

On the Properties of SIRMs Connected Type-1 and Type-2 Fuzzy Inference Systems

Chengdong Li and Guiqing Zhang

School of Information and Electrical Engineering
Shandong Jianzhu University
Jinan 250101, China
Email: lichengdong@sdjzu.edu.cn

Jianqiang Yi, *Senior Member, IEEE*

Institute of Automation
Chinese Academy of Sciences
Beijing 100190, China
Email: jianqiang.yi@ia.ac.cn

Tiechao Wang

College of Electrical Engineering
Liaoning University of Technology
Jinzhou 121001, China
Email: tiechao.wang@gmail.com

Abstract—This paper tries to show some important properties of the single input rule modules (SIRMs) connected fuzzy inference systems (FIS), including both type-1 (T1) and interval type-2 (IT2) FISs. Three kinds of properties – continuity, monotonicity and robustness – are explored. First, conditions on the parameters are derived to ensure that the SIRMs connected FISs are continuous and monotonic. Then, a methodology for the robustness analysis of the SIRMs connected FISs are presented. At last, an example is given to show the correctness of the theorems on the continuity and monotonicity and to demonstrate the effectiveness of the proposed methodology for robustness analysis. These results can not only deepen our understanding of the SIRMs connected FISs, but also provide us guidelines for the design of the SIRMs connected FISs.

Keywords — *type-2 fuzzy; continuity; monotonicity; robustness analysis*

I. INTRODUCTION

The single input rule modules (SIRMs) connected fuzzy inference system (FIS) is first proposed by J. Yi, et al. [1-3] to simplify the design process of the conventional type-1 (T1) FIS, and then studied by H. Seki, et al. [4, 5, 24]. Compared with the conventional FISs, the number of rules of the SIRMs connected FISs can be reduced greatly. And, this kind of FIS has been applied to many problems, such as the stabilization control of parallel-type double inverted pendulum systems [1], anti-swing and positioning control of overhead traveling crane [3], etc. Recently, we have extended the SIRMs connected T1 FISs to the interval type-2 (IT2) case [6]. Moreover, we have also utilized the SIRMs connected IT2 FISs to realize the stabilization control of the TORA system [7] and the inverted pendulum system [8].

However, little theoretical research has been conducted for the SIRMs connected FISs. To the best of the authors' knowledge, only two theoretical aspects of the SIRMs connected FISs have been explored. The first aspect is about the relationships between the SIRMs connected FISs and other types of FISs (e.g. Takagi-Sugeno FISs). Such issue has been explored by us [6] and Seki et al. [4, 5]. Recently, we have also studied another theoretical aspect – stability of the SIRMs connected FISs [9]. But, the theoretical research of the SIRMs connected FISs is still not profound enough. And, more properties of the SIRMs connected FISs should be derived to deepen our understanding of this kind of FIS.

Therefore, in this paper, we will study some new properties of the SIRMs connected FISs, such as continuity, monotonicity and robustness. First, we will derive parameter conditions to ensure the continuity and monotonicity of the SIRMs connected FISs. Then, we will present a methodology for the robustness analysis of the SIRMs connected FISs. At last, to show the correctness of the conditions for continuity and monotonicity and to demonstrate the effectiveness of the proposed methodology for robustness analysis, an example will be given. These results can help us to design more reasonable and robust SIRMs connected FISs.

The organization of this paper is as follows. In Section II, the SIRMs connected IT2 FIS is introduced briefly. In Section III, three properties – continuity, monotonicity and robustness – are explored. In Section IV, an example is given. At last, conclusions are drawn in Section V.

II. SIRMS CONNECTED FUZZY INFERENCE SYSTEM

As the SIRMs connected T1 FISs are special cases of the SIRMs connected IT2 FISs, in this paper, we will only consider the general case – the SIRMs connected IT2 FISs in detail. The presented results can be readily used for the SIRMs connected T1 FISs as well. In the following, we will give a brief introduction of the SIRMs connected IT2 FISs first.

A SIRMs connected IT2 FIS, which has n input variables $x_1, x_2, \dots, x_n$, is composed of n IT2 SIRMs. The IT2 SIRM for the input variable x_i (IT2SIRM- i) can be expressed as [1-3, 6-9]

$$\text{IT2SIRM-}i : \left\{ R_i^j : x_i = \tilde{A}_i^j \rightarrow y_i = \tilde{C}_i^j \right\}_{j=1}^{M_i}, \quad (1)$$

where $\tilde{A}_i^j$ s are IT2 fuzzy sets (FSs), $\tilde{C}_i^j$ s are interval weighting factors, and $\tilde{C}_i^j = [\underline{c}_i^j, \bar{c}_i^j]$. IT2FSs $\tilde{A}_i^j$ s and interval weighting factors $[\underline{c}_i^j, \bar{c}_i^j]$ s can be formed by blurring type-1 fuzzy sets A_i^j s and crisp values c_i^j s, respectively [10-16].

A IT2 SIRM can be seen as a single-input-single-output interval type-2 fuzzy logic system (IT2FLS) [10-16]. Therefore, the inference structure of the IT2 SIRM is the same as the inference structure of the IT2FLS, which consists of a fuzzifier, an inference engine, a rule base, a type-reducer and a defuzzifier [10-16].

Once a crisp input x_i is applied to IT2SIRM- i , the firing strength of the j th rule can be expressed as

$$F_i^j(x_i) = \left[\underline{f}_i^j(x_i), \bar{f}_i^j(x_i) \right], \quad (2)$$

where

$$\underline{f}_i^j(x_i) = \underline{\mu}_{\tilde{A}_i^j}(x_i), \quad \bar{f}_i^j(x_i) = \bar{\mu}_{\tilde{A}_i^j}(x_i), \quad (3)$$

in which $\underline{\mu}(\cdot), \bar{\mu}(\cdot)$ denote the grades of the lower membership functions (MFs) and upper MFs of IT2 FSs.

To generate a crisp output from IT2SIRM- i , different type-reduction and defuzzification methods can be used. Here, for the convenience of rigorous mathematical analysis, the following inference engine, which is an extension of the inference engine proposed in [16], is used:

$$y_i(x_i) = m_i \frac{\sum_{j=1}^{M_i} \underline{f}_i^j(x_i) \underline{c}_i^j}{\sum_{j=1}^{M_i} \underline{f}_i^j(x_i)} + n_i \frac{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i) \bar{c}_i^j}{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i)}, \quad (4)$$

where m_i and n_i , which satisfy that $m_i + n_i = 1$, are inference coefficient.

Once the fuzzy inference result $y_i(x_i)$ of IT2SIRM- i is already calculated, to express clearly the different role of each input variable on system performance, the output $y(\mathbf{x})$ of the SIRMs connected IT2 FIS will be computed as the summation of the products of the fuzzy inference result $y_i(x_i)$ and the importance degree w_i of the i th input variable x_i . In other words, the output $y(\mathbf{x})$ of the SIRMs connected IT2 FIS can be calculated as [6-9]

$$\begin{aligned} y(\mathbf{x}) &= \sum_{i=1}^n w_i y_i(x_i) \\ &= \sum_{i=1}^n w_i \left[m_i \frac{\sum_{j=1}^{M_i} \underline{f}_i^j(x_i) \underline{c}_i^j}{\sum_{j=1}^{M_i} \underline{f}_i^j(x_i)} + n_i \frac{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i) \bar{c}_i^j}{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i)} \right]. \end{aligned} \quad (5)$$

From (5), we can observe that the input-output mapping of the SIRMs connected FIS can be seen as a multi-variable function.

III. PROPERTIES OF SIRMs CONNECTED FISs

In this section, we will study the continuity, monotonicity and robustness of the SIRMs connected FISs. These properties always needs to be met in practical applications, e.g. the controller designed for the cart-pole system should be continuous, monotonic and robust. First, let us discuss the issue of continuity.

A. Continuity

As stated in [17], “in many cases, continuous and smooth input-output mapping is desired for a F S, because most physical systems are continuous”. Also in [17], the conditions have been derived to assure the continuity of the input-output mappings of IT2 FISs under six different type-reduction and defuzzification methods. From Section II, we can see that the type-reduction and defuzzification method adopted in the SIRMs is different from the six ones studied in [17], and the

inference process of the SIRMs connected IT2 FISs differs from the inference process of conventional IT2 FISs. Hence, there is a need for us to study the continuity of the SIRMs connected FISs.

To begin, let us introduce the definition of continuity.

Definition 1 [17, 18]: A multi-variable function $f(\mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{X}_1 \times \mathbb{X}_2 \times \dots \times \mathbb{X}_n$, is continuous at $\mathbf{x}^* = (x_1^*, \dots, x_n^*) \in \mathbb{X}_1 \times \mathbb{X}_2 \times \dots \times \mathbb{X}_n$ if and only if it is defined at $\mathbf{x}^*$, and $\lim_{\mathbf{x} \rightarrow \mathbf{x}^*} f(\mathbf{x}) = f(\mathbf{x}^*)$. And $f(\mathbf{x})$ is said to be continuous if it is continuous at each point of the universe of discourse $\mathbb{X}_1 \times \mathbb{X}_2 \times \dots \times \mathbb{X}_n$.

Definition 2 [17]: An IT2 FS A is continuous if and only if its lower MF $\underline{\mu}_{\tilde{A}}(x)$ and upper MF $\bar{\mu}_{\tilde{A}}(x)$ are continuous functions of x .

As T1 FSs are special cases of IT2 FSs, hence **Definition 2** is also valid for continuous T1 FSs. In this paper, all IT2 (T1) FSs used in the SIRMs connected IT2 (T1) FISs are supposed to be continuous IT2 (T1) FSs.

The following theorem and corollary show the conditions for the continuity of the SIRMs connected IT2 FISs and the SIRMs connected T1 FISs.

Theorem 1: The SIRMs connected IT2 FIS is continuous if and only if $\max_{j=1,2,\dots,M_i} \underline{\mu}_{\tilde{A}_i^j}(x_i) > 0$ for $i = 1, 2, \dots, n$, i.e., each point of $\mathbb{X}_i$ is covered by some lower MFs.

Proof: First, recall the following facts about continuous functions [17, 18]:

- a) If both $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ are continuous, then,
- b) for any constant k , the function $k \cdot f_1(\mathbf{x})$ is continuous;
- c) the sum of finite continuous functions is also continuous;
- d) $f_1(\mathbf{x})/f_2(\mathbf{x})$ is continuous if and only if $f_2(\mathbf{x}) \neq 0$.

Therefore, to prove that the input-output mapping of a SIRMs connected IT2 FIS is continuous, we just need to prove that $\frac{\sum_{j=1}^{M_i} \underline{f}_i^j(x_i) \underline{c}_i^j}{\sum_{j=1}^{M_i} \underline{f}_i^j(x_i)}$ and $\frac{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i) \bar{c}_i^j}{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i)}$ are continuous for $i = 1, 2, \dots, n$.

Since all lower and upper MFs are continuous, so $\underline{f}_i^j(x_i)$ and $\bar{f}_i^j(x_i)$ ($j = 1, 2, \dots, M_i$) are continuous for the input variable x_i . Hence, from the above fact b), both $\sum_{j=1}^{M_i} \underline{f}_i^j(x_i)$ and $\sum_{j=1}^{M_i} \bar{f}_i^j(x_i)$ are continuous.

Then, from the above fact c), $\frac{\sum_{j=1}^{M_i} \underline{f}_i^j(x_i) \underline{c}_i^j}{\sum_{j=1}^{M_i} \underline{f}_i^j(x_i)}$ and $\frac{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i) \bar{c}_i^j}{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i)}$ are continuous if and only if $\sum_{j=1}^{M_i} \underline{f}_i^j(x_i) > 0$ and $\sum_{j=1}^{M_i} \bar{f}_i^j(x_i) > 0$. This implies that $\max_{j=1,2,\dots,M_i} \underline{\mu}_{\tilde{A}_i^j}(x_i) > 0$.

From above discussions, we can conclude that the SIRMs connected IT2 FIS is continuous if and only if, for $i = 1, 2, \dots, n$, $\max_{j=1,2,\dots,M_i} \underline{\mu}_{\tilde{A}_i^j}(x_i) > 0$. □

Corollary 1: The SIRMs connected T1 FIS is continuous if and only if $\max_{j=1,2,\dots,M_i} \underline{\mu}_{\tilde{A}_i^j}(x_i) > 0$ for $i = 1, 2, \dots, n$, i.e., everywhere of the universe of discourse $\mathbb{X}_i$ is covered by some MFs.

In this subsection, conditions for the continuity of the SIRMs connected IT2 FISs and T1 FISs were derived. Such

results should be very useful in traditional control problems, where the input-output mappings of the designed controllers are usually required to be continuous.

B. Monotonicity

Many applications of fuzzy logic control requires monotonicity of the output with respect to inputs, e.g., in the cart-pole system, an appropriate controller for the cart-pole system needs to satisfy the monotonicity between the pole angle and the desired cart acceleration. Several papers have studied the monotonicity of traditional T1 FISs [19, 20] and traditional IT2 FISs [21-23]; however, only one paper has focused on the monotonicity of the SIRMs connected T1 FISs [24]. In the following, we will derive conditions to ensure the monotonicity between the inputs and outputs of the SIRMs connected IT2 FISs. As the SIRMs connected T1 FISs are the special SIRMs connected IT2 FISs, so our result is more general than that in [24].

To begin, let us give the definition of monotonicity.

Definition 2 [20]: A multi-variable function $f(\mathbf{x})$, where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{X}_1 \times \mathbb{X}_2 \times \dots \times \mathbb{X}_n$, is said to be monotonically increasing with respect to (w.r.t) x_k if $\forall x_k^1 \leq x_k^2 \in \mathbb{X}_k$ implies that $f(x_1, \dots, x_k^1, \dots, x_n) \leq f(x_1, \dots, x_k^2, \dots, x_n)$ for all combinations of $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$.

In this study, we just take increasing monotonicity into account. Similar results can be obtained for decreasing monotonicity.

Theorem 2: The SIRMs connected IT2 FIS is monotonically increasing w.r.t x_k if the following conditions can be satisfied:

- 1): $\frac{\partial \mu_{\tilde{A}_k^j}(x_k)}{\partial x_k} \mu_{\tilde{A}_k^j}(x_k) \leq \frac{\partial \mu_{\tilde{A}_k^i}(x_k)}{\partial x_k} \mu_{\tilde{A}_k^i}(x_k)$ and $\frac{\partial \bar{\mu}_{\tilde{A}_k^i}(x_k)}{\partial x_k} \bar{\mu}_{\tilde{A}_k^i}(x_k) \leq \frac{\partial \bar{\mu}_{\tilde{A}_k^j}(x_k)}{\partial x_k} \bar{\mu}_{\tilde{A}_k^j}(x_k)$, where $1 \leq i \leq j \leq M_k$;
- 2): $\underline{c}_k^1 \leq \underline{c}_k^2 \leq \dots \leq \underline{c}_k^{M_k}$ and $\bar{c}_k^1 \leq \bar{c}_k^2 \leq \dots \leq \bar{c}_k^{M_k}$.

Proof: Denote that $\mathbf{x}^1 = (x_1, \dots, x_k^1, \dots, x_n)$ and $\mathbf{x}^2 = (x_1, \dots, x_k^2, \dots, x_n)$.

From (4) and (5), we can obtain that

$$\begin{aligned} y(\mathbf{x}^2) - y(\mathbf{x}^1) &= w_k[y_k(x_k^1) - y_k(x_k^2)] \\ &= w_k m_k \alpha_k + w_k n_k \beta_k. \end{aligned} \quad (6)$$

where $w_k > 0, 0 \leq m_k = 1 - n_k \leq 1$ and

$$\alpha_k = \frac{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^2) \underline{c}_k^j}{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^2)} - \frac{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^1) \underline{c}_k^j}{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^1)}, \quad (7)$$

$$\beta_k = \frac{\sum_{j=1}^{M_k} \bar{f}_k^j(x_k^2) \bar{c}_k^j}{\sum_{j=1}^{M_k} \bar{f}_k^j(x_k^2)} - \frac{\sum_{j=1}^{M_k} \bar{f}_k^j(x_k^1) \bar{c}_k^j}{\sum_{j=1}^{M_k} \bar{f}_k^j(x_k^1)}. \quad (8)$$

Therefore, to prove that the SIRMs connected IT2 FIS is monotonically increasing w.r.t x_k , we only need to prove that $\alpha_k \geq 0$ and $\beta_k \geq 0$.

Below, we will only prove that $\alpha_k \geq 0$.

Note that

$$\alpha_k = \frac{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^2) \underline{c}_k^j}{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^2)} - \frac{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^1) \underline{c}_k^j}{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^1)}$$

$$\begin{aligned} &= \frac{\sum_{i=1}^{M_k} \sum_{j=1}^{M_k} \underline{f}_k^i(x_k^1) \underline{f}_k^j(x_k^2) \underline{c}_k^j - \sum_{i=1}^{M_k} \sum_{j=1}^{M_k} \underline{f}_k^i(x_k^1) \underline{f}_k^j(x_k^2) \underline{c}_k^i}{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^2) \sum_{i=1}^{M_k} \underline{f}_k^i(x_k^1)} \\ &= \frac{\sum_{i=1}^{M_k} \sum_{j=1}^{M_k} (\underline{c}_k^j - \underline{c}_k^i) \underline{f}_k^i(x_k^1) \underline{f}_k^j(x_k^2)}{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^2) \sum_{i=1}^{M_k} \underline{f}_k^i(x_k^1)} \\ &= \frac{\sum_{i=1}^{M_k-1} \sum_{j=i+1}^{M_k} (\underline{c}_k^j - \underline{c}_k^i) \left[\underline{f}_k^j(x_k^2) \underline{f}_k^i(x_k^1) - \underline{f}_k^i(x_k^2) \underline{f}_k^j(x_k^1) \right]}{\sum_{j=1}^{M_k} \underline{f}_k^j(x_k^2) \sum_{i=1}^{M_k} \underline{f}_k^i(x_k^1)}. \end{aligned} \quad (9)$$

Condition 2) implies that, if $j > i$, then

$$\underline{c}_k^j - \underline{c}_k^i \geq 0. \quad (10)$$

Condition 1) implies that $\frac{\partial}{\partial x_k} \left[\frac{\mu_{\tilde{A}_k^j}(x_k)}{\mu_{\tilde{A}_k^i}(x_k)} \right] \geq 0$, which means

that $\frac{\mu_{\tilde{A}_k^j}(x_k)}{\mu_{\tilde{A}_k^i}(x_k)}$ is monotonically increasing w.r.t x_k .

Therefore,

$$\frac{\mu_{\tilde{A}_k^j}(x_k^2)}{\mu_{\tilde{A}_k^i}(x_k^2)} - \frac{\mu_{\tilde{A}_k^j}(x_k^1)}{\mu_{\tilde{A}_k^i}(x_k^1)} = \frac{\underline{f}_k^j(x_k^2)}{\underline{f}_k^i(x_k^2)} - \frac{\underline{f}_k^j(x_k^1)}{\underline{f}_k^i(x_k^1)} \geq 0, \quad (11)$$

which implies that

$$\underline{f}_k^j(x_k^2) \underline{f}_k^i(x_k^1) - \underline{f}_k^i(x_k^2) \underline{f}_k^j(x_k^1) \geq 0. \quad (12)$$

From (9), (10) and (12), we can conclude that $\alpha_k \geq 0$.

The result that $\beta_k \geq 0$ can be similarly proved.

In conclusion, this theorem holds. $\square$

In [20], Won et al. have derived parameter conditions of triangular, trapezoidal and Gaussian T1 MFs to satisfy that $\frac{\partial \mu_{\tilde{A}_k^i}(x)}{\partial x} \mu_{\tilde{A}_k^j}(x) \leq \frac{\partial \mu_{\tilde{A}_k^j}(x)}{\partial x} \mu_{\tilde{A}_k^i}(x)$, where $j \geq i$.

Similarly, parameter conditions of triangular, trapezoidal and Gaussian lower MFs and upper MFs of IT2 FSs can be derived to satisfy the first condition in *Theorem 2*. The related conclusions are listed in the following:

D): Suppose that the lower and upper MFs of IT2 FS $\tilde{A}_k^t$ are triangular as depicted below

$$\mu_{\tilde{A}_k^t}(x_k) = \mu_{\tilde{A}_k^t}(x_k, \underline{a}_k^t, b_k^t, \underline{c}_k^t) = \begin{cases} \frac{x_k - \underline{a}_k^t}{b_k^t - \underline{a}_k^t}, & x_k \in (\underline{a}_k^t, b_k^t], \\ \frac{\underline{c}_k^t - x_k}{\underline{c}_k^t - b_k^t}, & x_k \in (b_k^t, \underline{c}_k^t], \\ 0, & \text{else} \end{cases} \quad (13)$$

$$\bar{\mu}_{\tilde{A}_k^t}(x_k) = \bar{\mu}_{\tilde{A}_k^t}(x_k, \bar{a}_k^t, b_k^t, \bar{c}_k^t) = \begin{cases} \frac{x_k - \bar{a}_k^t}{b_k^t - \bar{a}_k^t}, & x_k \in (\bar{a}_k^t, b_k^t], \\ \frac{\bar{c}_k^t - x_k}{\bar{c}_k^t - b_k^t}, & x_k \in (b_k^t, \bar{c}_k^t], \\ 0, & \text{else} \end{cases} \quad (14)$$

where $\bar{a}_k^t \leq \underline{a}_k^t \leq b_k^t \leq \underline{c}_k^t \leq \bar{c}_k^t$.

For triangular IT2 FSs, if $\underline{a}_k^i \leq \underline{a}_k^j$, $b_k^i \leq b_k^j$, $\underline{c}_k^i \leq \underline{c}_k^j$, $\bar{a}_k^i \leq \bar{a}_k^j$, and $\bar{c}_k^i \leq \bar{c}_k^j$, then the first condition in *Theorem 2* can be satisfied.

II): Suppose that the lower and upper MFs of IT2 FS $\tilde{A}_k^t$ are trapezoidal as depicted below

$$\underline{\mu}_{\tilde{A}_k^t}(x_k) = \underline{\mu}_{\tilde{A}_k^t}(x_k, \underline{a}_k^t, \underline{b}_k^t, \underline{c}_k^t, \underline{d}_k^t) = \begin{cases} \frac{x_k - \underline{a}_k^t}{\underline{b}_k^t - \underline{a}_k^t}, & x_k \in (\underline{a}_k^t, \underline{b}_k^t], \\ 1, & x_k \in (\underline{b}_k^t, \underline{c}_k^t], \\ \frac{\underline{d}_k^t - x_k}{\underline{d}_k^t - \underline{c}_k^t}, & x_k \in (\underline{c}_k^t, \underline{d}_k^t], \\ 0, & \text{else} \end{cases} \quad (15)$$

$$\bar{\mu}_{\tilde{A}_k^t}(x_k) = \bar{\mu}_{\tilde{A}_k^t}(x_k, \bar{a}_k^t, \bar{b}_k^t, \bar{c}_k^t, \bar{d}_k^t) = \begin{cases} \frac{x_k - \bar{a}_k^t}{\bar{b}_k^t - \bar{a}_k^t}, & x_k \in (\bar{a}_k^t, \bar{b}_k^t], \\ 1, & x_k \in (\bar{b}_k^t, \bar{c}_k^t], \\ \frac{\bar{d}_k^t - x_k}{\bar{d}_k^t - \bar{c}_k^t}, & x_k \in (\bar{c}_k^t, \bar{d}_k^t], \\ 0, & \text{else} \end{cases} \quad (16)$$

where $\bar{a}_k^t \leq \underline{a}_k^t$, $\bar{b}_k^t \leq \underline{b}_k^t$, $\bar{c}_k^t \leq \underline{c}_k^t$ and $\bar{d}_k^t \leq \underline{d}_k^t$.

For trapezoidal IT2 FSs, if $\underline{a}_k^i \leq \underline{a}_k^j$, $\underline{b}_k^i \leq \underline{b}_k^j$, $\underline{c}_k^i \leq \underline{c}_k^j$, $\underline{d}_k^i \leq \underline{d}_k^j$, $\bar{a}_k^i \leq \bar{a}_k^j$, $\bar{b}_k^i \leq \bar{b}_k^j$, $\bar{c}_k^i \leq \bar{c}_k^j$ and $\bar{d}_k^i \leq \bar{d}_k^j$, then the first condition in **Theorem 2** can be satisfied.

III): Suppose that the lower and upper MFs of IT2 FS $\tilde{A}_k^t$ are Gaussian as depicted below

$$\underline{\mu}_{\tilde{A}_k^t}(x_k) = \underline{\mu}_{\tilde{A}_k^t}(x_k, \underline{c}_k^t, \underline{\delta}_k^t) = \exp\left(-\frac{1}{2} \frac{(x_k - \underline{c}_k^t)^2}{\underline{\delta}_k^{t2}}\right), \quad (17)$$

$$\bar{\mu}_{\tilde{A}_k^t}(x_k) = \bar{\mu}_{\tilde{A}_k^t}(x_k, \bar{c}_k^t, \bar{\delta}_k^t) = \exp\left(-\frac{1}{2} \frac{(x_k - \bar{c}_k^t)^2}{\bar{\delta}_k^{t2}}\right), \quad (18)$$

where $0 < \underline{\delta}_k^t \leq \bar{\delta}_k^t$.

For Gaussian IT2 FSs, if $\underline{c}_k^i \leq \bar{c}_k^j$, $\underline{\delta}_k^i = \bar{\delta}_k^j$ and $\bar{\delta}_k^i = \underline{\delta}_k^j$, then the first condition in **Theorem 2** can be satisfied.

When IT2 FSs become to T1 FSs, the SIRMs connected IT2 FIS become to the SIRMs connected T1 FIS, and the result in **Theorem 2** can be rewritten for the SIRMs connected T1 FIS as follows.

Corollary 2: The SIRMs connected T1 FIS is monotonically increasing w.r.t x_k if the following conditions can be satisfied:

- 1): $\frac{\partial \mu_{\tilde{A}_k^i}(x_k)}{\partial x_k} \mu_{\tilde{A}_k^j}(x_k) \leq \frac{\partial \mu_{\tilde{A}_k^j}(x_k)}{\partial x_k} \mu_{\tilde{A}_k^i}(x_k)$, where $1 \leq i \leq j \leq M_k$;
- 2): $c_k^1 \leq c_k^2 \leq \dots \leq c_k^{M_k}$.

In this subsection, parameter conditions were derived to ensure the monotonicity of the SIRMs connected FISs. These conditions can be easily satisfied in the design process of the SIRMs connected FISs. But we should notice that the above derived conditions are sufficient ones. It means that, if such conditions are not fulfilled, the SIRMs connected FIS may still be monotonic.

C. Robustness

Robustness is an interesting and important issue of FISs as discussed in [25]:

When a system is subjected to small deviations around the sampling points (operating points), it is essential to find the maximum tolerance of the system with respect to those perturbations, referred to herein as the systems robustness.

Thus, in the context of modeling, robustness is a metric for measuring the impact of input deviations on the desired output.

In the following, we will first give the definition of the robustness of the SIRMs connected FISs as defined in [25], and then study how to compute the robustness of the SIRMs connected FISs.

Definition 3: the **robustness** of the SIRMs connected FIS at point $\mathbf{x}^*$ w.r.t x_k is defined as

$$R_k = \frac{\Delta_k}{\Delta y_d}, \quad (19)$$

where Δy_d is the given allowable output deviation, and Δ_k is the maximum allowable perturbation of the k th input variable at the sampling point $\mathbf{x}^*$ such that $|y(\mathbf{x}^* + \Delta \mathbf{x}) - y(\mathbf{x}^*)| \leq \Delta y_d$, i.e.

$$\Delta_k = \max \left\{ |\Delta x_k| \mid |y(\mathbf{x}^* + \Delta \mathbf{x}) - y(\mathbf{x}^*)| \leq \Delta y_d \right\}.$$

Definition 4: the **robustness** of the SIRMs connected FIS at points $\{\mathbf{x}^{*1}, \mathbf{x}^{*2}, \dots, \mathbf{x}^{*L}\}$ w.r.t x_k is defined as

$$R_k = \frac{\min_{j=1,2,\dots,L} \{\Delta_k^j\}}{\Delta y_d}, \quad (20)$$

where Δy_d is the given allowable output deviation, and Δ_k^j is the maximum allowable perturbation of the k th input variable at the sampling point $\mathbf{x}^{*j}$ such that $|y(\mathbf{x}^{*j} + \Delta \mathbf{x}^j) - y(\mathbf{x}^{*j})| \leq \Delta y_d$, i.e.

$$\Delta_k^j = \max \left\{ |\Delta x_k^j| \mid |y(\mathbf{x}^{*j} + \Delta \mathbf{x}^j) - y(\mathbf{x}^{*j})| \leq \Delta y_d \right\}.$$

One thing deserves to be mentioned is that the definition of robustness is mathematically equivalent to the sub-additivity or output resolution properties studied in [26-29]. Such properties are proposed for the assessment problems.

Below, we will discuss how to compute the robustness of the SIRMs connected FISs.

First, let us derive the formula for the computation of $\Delta y(\mathbf{x}^*) = y(\mathbf{x}^* + \Delta \mathbf{x}) - y(\mathbf{x}^*)$.

Note that

$$\begin{aligned} \Delta y(\mathbf{x}^*) &= y(\mathbf{x}^* + \Delta \mathbf{x}) - y(\mathbf{x}^*) \\ &= \sum_{i=1}^n w_i [y_i(x_i^* + \Delta x_i) - y_i(x_i^*)]. \end{aligned} \quad (21)$$

Define

$$\Delta y_i(x_i^*) = y_i(x_i^* + \Delta x_i) - y_i(x_i^*). \quad (22)$$

It follows

$$\Delta y(\mathbf{x}^*) = \sum_{i=1}^n w_i \Delta y_i(x_i^*). \quad (23)$$

Assuming Δx_i is small and using a Taylor-series expansion, $y_i(x_i^* + \Delta x_i)$ can be expressed around x_i^* as

$$y_i(x_i^* + \Delta x_i) = y_i(x_i^*) + \left. \frac{\partial y_i(x_i)}{\partial x_i} \right|_{x_i=x_i^*} \Delta x_i + h.o.t, \quad (24)$$

where *h.o.t* is short for higher order terms.

From (22) and (24), we obtain that

$$\Delta y_i(x_i^*) \simeq \left. \frac{\partial y_i(x_i)}{\partial x_i} \right|_{x_i=x_i^*} \cdot \Delta x_i. \quad (25)$$

From (4)

$$y_i(x_i) = m_i g_i(x_i) + n_i \bar{g}_i(x_i). \quad (26)$$

where

$$g_i(x_i) = \frac{\sum_{j=1}^{M_i} f_i^j(x_i) \underline{c}_i^j}{\sum_{j=1}^{M_i} f_i^j(x_i)}, \quad (27)$$

$$\bar{g}_i(x_i) = \frac{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i) \bar{c}_i^j}{\sum_{j=1}^{M_i} \bar{f}_i^j(x_i)}. \quad (28)$$

It follows that

$$\frac{\partial y_i(x_i)}{\partial x_i} = m_i \frac{\partial g_i(x_i)}{\partial x_i} + n_i \frac{\partial \bar{g}_i(x_i)}{\partial x_i}. \quad (29)$$

First, derive $\frac{\partial g_i(x_i)}{\partial x_i}$ and $\frac{\partial \bar{g}_i(x_i)}{\partial x_i}$.

$$\begin{aligned} \frac{\partial g_i(x_i)}{\partial x_i} &= \frac{\sum_{j=1}^{M_i} \sum_{s=1}^{M_i} \frac{\partial f_i^j(x_i)}{\partial x_i} \cdot f_i^s(x_i) \cdot [\underline{c}_i^j - \underline{c}_i^s]}{[\sum_{j=1}^{M_i} f_i^j(x_i)]^2} \\ &= \frac{\sum_{j=1}^{M_i} \sum_{s=1}^{M_i} \frac{\partial \mu_{\tilde{A}_i^j}(x_i)}{\partial x_i} \cdot \mu_{\tilde{A}_i^s}(x_i) \cdot [\underline{c}_i^j - \underline{c}_i^s]}{[\sum_{j=1}^{M_i} \mu_{\tilde{A}_i^j}(x_i)]^2}, \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{\partial \bar{g}_i(x_i)}{\partial x_i} &= \frac{\sum_{j=1}^{M_i} \sum_{s=1}^{M_i} \frac{\partial \bar{f}_i^j(x_i)}{\partial x_i} \cdot \bar{f}_i^s(x_i) \cdot [\bar{c}_i^j - \bar{c}_i^s]}{[\sum_{j=1}^{M_i} \bar{f}_i^j(x_i)]^2} \\ &= \frac{\sum_{j=1}^{M_i} \sum_{s=1}^{M_i} \frac{\partial \bar{\mu}_{\tilde{A}_i^j}(x_i)}{\partial x_i} \cdot \bar{\mu}_{\tilde{A}_i^s}(x_i) \cdot [\bar{c}_i^j - \bar{c}_i^s]}{[\sum_{j=1}^{M_i} \bar{\mu}_{\tilde{A}_i^j}(x_i)]^2}. \end{aligned} \quad (31)$$

From (23) (25), and (29)-(31), we can conclude that, at the sampling point (operating point) $\mathbf{x}^*$,

$$\Delta y(\mathbf{x}^*) \simeq \sum_{i=1}^n w_i \eta_i(x_i^*) \Delta x_i, \quad (32)$$

where

$$\begin{aligned} \eta_i(x_i^*) &= m_i \frac{\sum_{j=1}^{M_i} \sum_{s=1}^{M_i} \frac{\partial \mu_{\tilde{A}_i^j}(x_i^*)}{\partial x_i} \cdot \mu_{\tilde{A}_i^s}(x_i^*) \cdot [\underline{c}_i^j - \underline{c}_i^s]}{[\sum_{j=1}^{M_i} \mu_{\tilde{A}_i^j}(x_i^*)]^2} \\ &\quad + n_i \frac{\sum_{j=1}^{M_i} \sum_{s=1}^{M_i} \frac{\partial \bar{\mu}_{\tilde{A}_i^j}(x_i^*)}{\partial x_i} \cdot \bar{\mu}_{\tilde{A}_i^s}(x_i^*) \cdot [\bar{c}_i^j - \bar{c}_i^s]}{[\sum_{j=1}^{M_i} \bar{\mu}_{\tilde{A}_i^j}(x_i^*)]^2}. \end{aligned} \quad (33)$$

For the T1 case, (33) can be rewritten as

$$\eta_i(x_i^*) = \frac{\sum_{j=1}^{M_i} \sum_{s=1}^{M_i} \frac{\partial \mu_{\tilde{A}_i^j}(x_i^*)}{\partial x_i} \cdot \mu_{\tilde{A}_i^s}(x_i^*) \cdot [\underline{c}_i^j - \underline{c}_i^s]}{[\sum_{j=1}^{M_i} \mu_{\tilde{A}_i^j}(x_i^*)]^2}. \quad (34)$$

From Definition 3, to compute the robustness of the SIRMs connected IT2 FIS at point $\mathbf{x}^*$ w.r.t $x_1, x_2, \dots, x_n$, we need

to derive the maximum allowable perturbation $\Delta_1, \Delta_2, \dots, \Delta_n$ at point $\mathbf{x}^*$, which can be computed as $(\Delta_1, \Delta_2, \dots, \Delta_n) = \max \left\{ (|\Delta x_1|, |\Delta x_2|, \dots, |\Delta x_n|) \mid \left| \sum_{i=1}^n w_i \eta_i(x_i^*) \Delta x_i \right| \leq \Delta y_d \right\}$

Therefore, $(\Delta_1, \Delta_2, \dots, \Delta_n)$ can be obtained by solving the following optimization problem:

$$\begin{cases} \text{maximize} & (|\Delta x_1|, |\Delta x_2|, \dots, |\Delta x_n|) \\ \text{s.t.} & |w_1 \eta_1(x_1^*) \Delta x_1 + \dots + w_n \eta_n(x_n^*) \Delta x_n| \leq \Delta y_d \end{cases} \quad (35)$$

From Definition 4, to compute the robustness of the SIRMs connected IT2 FIS at points $\{\mathbf{x}^{*1}, \mathbf{x}^{*2}, \dots, \mathbf{x}^{*L}\}$ w.r.t $x_1, x_2, \dots, x_n$, we just need to derive the maximum allowable perturbation $(\Delta_1^j, \Delta_2^j, \dots, \Delta_n^j)$ s at each point $\mathbf{x}^{*j}$ using the aforementioned optimization problem.

IV. EXAMPLE

In this section, we will give an example to show the continuity and monotonicity of the SIRMs connected FISs. Also, the robustness of the SIRMs connected FISs will be computed in this example.

Consider the following SIRMs connected T1 and IT2 FISs:

SIRMs connected IT2 FIS

$$\begin{aligned} \text{IT2SIRM} - 1 &= \begin{cases} x_1 = \tilde{A}_1^1 \rightarrow y_1 = [8, 10] \\ x_1 = \tilde{A}_1^2 \rightarrow y_1 = [14, 16] \\ x_1 = \tilde{A}_1^3 \rightarrow y_1 = [20, 22] \end{cases} \\ \text{IT2SIRM} - 2 &= \begin{cases} x_2 = \tilde{A}_2^1 \rightarrow y_2 = [6, 10] \\ x_2 = \tilde{A}_2^2 \rightarrow y_2 = [16, 20] \\ x_2 = \tilde{A}_2^3 \rightarrow y_2 = [26, 30] \end{cases} \end{aligned}$$

SIRMs connected T1 FIS

$$\begin{aligned} \text{T1SIRM} - 1 &= \begin{cases} x_1 = \tilde{A}_1^1 \rightarrow y_1 = 9 \\ x_1 = \tilde{A}_1^2 \rightarrow y_1 = 15 \\ x_1 = \tilde{A}_1^3 \rightarrow y_1 = 21 \end{cases} \\ \text{T1SIRM} - 2 &= \begin{cases} x_2 = \tilde{A}_2^1 \rightarrow y_2 = 8 \\ x_2 = \tilde{A}_2^2 \rightarrow y_2 = 18 \\ x_2 = \tilde{A}_2^3 \rightarrow y_2 = 28 \end{cases} \end{aligned}$$

where the IT2 MFs and T1 MFs are shown in Fig. 1, and the importance degrees of IT2 (T1) SIRM-1 and IT2 (T1) SIRM-2 are 0.7 and 0.3, respectively. The coefficients used in the inference engine of the IT2 case are chosen as $m_1 = m_2 = 0.6$ and $n_1 = n_2 = 0.4$.

Fig. 2 shows the input-output mappings of the SIRMs connected T1 and IT2 FISs. We can observe from Fig. 2 that the corresponding input-output mappings are continuous and monotonically increasing. Such results are consistent with Theorem 1, Theorem 2 and their corollaries.

Now, let us compute the robustness of these two-input-one-output SIRMs connected FISs. To do so, we first need to solve the optimization problem below

$$\begin{cases} \text{maximize} & (|\Delta x_1|, |\Delta x_2|) \\ \text{s.t.} & |w_1 \eta_1(x_1^*) \Delta x_1 + w_2 \eta_2(x_2^*) \Delta x_2| \leq \Delta y_d \end{cases} \quad (36)$$

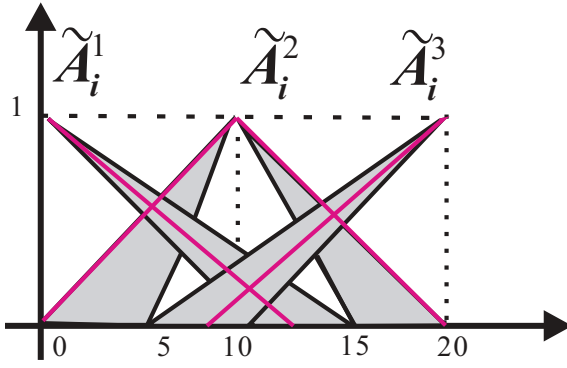
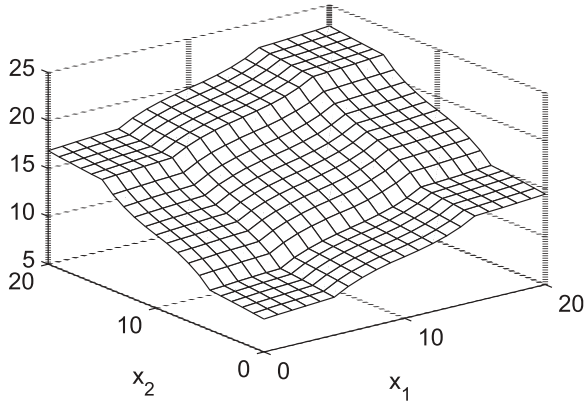
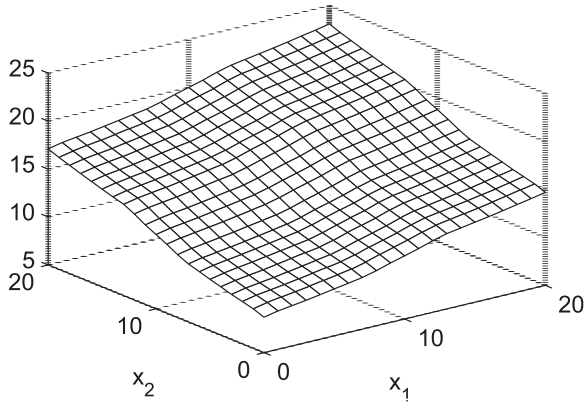


Fig. 1. IT2 MFs (grey areas) and T1 MFs (red lines)



(a) SIRMs Connected IT2 FIS



(b) SIRMs Connected T1 FIS

Fig. 2. Input-output mappings of the SIRMs connected FISs

The result of this optimization problem can be obtained as (details are omitted)

$$\Delta_1 = \frac{\Delta y_d}{2w_1|\eta_1(x_1^*)|}, \quad (37)$$

$$\Delta_2 = \frac{\Delta y_d}{2w_2|\eta_2(x_2^*)|}. \quad (38)$$

Hence, the robustness of such SIRMs connected FISs at

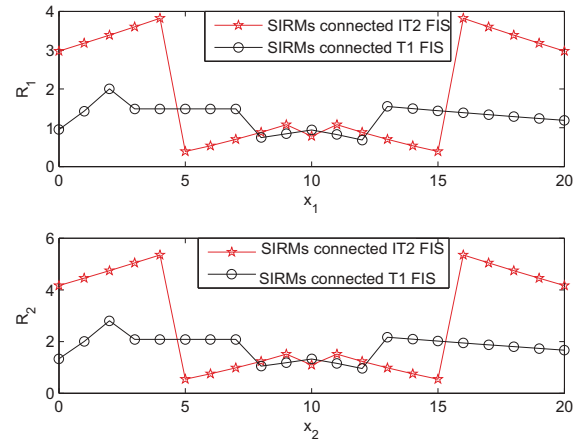


Fig. 3. Robustness indexes of the SIRMs connected FISs

point $\mathbf{x}^*$ w.r.t x_1, x_2 can be computed respectively as

$$R_1 = \frac{1}{2w_1|\eta_1(x_1^*)|}, \quad (39)$$

$$R_2 = \frac{1}{2w_2|\eta_2(x_2^*)|}. \quad (40)$$

It is easy to see that, in the two-input-one-output SIRMs connected FISs, R_1 and R_2 are only functions of the sampling points. Thus, the robustness R_i is constant for different desired output deviations.

The robustness of the SIRMs connected FISs at different sampling points w.r.t x_1, x_2 are shown in Fig. 3.

From Fig. 2 and Fig. 3, we can observe and conclude that:

1) The robustness of the SIRMs connected FISs w.r.t x_1, x_2 lies in the intervals $(0, 4]$ and $(0, 6]$, respectively. This means that, for the given allowable output deviation $\Delta y_d = 1$, the maximum allowable perturbation of the first and second input variables lies in $(0, 4]$ and $(0, 6]$ at different sampling points. This result is reasonable and also consistent with the input-output mappings.

2) The SIRMs connected IT2 FIS performs more robustly than the SIRMs connected T1 FIS when $x_1 \in [0, 4.5] \cup [15.5, 20]$ and $x_2 \in [0, 4.5] \cup [15.5, 20]$, although the input-output mapping of the SIRMs connected T1 FIS seems smoother than the IT2 one. The reason for this may be that the output of the SIRMs connected IT2 FIS changes more slowly in the neighbor of these sampling points.

3) For both SIRMs connected FISs, if their outputs change slowly in the neighbor of one sampling point, then the robustness at this point will be large. On the contrary, if the outputs change fast in the neighbor of one sampling point, then the robustness at this point will become small. This is also consistent with our intuition.

V. CONCLUSION

In this paper, new properties – continuity, monotonicity and robustness – of the SIRMs connected FISs, including both T1 and IT2 cases, are explored. Parameter conditions are derived to ensure the continuity and monotonicity of the

SIRMs connected FISs. These conditions can be easily satisfied in the design process of the SIRMs connected FISs. Also, a methodology for the robustness analysis of the SIRMs connected FISs is presented. The given example demonstrate that the proposed robustness analysis methodology is reasonable. However, how to design SIRMs connected FISs using the proposed methodology has not been studied. This will be one of our future research directions.

ACKNOWLEDGMENT

This work is supported by the National Natural Science Foundation of China under Grant No. 60975060 and 61074149. We also gratefully acknowledge the supports of the Shandong Provincial Key Laboratory of Intelligent Buildings Technology, Jinan 250101, China.

REFERENCES

- [1] J. Yi, N. Yubazaki and K. Hirota, "A new fuzzy controller for stabilization of parallel-type double inverted pendulum system," *Fuzzy Sets and Systems*, vol. 126, pp. 105–119, 2002.
- [2] J. Yi, N. Yubazaki and K. Hirota, "A proposal of SIRMs dynamically connected fuzzy inference model for plural input fuzzy control," *Fuzzy Sets and Systems*, vol. 125, pp. 79–92, 2002.
- [3] J. Yi, N. Yubazaki and K. Hirota, "Anti-swing and positioning control of overhead traveling crane," *Information Sciences*, vol. 155, pp. 19–42, 2002.
- [4] H. Seki, M. Mizumoto and N. Yubazaki, "On the property of single input rule modules connected type fuzzy reasoning method," *IEICE Transactions on Fundamentals*, vol. J89-A, pp. 557–565, June 2006.
- [5] H. Seki, H. Ishii and M. Mizumoto, "On the generalization of single input rule modules connected type fuzzy reasoning method," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 5, pp. 1180–1187, 2008.
- [6] C. Li and J. Yi, "SIRMs based interval type-2 fuzzy inference systems: properties and application," *International Journal of Innovative Computing, Information and Control*, vol. 6, no. 9, pp. 4019–4028, 2010.
- [7] C. Li, J. Yi and D. Zhao, "Control of the TORA system using SIRMs based type-2 fuzzy logic," *Proc. 2009 IEEE International Conference on Fuzzy Systems*, Jeju Island, Korea, 2009, pp. 694–699.
- [8] C. Li and J. Yi, "Design of SIRMs connected interval type-2 fuzzy logic controller based on prior knowledge," *Journal of Southeast University (Natural Science Edition)*, vol. 39, no. suppl. 1, pp. 57–61, 2009.
- [9] C. Li, J. Yi and T. Wang, "Stability analysis of SIRMs based type-2 fuzzy logic control systems," *Proc. 2010 IEEE World Congress on Computational Intelligence*, Barcelona, Spain, 2010, pp. 2913–2918.
- [10] Q. Liang and J. M. Mendel, "Interval type-2 fuzzy logic systems: theory and design," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 5, pp. 535–550, 2000.
- [11] J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New directions*, New Jersey : Prentice-Hall, 2001.
- [12] J. M. Mendel, R. I. John and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 6, pp. 808–821, 2006.
- [13] J. M. Mendel, "Advances in type-2 fuzzy sets and systems," *Information Sciences*, vol. 177, pp. 84–110, 2007.
- [14] J. M. Mendel, "On answering the question 'where do I start in order to solve a new problem involving interval type-2 fuzzy sets?'," *Information Sciences*, vol. 179, pp. 3418–3431, 2009.
- [15] S. Coupland and R. John, "Geometric type-1 and type-2 fuzzy logic systems," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 1, pp. 3–15, 2007.
- [16] M. Biglarbegian, W.W. Melek and J. M. Mendel, "On the stability of interval type-2 TSK fuzzy logic control systems," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 4, no. 3, pp. 798–818, 2010.
- [17] D. Wu and J. M. Mendel, "On the continuity of type-1 and interval type-2 fuzzy logic systems," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 1, pp. 179–192, 2011.
- [18] R. Courant and F. John, *Introduction to Calculus and Analysis*. vol. II/I, New York: Springer, 1999.
- [19] V. S. Kouikoglou and Y. A. Phillis, "On the monotonicity of hierarchical sum-product fuzzy systems", *Fuzzy Sets and Systems*, vol. 160, pp. 3530–3538, 2009.
- [20] J. Won, S. Park and J. Lee, "Parameter conditions for monotonic Takagi–Sugeno–Kang fuzzy system," *Fuzzy Sets and Systems*, vol. 132, pp. 135–146, 2002.
- [21] C. Li, J. Yi and T. Wang, "Combing prior knowledge with data driven design of interval type-2 fuzzy logic systems," *International Journal of Innovative Computing, Information and Control*, vol. 7, no. 3, pp. 1133–1144, 2011.
- [22] C. Li, J. Yi and D. Zhao, "Analysis and design of monotonic type-2 fuzzy inference systems," *Proc. 2009 IEEE International Conference on Fuzzy Systems*, Jeju Island, Korea, 2009, pp. 1193–1198.
- [23] T. Wang, J. Yi and C. Li, "Fuzzy identification based on monotonicity and interval type-2 fuzzy logic," *ICIC Express Letters – An International Journal of Research and Surveys*, vol. 5, no. 4(A), pp. 1107–1113, 2011.
- [24] H. Seki, H. Ishii and M. Mizumoto, "On the monotonicity of single input type fuzzy reasoning methods," *IEICE Transactions on Fundamentals*, vol. E90-A, pp. 1462–1468, July 2007.
- [25] M. Biglarbegian, W. Melek and J. M. Mendel, "On the robustness of type-1 and interval tpe-2 fuzzy logic systems in modeling," *Information Sciences*, vol. 181, pp. 1325–1347, 2011.
- [26] K. M. Tay and C. P. Lim, "A fuzzy inference system-based criterion-referenced assessment model," *Expert Systems With Applications*, in press.
- [27] K. M. Tay and C. P. Lim, "Enhancing the failure mode and effect analysis methodology with fuzzy inference techniques," *Journal of Intelligent and Fuzzy systems*, vol. 21, no. 1-2, pp. 135–146, 2010.
- [28] K. M. Tay and C. P. Lim, "On the use of fuzzy inference techniques in assessment models: Part I: theoretical," *Fuzzy Optimization and Decision Making: A Journal of Modeling and Computation Under Uncertainty*, vol. 7, no. 3, pp. 269–281, 2008.
- [29] K. M. Tay and C. P. Lim, "On the use of fuzzy inference techniques in assessment models: Part II: industrial applications," *Fuzzy Optimization and Decision Making: A Journal of Modeling and Computation Under Uncertainty*, vol. 7, no. 3, pp. 283–302, 2008.