

An Approach for Stability Analysis of Polynomial Fuzzy Model-Based Control Systems

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Abstract—Stability analysis of polynomial fuzzy model-based (PFMB) control systems under the parallel distributed compensation (PDC) design technique is investigated. A new polynomial fuzzy controller (PFC) is introduced to release conservativeness in the existing approaches. Compared to the conventional (PFC), the controller under consideration in this paper has a favorable property which introduces some more variables in the stability conditions such that, the solution of the derived stability conditions can be explored in a larger group of potential solutions. This property may lead to further relax stability conditions. However, the stability conditions will not be in the form of SOS conditions. Hence, particle swarm optimization (PSO) is employed to explore the solution of the derived stability conditions which are in the form of state-dependent BMI. It is shown that the conventional PFC is a special case of that of proposed one. Finally, a simulation example is given to illustrate the effectiveness of the proposed approach.

Index Terms—polynomial fuzzy model(PFM), state-dependent Bilinear Matrix Inequality (BMI), stability conditions, sum of squares (SOS), particle swarm optimization (PSO).

I. INTRODUCTION

Recently, fuzzy model-based (FMB) control approach has become a popular system control analysis methodology. It provides a systematic way for handling of nonlinear control systems [1]. To demonstrate complex nonlinear systems in a fuzzy frame, T-S fuzzy model [2], [3], introduces a mathematical framework. This model portrays a nonlinear system by a set of fuzzy rules. Each rule contains a linear model, so-called subsystem, that represents the dynamics of the nonlinear plant in a specific local operating domain. The overall T-S fuzzy model is constructed as an average weighted sums of some subsystems. To control a nonlinear plant, a fuzzy controller is employed to close the feedback loop. The design strategy of fuzzy controller is based on the T-S fuzzy model so that, a FMB control system [4]–[6] is formed.

At early stage of research on the FMB control approach, lack of a rigorous theoretical basis for stability analysis had drawn a great deal of attention to this area. Recently, based on the Lyapunov stability theory [7] for nonlinear control systems, sufficient conditions to guarantee the stability of the FMB control system are derived. In most of the proposed approaches,

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stability analysis and control design problems are formulated to Linear Matrix Inequality (LMI) problems [8]. Then, utilizing convex programming techniques, LMI MATLAB Toolbox [9], numerical solutions are obtained. Some of the well-known LMI-based stability conditions have been reported in [1], [5], [10]–[18]. *Parallel Distributed Compensation* (PDC) [5] design technique, for which the fuzzy controller shares the same membership functions as T-S fuzzy model, is employed to further relax the stability conditions [1], [11], [12], [18]. When the fuzzy controller does not follow the PDC design, it is called non-PDC fuzzy controller.

Very recently, nonlinear control systems have been represented by polynomial fuzzy model (PFM) [19], which is a generalized form of the T-S fuzzy model. PFMs have the same structure as the T-S fuzzy model, but polynomials are allowed to be considered in the subsystems. To close the feedback loop, polynomial fuzzy controller (PFC) which is similar to the traditional fuzzy controller [6], but with polynomials in its structure, is employed and polynomial fuzzy model-based (PFMB) control system is constructed.

To deal with the stability analysis and performance design of the PFMB control systems, sum of squares (SOS) approach is employed [19]. Based on the SOS approach, polynomials are allowed to be incorporated in the stability conditions. These conditions can be regarded as state-dependent LMI conditions. Semi-definite programming software SOSTOOLS [20] is the computational method to be employed to solve numerically these SOS (state-dependent LMI) conditions, based on the SOS decomposition. Compared to the LMI approach, SOS approach has some advantages in dealing with stability analysis of fuzzy control systems. Since, polynomials are allowed to be incorporated in the PFM, it is more effective in representing nonlinear control systems. Moreover, when various polynomials of state variables are allowed to be considered in the PFC, solution of stability conditions can be searched in a larger group of candidates, which may result in the relaxation of stability conditions. In addition, to investigate stability of these systems, polynomial Lyapunov function (PLF) which is a generalized form of quadratic Lyapunov function, can be implemented. It should be noted that PLF can be utilized in the LMI-based stability conditions.

Conservatism is one of the drawbacks that most of the existing stability conditions suffer from it. In [21], to relax

the stability conditions under the non-PDC paradigm for FMB control systems, a fuzzy controller has been considered, for which the feedback gains were time-varying nonlinear functions. Due to introducing more decision variables in the structure of the stability conditions, feedback gains can be searched in a wider domain of potential solution for the conditions under consideration. However, the derived stability conditions were in the form of bilinear matrix inequality (BMI). Therefore, numerical solution were found by employing genetic algorithm (GA) [22] and LMI MATLAB Toolbox.

In this paper, stability of PFMB control systems under the PDC design technique is investigated. To release conservativeness resulting from considering conventional form for PFC [19], PFC with similar structure to the controller in [21] is introduced. Compared to the conventional PFC, the PFC under consideration introduces some more decision variables in the stability conditions such that, the solution of the derived stability conditions can be explored in a larger group of potential solutions which may lead to relaxed stability conditions. However, the resulting conditions will be in the form of state-dependent BMI. Then, utilizing searching algorithms say, particle swarm optimization (PSO) [23], the solution of derived conditions can be searched in a wider domain.

The rest of this chapter is organized as follows. In section II, PFM, PFC and PFMB control system are presented. In section III, stability analysis of PFMB control system is presented. In section IV, a simulation example is given to illustrate the validity of the proposed approach. Section V concludes this paper.

II. PFM, PFC AND PFMB CONTROL SYSTEM

Consider a nonlinear system described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad (1)$$

where $\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ is a nonlinear function in which $\mathbf{x}(t) \in \mathbb{R}^\rho$ is the system state vector and $\mathbf{u}(t) \in \mathbb{R}^\omega$ is the input vector. Using the sector nonlinearity concept [6], the nonlinear system (1) is represented by the following polynomial fuzzy model [19], [24],

Plant Rule i :

$$\begin{aligned} \text{IF } z_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \dots \text{ AND } z_p(\mathbf{x}(t)) \text{ is } M_p^i \\ \text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \end{aligned} \quad (2)$$

$$i = 1, \dots, r,$$

for which $z_\alpha(\mathbf{x}(t))$ is the premise variable; M_α^i is a fuzzy term corresponding to i^{th} rule; $\mathbf{A}_i(\mathbf{x}(t)) \in \mathbb{R}^{\rho \times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \mathbb{R}^{\rho \times \omega}$ are the polynomial system and polynomial input matrices respectively, for $i = 1, \dots, r$, $\alpha = 1, \dots, p$; In addition, $\hat{\mathbf{x}}(t) = [\hat{x}_1(t), \hat{x}_2(t), \dots, \hat{x}_N(t)]^T \in \mathbb{R}^N$ is a vector of monomials in $\mathbf{x}(t)$ with assumption that $\hat{\mathbf{x}}(t) = \mathbf{0}$ iff $\mathbf{x}(t) = \mathbf{0}$. A monomial in $\mathbf{x}(t)$ is a function of the form $x_1^{\alpha_1}(t)x_2^{\alpha_2}(t)\dots x_n^{\alpha_n}(t)$, where $\alpha_1, \alpha_2, \dots, \alpha_p$ are nonnegative

integers. In this case, the degree of the monomial is given by $\alpha_1 + \alpha_2 + \dots + \alpha_p$.

The overall system dynamics is represented by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^r \mu_i(\mathbf{z}(\mathbf{x}(t))) \left(\mathbf{A}_i(\mathbf{x}(t))\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \right). \quad (3)$$

where

$$\begin{aligned} \mu_i(\mathbf{z}(\mathbf{x}(t))) &\in [0, 1], \\ \sum_{i=1}^r \mu_i(\mathbf{z}(\mathbf{x}(t))) &= 1. \end{aligned}$$

To control nonlinear system represented by the polynomial fuzzy model (3), a polynomial fuzzy controller is introduced, which is described by following fuzzy rules.

Control Rule j :

$$\begin{aligned} \text{IF } g_1(\mathbf{x}(t)) \text{ is } N_1^j \text{ AND } \dots \text{ AND } g_q(\mathbf{x}(t)) \text{ is } N_q^j \\ \text{THEN } \mathbf{u}(t) = \frac{\mathbf{F}_j(\mathbf{x}(t))}{H(\mathbf{x}(t))} \hat{\mathbf{x}}(\mathbf{x}(t)) \end{aligned} \quad (4)$$

$$j = 1, \dots, c,$$

where $g_\theta(\mathbf{x}(t))$ is the premise variable; N_θ^j is a fuzzy term corresponding to the j^{th} control rule, $j = 1, \dots, c$, $\theta = 1, \dots, q$; $\mathbf{F}_j(\mathbf{x}(t)) \in \mathbb{R}^{\omega \times N}$ is the polynomial feedback gain of the j^{th} rule to be determined and $H(\mathbf{x}(t))$ is defined as follows.

$$H(\mathbf{x}(t)) = \sum_{k=1}^c \lambda_k(\mathbf{g}(\mathbf{x}(t))) a_k(\mathbf{x}) \quad (5)$$

in which

$$a_k(\mathbf{x}(t)) = a_k^1 \hat{x}_1 + \dots + a_k^{l-1} \hat{x}_{l-1} + a_k^l,$$

where $\hat{x}_1, \dots, \hat{x}_{l-1}$ are monomials and $a_k^1, \dots, a_k^l$ are scalars.

Then, regarding to (5), the overall PFC is defined as

$$\begin{aligned} \mathbf{u}(t) &= \sum_{j=1}^c \lambda_j(\mathbf{g}(\mathbf{x}(t))) \frac{\mathbf{F}_j(\mathbf{x}(t))}{H(\mathbf{x}(t))} \hat{\mathbf{x}}(\mathbf{x}(t)) \\ &= \sum_{j=1}^c \lambda_j(\mathbf{g}(\mathbf{x}(t))) \frac{\mathbf{F}_j(\mathbf{x}(t))}{\sum_{k=1}^c \lambda_k(\mathbf{g}(\mathbf{x}(t))) a_k(\mathbf{x}(t))} \hat{\mathbf{x}}(\mathbf{x}(t)) \end{aligned} \quad (6)$$

where

$$\begin{aligned} \lambda_j(\mathbf{g}(\mathbf{x}(t))) &\in [0, 1], \\ \sum_{j=1}^c \lambda_j(\mathbf{g}(\mathbf{x}(t))) &= 1. \end{aligned} \quad (7)$$

$$\dot{\mathbf{x}}(t) = \frac{\sum_{i=1}^r \sum_{j=1}^c \mu_i(\mathbf{x}(t)) \lambda_j(\mathbf{x}(t)) \left(a_j(\mathbf{x}(t)) \mathbf{A}_i(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t)) \mathbf{F}_j(\mathbf{x}(t)) \right)}{\sum_{k=1}^c \lambda_k(\mathbf{x}(t)) a_k(\mathbf{x}(t))} \dot{\mathbf{x}}(\mathbf{x}(t)). \quad (8)$$

Considering the PFM (3) and the PFC (6) connected in a closed loop, we have

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{i=1}^r \mu_i(\mathbf{x})(t) \left(\mathbf{A}_i(\mathbf{x}(t)) \hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t)) \sum_{j=1}^c \lambda_j(\mathbf{x})(t) \right. \\ &\quad \times \left. \frac{\mathbf{F}_j(\mathbf{x}(t))}{\sum_{k=1}^c \lambda_k(\mathbf{x}(t)) a_k(\mathbf{x}(t))} \hat{\mathbf{x}}(\mathbf{x}(t)) \right) \end{aligned}$$

Then, the PFMB control system is obtained as (8).

For brevity, $\mathbf{x}$, $\hat{\mathbf{x}}(\mathbf{x})$, $\mu_i(\mathbf{x})$, $\lambda_j(\mathbf{x})$ and $\mathbf{u}$ are used to denote $\mathbf{x}(t)$, $\hat{\mathbf{x}}(\mathbf{x}(t))$, $\mu_i(\mathbf{z}(t))$, $\lambda_j(\mathbf{s}(\mathbf{x}(t)))$, and $\mathbf{u}(t)$, respectively, in the following analysis.

III. STABILITY ANALYSIS OF PFMB CONTROL SYSTEMS

Based on the Lyapunov stability theory, the stability of the PFMB control system (8) is investigated. Consider the following polynomial quadratic Lyapunov function candidate [19]

$$V(\mathbf{x}) = \hat{\mathbf{x}}(\mathbf{x})^T \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \hat{\mathbf{x}}(\mathbf{x}), \quad (9)$$

where $\mathbf{X}^{-1}(\tilde{\mathbf{x}}) \in \mathbb{R}^{N \times N}$ is a symmetric positive definite polynomial matrix and $\tilde{\mathbf{x}} = [x_{k_1}, x_{k_2}, \dots, x_{k_d}]$ is a vector to be chosen such that $\mathbf{K} = \{k_1, k_2, \dots, k_d\}$ denotes the indices of the corresponding zero rows in $\mathbf{B}_i(\mathbf{x})$ for all i . Denoting the k^{th} row in $\mathbf{A}_i(\mathbf{x})$ and $\mathbf{B}_i(\mathbf{x})$ as $\mathbf{A}_i^k(\mathbf{x})$ and $\mathbf{B}_i^k(\mathbf{x})$ respectively, we have

$$\mathbf{B}_i^k(\mathbf{x}) = \mathbf{0} \quad \text{for } k \in \mathbf{K}. \quad (10)$$

To avoid introducing non-convex condition, we assume that $\mathbf{X}(\tilde{\mathbf{x}})$ only depends on state vector $\tilde{\mathbf{x}}$. Hence $\frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k} = \mathbf{0}$ for $k \notin \mathbf{K}$. For more details readers are referred to [19], [25]. According to the Lyapunov stability theory, if $\mathbf{X}(\tilde{\mathbf{x}})$ and $\mathbf{X}^{-1}(\tilde{\mathbf{x}})$ are positive definite matrices in $\tilde{\mathbf{x}}$ and $\dot{V}(t) \leq 0$ (equality holds when $\mathbf{x} = \mathbf{0}$), then the polynomial FMB control system (8) is asymptotically stable. Following the approach in [19]:

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T(\mathbf{x}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}^T(\mathbf{x}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \dot{\hat{\mathbf{x}}}(\mathbf{x}) \\ &\quad + \hat{\mathbf{x}}^T(\mathbf{x}) \dot{\mathbf{X}}^{-1}(\tilde{\mathbf{x}}) \hat{\mathbf{x}}(\mathbf{x}) \end{aligned} \quad (11)$$

in which

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(\mathbf{x}) &= \begin{pmatrix} \dot{\hat{\mathbf{x}}}_1(\mathbf{x}) \\ \vdots \\ \dot{\hat{\mathbf{x}}}_N(\mathbf{x}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial \hat{x}_1}{\partial x_1} & \cdots & \frac{\partial \hat{x}_1}{\partial x_\rho} \\ \vdots & \ddots & \vdots \\ \frac{\partial \hat{x}_N}{\partial x_1} & \cdots & \frac{\partial \hat{x}_N}{\partial x_\rho} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \vdots \\ \frac{\partial x_\rho}{\partial t} \end{pmatrix} \\ &= \Phi(\mathbf{x}) \dot{\mathbf{x}} \end{aligned}$$

where $\Phi(\mathbf{x}) \in \mathbb{R}^{N \times \rho}$ is a polynomial matrix defined as follows

$$\Phi(\mathbf{x}) = \begin{pmatrix} \frac{\partial \hat{x}_1}{\partial x_1} & \cdots & \frac{\partial \hat{x}_1}{\partial x_\rho} \\ \vdots & \ddots & \vdots \\ \frac{\partial \hat{x}_N}{\partial x_1} & \cdots & \frac{\partial \hat{x}_N}{\partial x_\rho} \end{pmatrix}. \quad (12)$$

Then, (11) is derived as follows.

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T \Phi^T(\mathbf{x}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}^T(\mathbf{x}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \Phi(\mathbf{x}) \dot{\mathbf{x}} + \hat{\mathbf{x}}^T(\mathbf{x}) \\ &\quad \times \left(\sum_{k=1}^{\rho} \frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k} \dot{x}_k \right) \hat{\mathbf{x}}(\mathbf{x}). \end{aligned} \quad (13)$$

Since $\frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k} = \mathbf{0}$ for $k \notin \mathbf{K}$, and according to (10), $\dot{V}(\mathbf{x})$ is written as follows.

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T \Phi^T(\mathbf{x}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \hat{\mathbf{x}}(\mathbf{x}) + \hat{\mathbf{x}}^T(\mathbf{x}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \Phi(\mathbf{x}) \dot{\mathbf{x}} + \hat{\mathbf{x}}^T(\mathbf{x}) \\ &\quad \times \left(\sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial x_k} \sum_{i=1}^r \mu_i(\mathbf{x}) \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \right) \hat{\mathbf{x}}(\mathbf{x}). \end{aligned} \quad (14)$$

Also, $\mathbf{X}(\tilde{\mathbf{x}})$ is invertible, then we have $\mathbf{X}^{-1}(\tilde{\mathbf{x}}) \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{I}$. Differentiating both sides with respect to x_k yields

$$\frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial t} \mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial t} = \mathbf{0}.$$

Hence the following relation holds.

$$\frac{\partial \mathbf{X}^{-1}(\tilde{\mathbf{x}})}{\partial t} = -\mathbf{X}^{-1}(\tilde{\mathbf{x}}) \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial t} \mathbf{X}^{-1}(\tilde{\mathbf{x}}) \quad (15)$$

Denoting $\mathbf{y}(\mathbf{x}) = \mathbf{X}^{-1}\hat{\mathbf{x}}(\mathbf{x})$ and according to (15), $\dot{V}(\mathbf{x})$ can be written in the following form.

$$\dot{V}(\mathbf{x}) = \sum_{i=1}^r \sum_{j=1}^c \mu_i(\mathbf{x}) \lambda_j(\mathbf{x}) \mathbf{y}^T(\mathbf{x}) \frac{\mathbf{T}_{ij}(\mathbf{x})}{\sum_{k=1}^c \lambda_k(\mathbf{x}) a_k(\mathbf{x})} \mathbf{y}(\mathbf{x}), \quad (16)$$

where

$$\begin{aligned} \mathbf{T}_{ij}(\mathbf{x}) = & \left(a_j(\mathbf{x}) \mathbf{A}_i(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{B}_i(\mathbf{x}) \mathbf{M}_j(\mathbf{x}) \right)^T \Phi^T(\mathbf{x}) \\ & + \Phi(\mathbf{x}) \left(a_j(\mathbf{x}) \mathbf{A}_i(\mathbf{x}) \mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{B}_i(\mathbf{x}) \mathbf{M}_j(\mathbf{x}) \right) \\ & - \sum_{k \in \mathbf{K}} \frac{\partial \mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k} a_j(\mathbf{x}) \mathbf{A}_i^k(\mathbf{x}) \hat{\mathbf{x}}(\mathbf{x}) \end{aligned} \quad (17)$$

in which the local feedback gains are defined as

$$\mathbf{F}_j(\mathbf{x}) = \mathbf{M}_j(\mathbf{x}) \mathbf{X}^{-1}(\tilde{\mathbf{x}}), \quad j = 1, \dots, c. \quad (18)$$

Utilizing the PDC design technique, (16) can be written as

$$\begin{aligned} \dot{V}(\mathbf{x}) = & \sum_{i=1}^r \sum_{j=1}^r \mu_i(\mathbf{x}) \mu_j(\mathbf{x}) \mathbf{y}^T(\mathbf{x}) \frac{\mathbf{T}_{ij}(\mathbf{x})}{\sum_{k=1}^c \mu_k(\mathbf{x}) a_k(\mathbf{x})} \mathbf{y}(\mathbf{x}) \\ = & \sum_{i=1}^r \mu_i(\mathbf{x}) \mu_i(\mathbf{x}) \mathbf{y}^T(\mathbf{x}) \frac{\mathbf{T}_{ii}(\mathbf{x})}{\sum_{k=1}^c \mu_k(\mathbf{x}) a_k(\mathbf{x})} \mathbf{y}(\mathbf{x}) \\ & + \sum_{i=1}^r \sum_{i < j} \mu_i(\mathbf{x}) \mu_j(\mathbf{x}) \mathbf{y}^T(\mathbf{x}) \frac{\mathbf{T}_{ij}(\mathbf{x}) + \mathbf{T}_{ji}(\mathbf{x})}{\sum_{k=1}^c \mu_k(\mathbf{x}) a_k(\mathbf{x})} \mathbf{y}(\mathbf{x}) \end{aligned} \quad (19)$$

Choosing $a_j(\mathbf{x}) > 0$, $j = 1, \dots, c$ and regarding to the properties of the membership functions in (4),

$$\sum_{k=1}^c \lambda_k(\mathbf{x}) a_k(\mathbf{x}) > 0. \quad (20)$$

Referring to the Lyapunov stability theory, the PFMB control system (8) is asymptotically stable if $\dot{V}(\mathbf{x}) < 0$. If $\mathbf{T}_{ii}(\mathbf{x}) < 0$ for $i = 1, \dots, r$ and $\mathbf{T}_{ij}(\mathbf{x}) < 0$ for $i = 1, \dots, r, i < j$ in (19), then $\dot{V}(\mathbf{x}) < 0$ and thus, resulting in asymptotically stability of PFMB control system (8).

The stability analysis result is proposed as the following theorem.

Theorem 1. *The PFMB control system (8) consisting of the PFM (3) and the PFC (6) is asymptotically stable if there exist pre-defined polynomials $a_k(\mathbf{x}) > 0$ $k = 1, \dots, r$, and matrices $\mathbf{X}(\tilde{\mathbf{x}})$ and $\mathbf{M}_j(\mathbf{x})$ such that, the following SOS-based conditions*

are satisfied,

$$v^T \left(\mathbf{X}(\tilde{\mathbf{x}}) - \epsilon_1(\mathbf{x}) \mathbf{I} \right) v \text{ is SOS}, \quad (21)$$

$$-v^T \left(\mathbf{T}_{ii}(\mathbf{x}) + \epsilon_{2ii}(\mathbf{x}) \mathbf{I} \right) v \text{ is SOS}, \quad \text{for } i = 1, \dots, r, \quad (22)$$

$$-v^T \left(\mathbf{T}_{ij}(\mathbf{x}) + \mathbf{T}_{ji}(\mathbf{x}) + \epsilon_{2ij}(\mathbf{x}) \mathbf{I} \right) v \text{ is SOS}, \quad \forall i < j, \quad (23)$$

where $\mathbf{T}_{ij}(\mathbf{x})$ is defined in (17), $v(t) \in \mathbb{R}^p$ is an arbitrary vector independent of $\mathbf{x}$ and $\epsilon_1(\mathbf{x}) > 0$, $\epsilon_{2ij}(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$ are pre-defined polynomial scalars and the local feedback gains are defined in (18).

Remark 1: To derive the numerical solution of state-dependent BMI conditions in Theorem 1, the searching algorithm, PSO technique, is exploited. The procedure for finding numerical solution is illustrated in Fig. 1 and summarized in the following stages:

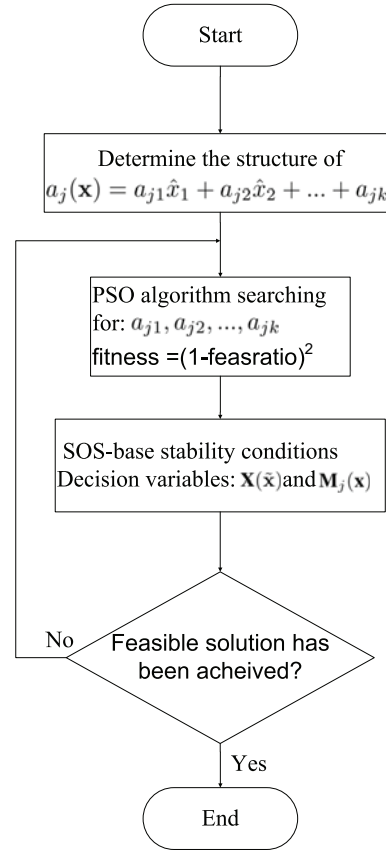


Fig. 1. Flowchart of the proposed PSO and SOS-based technique

Stage 1: The $a_j(\mathbf{x})$ $j = 1, \dots, r$, are polynomials containing monomials with various degrees. As the first step, proper monomials which have contribution in relaxation of stability conditions are determined. The coefficients of these monomials

are considered as decision variables to be determined by the PSO in the next steps.

Stage 2: The PSO assigns initial values to the coefficients of $a_j(\mathbf{x})$ $j = 1, \dots, r$. These values are kept fixed during stage 3.

Stage 3: Since the values of $a_j(\mathbf{x})$ are fixed then, the stability conditions in Theorem 1 convert to SOS-based stability conditions. Numerical solution for the derived SOS-based stability conditions are obtained by SOSTOOLS [25].

Stage 4: If the derived parameters for the SOSTOOLS show feasibility of solution for the SOS-based stability conditions then, the searching procedure will stop. Otherwise, if the stopping criterion, which is a predefined number of iteration has not been reached, return to *Step 2*. It should be noted that, in the PSO searching algorithm, finding of solution is formulated as a minimization problem for a fitness function. In this paper, fitness function for the PSO algorithm is chosen as $-(1 - \text{feasratio})^2$. A more closer value of 'feasratio' to '1' leads to minimization of the fitness function and implies more accurate solution for the SOS-based stability conditions.

IV. SIMULATION EXAMPLE

The effectiveness of the proposed stability analysis approach is illustrated by following simulation example.

Consider a three-rule PFM with the following system and input relations.

$$\mathbf{A}_1(\mathbf{x}) = \begin{pmatrix} 1.59 + x_1^2 - 2x_2^2 - x_1x_2 & -7.29 + 2x_1x_2 \\ 0.01 & -x_1^2 - x_2^2 \end{pmatrix},$$

$$\mathbf{A}_2(\mathbf{x}) = \begin{pmatrix} 0.02 + x_1^2 - 2x_2^2 - x_1x_2 & -4.64 + 2x_1x_2 \\ 0.35 & 0.21 - x_1^2 - x_2^2 \end{pmatrix},$$

$$\mathbf{A}_3(\mathbf{x}) = \begin{pmatrix} -\alpha + x_1^2 - 2x_2^2 - x_1x_2 & -4.33 + 2x_1x_2 \\ 0 & 0.05 - x_1^2 - x_2^2 \end{pmatrix},$$

$$\mathbf{B}_1(\mathbf{x}) = \begin{pmatrix} 1 + x_1 + x_1^2 \\ 0 \end{pmatrix},$$

$$\mathbf{B}_2(\mathbf{x}) = \begin{pmatrix} 8 + x_1 + x_1^2 \\ 0 \end{pmatrix},$$

$$\mathbf{B}_3(\mathbf{x}) = \begin{pmatrix} -\beta + 6 + x_1 + x_1^2; 0 \\ -1 \end{pmatrix},$$

where α and β are constant parameters. Based on the PDC design technique, a three-rule PFC in the form of (6) is employed to close the feedback loop. Then, the PFMB control system in the form of (8) is obtained.

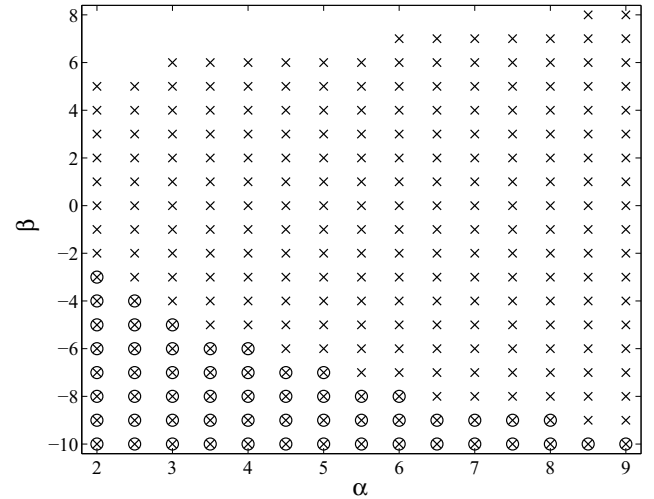


Fig. 2. Stability regions given by [19] (x), and Theorem 1 (o) for the Simulation Example.

To investigate the stability of the aforementioned PFMB control system, the stability conditions proposed in Theorem 1 are employed. However, as it mentioned in Remark 1, $a_j(\mathbf{x})$ are decision variables that means the derived stability conditions are in the form of state-dependent BMI. Then, implementing algorithm shown in Fig. 1, the solution for the derived stability conditions are achieved.

In this simulation example the following format is chosen for $a_j(\mathbf{x})$

$$a_j(\mathbf{x}) = a_j^1 x_1^2 + a_j^2, \quad (24)$$

where a_j^1 and a_j^2 are scalars to be chosen via PSO searching algorithms. The lower and upper bounds of a_j^1 and a_j^2 , are chosen as 10^{-3} and 3 respectively.

For comparison purposes the stability conditions proposed in [19], is employed to investigate the system stability for $2 \leq a \leq 9$, $7.4 \leq b \leq 9$. The stability regions given by SOS-based stability conditions in [19] and Theorem 1 are demonstrated in Fig. 2. It can be seen that comparing to the stability conditions in [19], the proposed SOS-based stability conditions in Theorem 1 are able to offer a larger stability region.

For reference, according to Theorem 1 with considering the relaxation expressions, for $a = 5$ and $b = 6$, $\mathbf{X}(\bar{\mathbf{x}})$ and the feedback gains are obtained as the above of this page.

V. CONCLUSION

Stability analysis of PFMB control systems has been proposed. Introducing a new form of PFC which is the generalized form of the conventional one, relaxed stability conditions have been achieved. Compared to the conventional PFC, the fuzzy controller under consideration has a favorable property which introduces some more variables in the stability conditions such that, the solution of the derived stability conditions can be explored in a larger group of potential solutions. The

$$\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix} 10.807 & -1.5579 \\ -1.5579 & 1.4054 \end{bmatrix}$$

$$\begin{aligned} \mathbf{F}_1 &= [-1.0414 - 0.5533x_1 + 0.0287x_1^2 + 0.1159x_1^3 - 0.1741x_1^4 - 0.0201x_1x_2 + 0.0450x_2 - 0.2055x_2^2 \\ &\quad 0.9848 - 1.6206x_1 + 0.0919x_1^2 + 0.1286x_1^3 - 0.1929x_1^4 - 0.0435x_1x_2 + 0.0393x_2 - 0.2276x_2^2], \\ \mathbf{F}_2 &= [-2.7283 + 0.2983x_1 + 0.2779x_1^2 - 0.0339x_1^3 - 0.3559x_1^4 + 0.3844x_1x_2 - 0.0088x_2 - 0.3004x_2^2 \\ &\quad - 0.3482 - 0.0861x_1 + 0.2112x_1^2 - 0.0380x_1^3 - 0.3945x_1^4 + 0.0470x_1x_2 + 0.0137x_2 - 0.3329x_2^2], \\ \mathbf{F}_3 &= [-0.7934 + 0.0610x_1 + 0.0461x_1^2 - 0.1019x_1^3 - 0.1316x_1^4 - 0.0088x_1x_2 - 0.0019x_2 - 0.1139x_2^2 \\ &\quad 0.5162 - 0.6742x_1 + 0.3559x_1^2 - 0.1129x_1^3 - 0.1459x_1^4 - 0.0731x_1x_2 + 0.0594x_2 - 0.1263x_2^2], \end{aligned}$$

achieved stability conditions are in the form of state-dependent BMI conditions. The PSO algorithm has been employed to search for the numerical solution of the derived conditions. A simulation example has been presented to confirm the validity of the proposed approach.

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