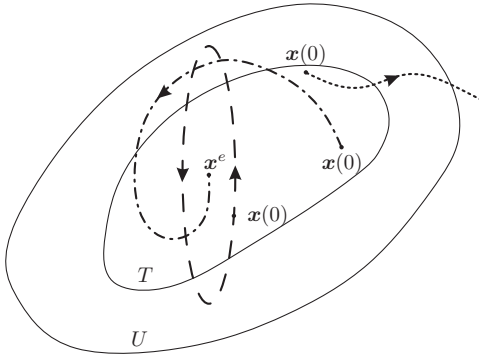


Fig. 2. Stability definitions

Definition 3.1: An equilibrium state $\mathbf{x}^e$ is *marginally stable* (Fig. 2, dashed line) if for every neighborhood $U > 0$ of $\mathbf{x}^e$ there is a neighborhood $T > 0$, $T \subseteq U$ of $\mathbf{x}^e$ such that every solution $\mathbf{x}(k)$ starting within $T(\mathbf{x}(0) \in T)$ remains within U for all $k > 0$. Otherwise the equilibrium point $\mathbf{x}^e$ is *unstable* (Fig. 2, dotted line). Note that $\mathbf{x}(k)$ need not approach $\mathbf{x}^e$.

Definition 3.2: An equilibrium state $\mathbf{x}^e$ is *asymptotically stable* (Fig. 2, dash-dotted line) if it is marginal stable and additionally T can be chosen so that $\|\mathbf{x}(k) - \mathbf{x}^e\| \rightarrow 0$ as $k \rightarrow \infty$ for all $\mathbf{x}(0) \in T$.

B. Lyapunov Stability for Discrete-Time Local Model Networks

The stability of LMN can be proved by Lyapunov's direct method. This general approach is based on a state space formulation of the system, such as (16), and one has to find a suitable Lyapunov function, [9]. Since this is generally difficult

it is common to restrict the search to the class of quadratic Lyapunov functions

$$V(k) = \mathbf{x}^T(k) \mathbf{P} \mathbf{x}(k) > 0, \quad \mathbf{P} \succ 0 \quad (18)$$

where $\mathbf{P}$ is a positive definite matrix and $\mathbf{x}(k)$ is the state of the model according to (16). For asymptotic stability, the Lyapunov function must be strictly monotonically decreasing over time k :

$$V(k+1) - V(k) < 0, \quad \forall k. \quad (19)$$

C. Common Quadratic Lyapunov Approach for Discrete-Time Systems

The common quadratic Lyapunov approach was the first stability criterion to be proposed for local model networks, [2]. It aims to find one common quadratic Lyapunov function for all local models. Using (16), (18) and (19) the common quadratic Lyapunov approach can be formulated as:

Theorem 1: A dynamic local model network is globally asymptotically stable if there exists a *common positive definite matrix* $\mathbf{P}$ for all local models:

$$\mathbf{P} \succ 0 \quad (20)$$

such that the following condition is satisfied:

$$\mathbf{A}_i^T \mathbf{P} \mathbf{A}_i - \mathbf{P} \prec 0, \quad \forall i \in \mathcal{I} \quad (21)$$

with $\mathbf{A}_i$ taken from (16), [2].

The proof of Theorem 1 is given in [2].

The stability condition (21) constitutes a set of *linear matrix inequalities* (LMIs) which is usually solved by numerical optimization.

In (21) a *common* matrix $\mathbf{P}$ must be found for *all* local system matrices $\mathbf{A}_i$. This is a much stronger condition than the individual stability of the local linear models, for which only $\mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i - \mathbf{P}_i \prec 0$ with individual $\mathbf{P}_i$ would be required, [9]. Note that it may not be possible to find a $\mathbf{P}$ that fulfills (21), although the global model may be stable. The condition (21) is therefore sufficient but not necessary.

D. Fuzzy Lyapunov Approach for Discrete-Time Systems

As the common quadratic Lyapunov approach tends to be conservative, it is desirable to have less conservative stability methods available. The fuzzy Lyapunov approach [3], [4], [5], [6], [7], [17] is one of the options for this purpose.

This approach aims to find a non-quadratic Lyapunov function for the global model

$$V(k) = \mathbf{x}^T(k) \left(\sum_{\mathcal{I}} \Phi_i(\tilde{\mathbf{z}}(k)) \mathbf{P}_i \right) \mathbf{x}(k) > 0, \quad (22)$$

where each local model independently fulfills (18) (there is an individual $\mathbf{P}_i$ matrix associated to each local model).

Remark 1: The validity functions of the $\mathbf{P}_i$ matrices in (22) are equivalent to the validity functions of the local models in (16).

To fulfill (19) all local models must be stable as a prerequisite. Second, every possible trajectory must also satisfy (19). This

shows, that the occurring model transitions are of high interest when using the fuzzy Lyapunov approach:

Theorem 2: The dynamic local model network is asymptotically stable in the region U if there exists a *set of positive definite matrices* P_i :

$$P_i \succ 0, \quad \forall i \in \mathcal{I} \quad (23)$$

such that the following condition is satisfied:

$$A_i^T P_j A_i - P_i \prec 0, \quad \forall i \in \mathcal{I}, j \in \Omega_i \quad (24)$$

where the set Ω_i defines all possible model transitions of model i within region U , [4]. Defining the dominant region of a local model, according to [18], as

$$S_i = \{\tilde{x} | \Phi_i(\tilde{x}) > \Phi_l(\tilde{x}), l \in \mathcal{I}, l \neq i\}, i \in \mathcal{I} \quad (25)$$

the possible transitions can be defined, as

$$\Omega_i = \{j | (\tilde{x}(k) \in S_i \wedge \tilde{x}(k+1) \in S_j) \vee (\tilde{x}(k) \in S_j \wedge \tilde{x}(k+1) \in S_i), j \in \mathcal{I}\}, \forall i \in \mathcal{I}. \quad (26)$$

This includes the case $i = j$ which requires that every local model must have stable dynamics as a necessary condition. The proof of Theorem 2 is given in [4].

IV. TRANSITION DETERMINATION

To reduce the conservatism of the fuzzy Lyapunov approach it is essential to determine all occurring model transition Ω_i . If no information about the model transitions is available, then transitions to all other models must be taken into account:

$$\Omega_i = \mathcal{I}, \quad \forall i \in \mathcal{I}. \quad (27)$$

Due to (26) each model transition results in two additional LMIs. Thus, the difficulty of finding a suitable Lyapunov function for the considered system is increased when additional model transitions are taken into account. Therefore, a proper determination of the occurring model transitions will help to reduce conservatism on the one hand, and often even makes a stability check on complex LMNs possible at all. Thus, in this section a method for the determination of all occurring model transitions is presented. One important point is the independency regarding the partitioning strategy of the considered local model network. Fig. 3 illustrates four common partitioning strategies of local model networks.

Obviously the determination of the model transitions by using the information of the partitioning can become difficult or impossible particularly for high order partition spaces and complex partitioning strategies as [10], [12] and [14]. Further, different types of validity functions are commonly used, see Fig. 4.

Thus, a method for the determination of the occurring model transitions, which is independent regarding the partitioning strategy and the type of validity function, is essential for the application of the fuzzy Lyapunov approach to local model networks.

The methodology proposed in this paper determines the possible transitions in a local model network by utilizing a simulation

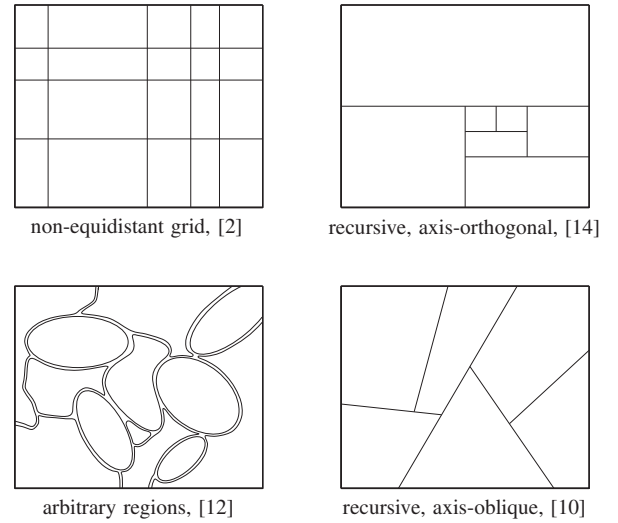


Fig. 3. Common partitioning strategies of local model networks

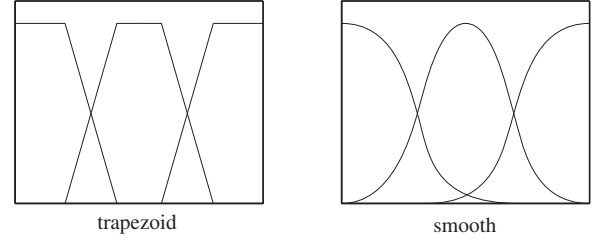


Fig. 4. Common validity functions of local model networks

result with an optimally designed excitation data sequence and is independent regarding the partitioning strategy and the used validity function. As a consequence the stability region U (Fig. 2) in the present paper is aimed to be identical to the convex hull of the used data sequence. In contrast, most available stability criteria for LMN aim to show *unbounded* global stability which can become too expensive when only a certain part of the LMN describes the actual dynamic operating range of the process. On the other hand, when using the approach of this paper all detected model transitions take place within the convex hull U of the identification data sequence. It can not be ruled out that outside of this hull unnoticed transitions could occur. Especially for the design of controllers or observers stability close to or beyond that boundary is of only limited interest.

Note that stable local dynamics are a prerequisite of the fuzzy Lyapunov approach. Thus, for unstable local dynamics, which could lead to an improper data sequence for the determination of the model transitions, the fuzzy Lyapunov approach will fail anyway, independent to the investigated model transitions.

The information about the model transitions for the local model network under consideration will be represented by a so-called *transition matrix* Ξ . It is based on the simulation result with an excitation data sequence and the method to generate it will be described in the following.

The excitation signals $u_{\mathcal{J}}(\mathcal{K})$, which are generated by a proper DoE [13] and the model output $\hat{y}(\mathcal{K})$, which is generated by simulating the process using the considered LMN with the

excitation signal, are merged to a matrix

$$\mathbf{L}(k) = [\mathbf{u}_{\mathcal{J}}(k - \mathcal{M}) \ \hat{\mathbf{y}}(k - \mathcal{N})], \forall k \in \mathcal{K}, \mathbf{L} \in \mathbb{R}^{K \times O} \quad (28)$$

which rows correspond to the input vector $\mathbf{z}(k)$. The ordered set for the samples of the data sequence is given by

$$\mathcal{K} = \{k \in \mathbb{N} | 1 \leq k \leq K\} \quad (29)$$

where K is the number of samples. Those signals which are used as input arguments to the validity function form a subset of the data sequence. The choice of such a particular subset is strongly dependent on the system's nonlinearities and is thus usually done by the user. The input arguments to the validity function are denoted by $\tilde{\mathbf{L}}$ indicating that it is a sub-matrix of $\mathbf{L}$:

$$\tilde{\mathbf{L}} = \mathbf{L}(\mathcal{K}, \tilde{\mathcal{O}}), \tilde{\mathbf{L}} \in \mathbb{R}^{K \times \tilde{O}} \quad (30)$$

which rows correspond to $\tilde{\mathbf{z}}(k)$. Consequently the output of a validity function $\mathbf{C}$ can be formulated as

$$\mathbf{C}(\mathcal{K}, \mathcal{I}) = \Phi_{\mathcal{T}}(\tilde{\mathbf{L}}(\mathcal{K})), \mathbf{C} \in \mathbb{R}^{K \times I}. \quad (31)$$

When the state trajectory evolves over time and transits through different local models a matrix of indices of the two most dominant local models $\mathbf{D}$ (= local models with maximum Φ) can be generated in a similar way as $\mathbf{L}$ by using $\mathbf{C}$:

$$\begin{aligned} \mathbf{D}(k) = & \{ \{i_1 | \mathbf{C}(k, i_1) = \max(\mathbf{C}(k, \mathcal{I}))\} \\ & \{i_2 | \mathbf{C}(k, i_2) = \max(\mathbf{C}(k, \mathcal{I} \setminus i_1))\} \}, \\ & \forall k \in \mathcal{K}, \mathbf{D} \in \mathbb{N}^{K \times 2}. \end{aligned} \quad (32)$$

The matrices $\mathbf{C}$ and $\mathbf{D}$ inherently contains the relevant information required to determine the actual model transitions. They are used to generate the *transition matrix* Ξ which compactly describes possible local model transits for the local model network under consideration. Ξ is a square matrix with boolean entries and has the following properties:

$$\dim \Xi = I \times I \quad (33)$$

$$\Xi(\xi_r, \xi_c) \in \{0, 1\}, \forall \xi_r, \xi_c \in \mathcal{I} \quad (34)$$

The elements of Ξ can be interpreted as follows:

$$\Xi(\xi_r, \xi_c) = 1 \text{ if there is a transition from model } \xi_r \text{ to } \xi_c \\ \text{and zero otherwise}$$

Note that Ξ is generally *symmetric*, because the fuzzy Lyapunov approach requires that (24) is fulfilled for both interacting models.

To generate the transition matrix Ξ two steps are necessary:

- ① *Initialize Ξ as identity matrix*

$$\Xi = \mathbf{I}_I \quad (35)$$

The main diagonal entries which formally refer to an interaction of a local model with itself must be set to one since the fuzzy Lyapunov approach requires stable local models.

- ② *Set the occurring model transitions*

From (32) the two most dominant local models are

determined for each sample. A valid transition is detected whenever the dominant models change or when the validity functions do not change:

$$\begin{aligned} \mathbf{D}(k, 1) &= \mathbf{D}(k-1, 2) \wedge \mathbf{D}(k, 2) = \mathbf{D}(k-1, 1) \vee \\ \mathbf{C}(k, \mathcal{I}) &= \mathbf{C}(k-1, \mathcal{I}) \wedge \mathbf{D}(k, \mathcal{I}) = \mathbf{D}(k-1, \mathcal{I}), \\ &\forall k \in \mathcal{K} \end{aligned} \quad (36)$$

Condition (36) is examined for all rows of $\mathbf{C}$ and $\mathbf{D}$, respectively. The detected transition directly follows when (36) is true for a k and the respective entry is set in the transition matrix:

$$\begin{aligned} \Xi(\mathbf{D}(k-1, 1), \mathbf{D}(k, 1)) &= \\ &= \Xi(\mathbf{D}(k, 1), \mathbf{D}(k-1, 1)) = 1 \end{aligned} \quad (37)$$

Using Ξ for the fuzzy Lyapunov approach (24) the set Ω_i corresponds to the i -th row of Ξ :

$$\Omega_i = \{j | \Xi(i, j) = 1, \forall j \in \mathcal{I}\}, \forall i \in \mathcal{I} \quad (38)$$

V. EXAMPLES

In section V-A a simple example exemplarily shows a single transition from one model to another. The effectiveness of the fuzzy Lyapunov approach in conjunction with the proposed method is demonstrated by means of an illustrative nonlinear process in section V-B.

A. Single Transition

Fig. 5 illustrates an exemplary trajectory of an identification data sequence with constant input $u(k)$ and the 50% contour lines of the validity functions of the local models. The timestep from $k = 2$ to $k = 3$ is considered in detail.

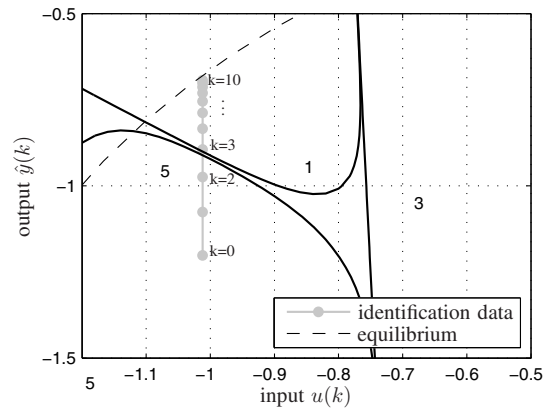


Fig. 5. Trajectory within partition space

The analysis of the trajectory by the method proposed in this paper is illustrated in Fig. 6.

The upper plot of Fig. 6 shows the system output over time, the middle one shows how the validity function outputs of local models 1 and 5, and the bottom one illustrates how the two most dominant models were determined by (37).

At $k = 2$ local model 5 is most dominant ($\mathbf{D}(2, 1) = 5$) and

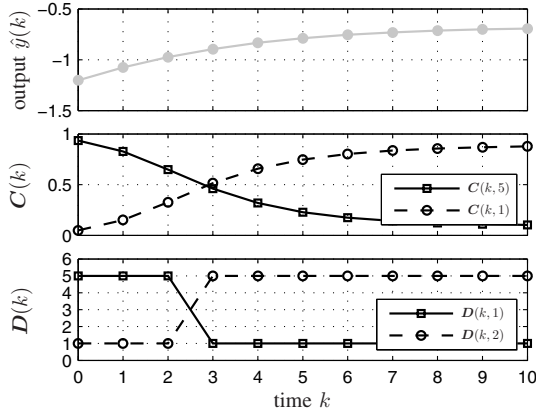


Fig. 6. Validity sequence

at $k = 3$ ($D(3,1) = 1$) which indicates that model 1 is now dominating:

$$D(2,1) = 5, D(2,2) = 1, D(3,1) = 1, D(3,2) = 5 \quad (39)$$

Thus, according to (36) a valid transition is detected and the corresponding entry is set in the transition matrix according to (37):

$$\Xi(5,1) = \Xi(1,5) = 1$$

B. Nonlinear Dynamic Process

For illustrative purposes a nonlinear dynamic process is considered, which possesses strongly operating point dependent dynamics. This process is adopted from [9], [19] and can be described by the following difference equation:

$$y(k) = 0.133u(k-1) - 0.0677u(k-2) + 1.5y(k-1) - 0.7y(k-2) + u(k)[0.1y(k-1) - 0.2y(k-2)] \quad (40)$$

In this example the input $u(k)$ is bounded to the interval $[-1.5 \ 0.5]$.

Fig. 7 demonstrates that the system has weakly damped oscillatory behavior for large inputs whereas for small inputs strongly damped aperiodic behavior is present. For system

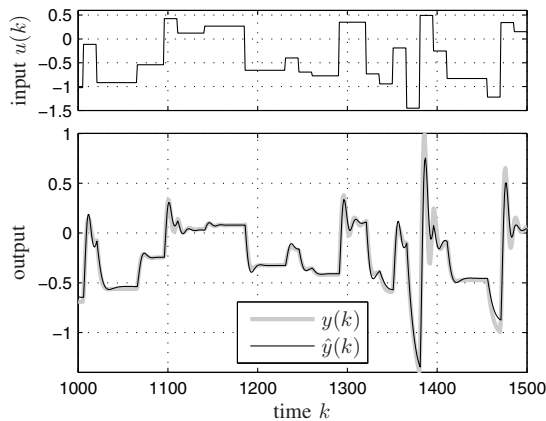


Fig. 7. In- and output of the system and the LMN

identification an excitation signal was generated to capture both the nonlinear dynamic and static behavior, [13].

From these data a local model network comprising six local models was generated by the algorithm presented in [10], where the local models are defined by an axis oblique decomposition of the partition space. Fig. 8 shows the partition space

$$\tilde{x}(k) = [u(k-1) \ \hat{y}(k-1)] \quad (41)$$

as well as the identification data. The partitioning strategy of [10] uses statistical methods to avoid overfitting by local models generated with only few points.

The comparison of the system output and the output of the LMN in Fig. 7 shows that the LMN is accurately approximating the system within the whole operating range. The following system

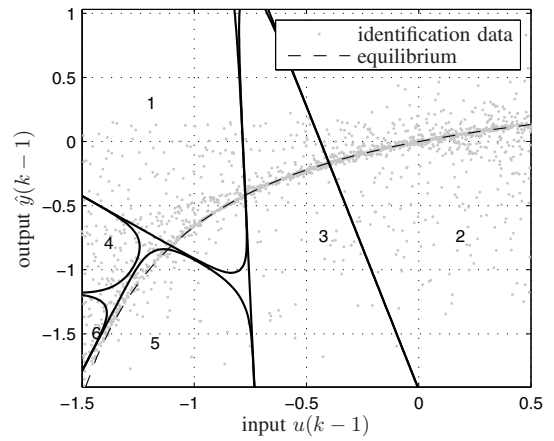


Fig. 8. Partition space and identification data sequence

matrices are obtained from the LMN when the discrete-time transfer functions of the local linear dynamics are represented in controllability canonical form:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ -0.71 & 1.668 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -0.736 & 1.525 \end{bmatrix} \\ A_3 &= \begin{bmatrix} 0 & 1 \\ -0.622 & 1.488 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 1 \\ -0.844 & 1.832 \end{bmatrix} \\ A_5 &= \begin{bmatrix} 0 & 1 \\ -0.767 & 1.74 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 1 \\ -0.733 & 1.702 \end{bmatrix} \end{aligned} \quad (42)$$

The local models have very different damping ratios, Fig. 9 shows their eigenvalues for illustration.

Fuzzy Lyapunov Approach with Transition Determination

The following transition matrix is generated using the proposed method of section IV):

$$\Xi = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad (43)$$

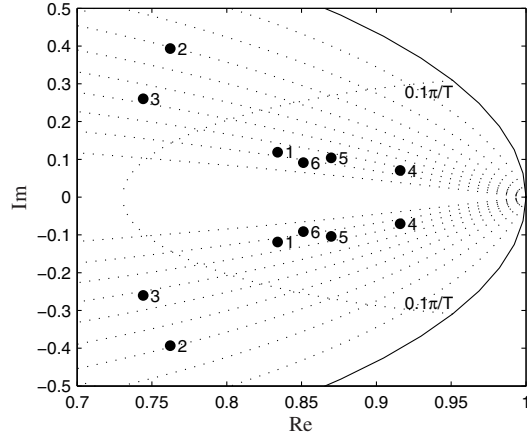


Fig. 9. Eigenvalues of the local models

Defining the model transitions Ω_i , according to (38) with (43):

$$\begin{aligned}\Omega_1 &= \{1, 3, 4, 5\}, \Omega_2 = \{2, 3\}, \\ \Omega_3 &= \{1, 2, 3, 5\}, \Omega_4 = \{1, 4, 5, 6\}, \\ \Omega_5 &= \{1, 3, 4, 5, 6\}, \Omega_6 = \{4, 5, 6\}\end{aligned}\quad (44)$$

for the fuzzy Lyapunov approach, according to Theorem 2 and using the MATLAB LMI Toolbox[®] the following positive definite P_i matrices are determined:

$$\begin{aligned}P_1 &= \begin{bmatrix} 0.14 & -0.152 \\ -0.152 & 0.175 \end{bmatrix}, P_2 = \begin{bmatrix} 0.715 & -0.726 \\ -0.726 & 0.856 \end{bmatrix} \\ P_3 &= \begin{bmatrix} 0.082 & -0.086 \\ -0.086 & 0.1 \end{bmatrix}, P_4 = \begin{bmatrix} 0.191 & -0.209 \\ -0.209 & 0.237 \end{bmatrix} \\ P_5 &= \begin{bmatrix} 0.148 & -0.162 \\ -0.162 & 0.188 \end{bmatrix}, P_6 = \begin{bmatrix} 0.139 & -0.153 \\ -0.153 & 0.177 \end{bmatrix}\end{aligned}$$

which satisfy (23) and (24). Thus it is guaranteed, that the considered LMN is asymptotically stable within U .

Common Quadratic- and Fuzzy Lyapunov Approach without Transition Determination

In this section it is assumed that no information about the model transitions is available for the fuzzy Lyapunov approach. Consequently all entries of Ξ are set to one. This leads to a more conservative result, because of additional LMIs. Thus, the classic fuzzy Lyapunov approach fails to prove stability of the considered LMN with six local models.

Using the common quadratic Lyapunov approach

$$A_i^T P A_i - P \prec 0 \quad (45)$$

for all A_i of (42) it is not possible to prove stability of the nonlinear dynamic process neither because of the different dynamic behavior. This is independent of the number of local models.

Influence of the number of local models

For the considered local model network with six local models it is only possible to prove stability using the proposed

concept. Using the common quadratic Lyapunov approach or the fuzzy Lyapunov approach without the proposed concept increasing the number of local models result in an increased complexity of the stability criterion because of additional LMIs. Thus, in this section the influence of the number of local models and the number of the considered transitions on the stability criteria will be shown. For that purpose four different local model networks, comprising tree to six local models, are compared in the following. These local model networks are generated utilizing the same identification data sequence. Fig. 10 illustrated that the complexity of the local model network is generally increased by the number of local models.

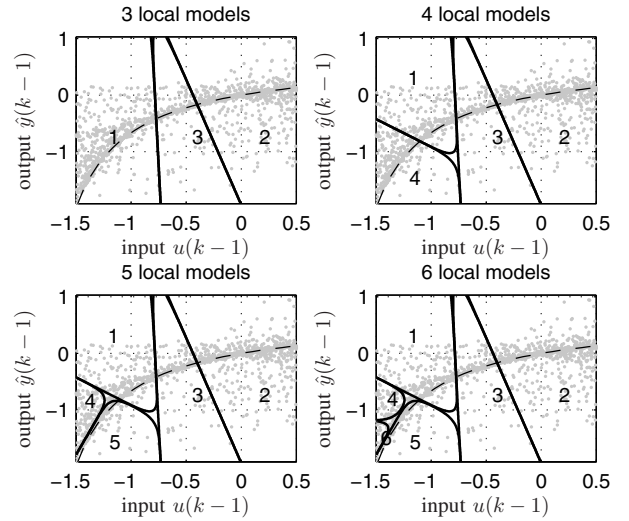


Fig. 10. Partition Space of local model networks with different number of local models

Fig. 11 shows that the complexity of the fuzzy Lyapunov approach can be significantly reduced by using the proposed method, particularly for local model networks with an high number of local models.

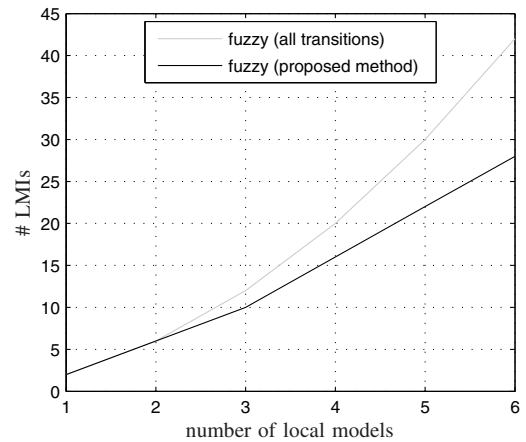


Fig. 11. Number of LMIs for different stability criteria and different number of local models

Table I shows an overview of the common quadratic Lyapunov approach, the fuzzy Lyapunov approach with all transitions and the fuzzy Lyapunov approach using the proposed method. Further, the influence of the number of local models on the stability criteria is presented utilizing the considered nonlinear dynamic process.

TABLE I
NONLINEAR DYNAMIC PROCESS: STABILITY CRITERIA

Method	# LMI variables	# LMIs	stable
6 local models			
common quadratic	1	7	
fuzzy (all transitions)	6	42	
fuzzy (proposed method)	6	28	✓
5 local models			
common quadratic	1	6	
fuzzy (all transitions)	5	30	✓
fuzzy (proposed method)	5	22	✓
4 local models			
common quadratic	1	5	
fuzzy (all transitions)	4	20	✓
fuzzy (proposed method)	4	16	✓
3 local models			
common quadratic	1	3	
fuzzy (all transitions)	3	12	
fuzzy (proposed method)	3	10	✓

The analysis of Table I shows that the conservatism of the Lyapunov approach can be generally reduced by using the proposed method because the number of LMIs is reduced in all cases. Further, it is even possible, that the conservatism decreases when the number of local models is increased. Using the common quadratic Lyapunov approach no stability proof is possible.

VI. CONCLUSION

This article addresses the problem of stability analysis of dynamic local model networks. In order to reduce the conservatism of established Lyapunov approaches a new method for the determination of the local model transitions in a dynamic local model network is proposed. The concept can be used for arbitrary partitioning strategies and high dimensional partition spaces and uses identification data to determine feasible model transitions. For artificial local model networks, which are not generated from identification data, the concept is also applicable using a proper excitation signal.

The effectiveness of the proposed method is shown by a simulation example with the fuzzy Lyapunov approach as a stability criterion. It is demonstrated how the information inherently contained in a proper simulation data sequence can be utilized to reduce conservatism and thus to extend the applicability of established stability criteria to more sophisticated dynamic local model networks.

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