

Relationship between Intuitionistic Fuzzy Similarity Measures

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Abstract—Numerous similarity measures between intuitionistic fuzzy sets are proposed in literature using different approaches. In this paper, relationship between some existing intuitionistic fuzzy similarity and distance measures are investigated. These relations are paramount for the choice of a similarity or a distance measure and its application for any research topic.

keywords—Similarity measures, distance measures, intuitionistic fuzzy sets

I. INTRODUCTION

Intuitionistic fuzzy set (IFSs) [1], [2] which are generalizations of fuzzy sets [3] are showed in [4] to be similar to vague sets proposed by Gau and Buehrer [5] and demonstrated in [6] to be equivalent to interval valued fuzzy sets [7]. These theories were the subject of operators' propositions. Similarity and distance measures between IFSs are widely studied and discussed. Chen [8], [9] proposed measures of similarity between vague sets. Hong and Kim [10] demonstrated that these measures are not effective in some cases and proposed to rectify them by giving another similarity measure. Fan and Zhangian [11] proposed an intuitionistic fuzzy similarity measure (IFSM) which resolves some problems of the precedent measures but still contains some counter intuitive cases [12]. Dengfeng and Chuntian [13] proposed some intuitionistic fuzzy similarity measures (IFSMs) and applied them to pattern recognition. However, Mitchell showed in [14] some counter intuitive cases of these measures and proposed an IFSM to solve their weakness by adopting a statistical viewpoint. Other IFSMs are proposed in [15], [16].

In addition to IFSMs, many measures of distances between IFSs are proposed. Szmidt and Kacprzyk [17] proposed Hamming and Euclidean distances based on a geometric interpretation of IFSs. Wang and Xin [18], proposed other distance measures and applied them to pattern recognition showing that the found results are similar to those proposed in [15]. Hung and Yung [19] proposed IFSMs based on Hausdorff distance. The same authors proposed some distance measures and IFSMs in [20] based on L_p metric. More details about IFSMs and distance measures between IFSs are presented in

[21], [22].

In section two, we present IFSs definition followed by the definition of distance measures and similarity measures between IFSs from literature. Therefore, we provide relationship between them.

II. INTUITIONISTIC FUZZY SETS AND INTERVAL VALUED FUZZY SETS

A. Definition 1

An intuitionistic fuzzy set A in a discourse universe X is defined in [1] as:

$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$ where $\mu_A(x) : X \rightarrow [0, 1]$ and $\nu_A(x) : X \rightarrow [0, 1]$ with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1 \forall x \in X$.

The numbers $\mu_A(x)$, $\nu_A(x)$ denote respectively the degrees of membership and of non-membership of the element $x \in X$ to A . Also, μ_A and ν_A are called respectively truth membership function and false membership function.

In addition to membership and non-membership functions, a function of hesitancy or uncertainty of x to A denoted by $\pi_A(x)$ must be taken into consideration. $\pi_A(x)$ is computed as:

$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ with $0 \leq \pi_A(x) \leq 1$

B. Definition 2

In [7], an interval-valued fuzzy set in a universe X is a mapping $A : X \rightarrow \text{Int}([0, 1])$

$x \mapsto [\underline{A}(x), \overline{A}(x)] \subseteq [0, 1]. \forall x \in X.$

where $\text{Int}([0, 1])$ is the set of all closed subintervals of $[0, 1]$.

C. Notations

We note $\text{IFSs}(X)$ the set of all IFSs in X and we use μ_A , ν_A and π_A instead of $\mu_A(x_i)$, $\nu_A(x_i)$ and $\pi_A(x_i)$ respectively, $\forall x_i \in A, \forall A \in \text{IFSs}(X)$.

III. INTUITIONISTIC FUZZY DISTANCE MEASURES PROPOSED IN LITERATURE

In this section, we present some distance measures, existing in literature, between two IFSs A and B .

A. Definition 1

Atanassov [2] proposed the following distance measures:

- Hamming Distance

$$H(A, B) = \frac{1}{2n} \sum_{i=1}^n |\mu_A - \mu_B| + |\nu_A - \nu_B| \quad (1)$$

- Euclidean Distance

$$E(A, B) = \sqrt{\frac{1}{2n} \sum_{i=1}^n (\mu_A - \mu_B)^2 + (\nu_A - \nu_B)^2} \quad (2)$$

- [2] present these distances using the max operator

$$J_1(A, B) = \max_i |\mu_A(x_i) - \mu_B(x_i)| \quad (3)$$

J_1 defines the Hamming distance between fuzzy sets

$$J_2(A, B) = \max_i |\nu_A(x_i) - \nu_B(x_i)| \quad (4)$$

$$J(A, B) = \frac{1}{2} (J_1(A, B) + J_2(A, B)) \quad (5)$$

B. Definition 2

Szmidt and Kacprzyk [17] interpreted the precedent measures (1, 2) by a geometric perspective. Thus, they concluded that the hesitation degree must be taken into consideration to compute Euclidean and hamming distances and proposed these measures.

- Hamming Distance

$$l_{IFS}(A, B) = \frac{1}{2n} \sum_{i=1}^n |\mu_A - \mu_B| + |\nu_A - \nu_B| + |\pi_A - \pi_B| \quad (6)$$

- Euclidean Distance

$$q_{IFS}(A, B) = \frac{1}{\sqrt{2n}} \times \sqrt{\sum_{i=1}^n (\mu_A - \mu_B)^2 + (\nu_A - \nu_B)^2 + (\pi_A - \pi_B)^2} \quad (7)$$

C. Definition 3

In the real space $\mathbb{R}$, for any two intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$, the Hausdorff distance $H(A, B)$ is defined as:

$$Hd(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\} \quad (8)$$

Based on the definition of $H(A, B)$, Hung and Yang [19] proposed Hausdorff distance between two subintervals $I_A(x_i) = [\mu_B(x_i), 1 - \nu_B(x_i)]$ and $I_B(x_i) = [\mu_A(x_i), 1 - \nu_A(x_i)]$, $i = 1, 2, \dots, n$ of the IFSs A and B defined in $[0, 1]$ by this formula:

$$d_{Hd}(A, B) = \frac{1}{n} \sum_{i=1}^n \max(|\mu_A - \mu_B|, |\nu_A - \nu_B|) \quad (9)$$

Using the $d_{Hd}(A, B)$ definition, the authors proposed some similarity measures between IFSs in [19].

D. Definition 4

Grzegorzewski [23] proposed the following formulas for Euclidean and hamming distances for IFSs based on Hausdorff metric.

- Hamming Distance

$$l_h(A, B) = \frac{1}{n} \sum_{i=1}^n \max(|\mu_A - \mu_B|, |\nu_A - \nu_B|) \quad (10)$$

- Euclidean Distance

$$q_h(A, B) = \sqrt{\frac{1}{n} \sum_{i=1}^n \max((\mu_A - \mu_B)^2, (\nu_A - \nu_B)^2)} \quad (11)$$

E. Definition 5

Wang and Xin [18] showed by examples, that the normalized Euclidean distance (7) may not classify patterns in some cases and proposed the following distance measures:

$$d_1(A, B) = \frac{1}{n} \sum_{i=1}^n \left[\frac{|\mu_A - \mu_B| + |\nu_A - \nu_B|}{4} \right] + \frac{1}{n} \sum_{i=1}^n \left[\frac{\max(|\mu_A - \mu_B|, |\nu_A - \nu_B|)}{2} \right] \quad (12)$$

$$d_2^p(A, B) = \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^n (\varphi_\mu(x_i) + \varphi_\nu(x_i))^p} \quad (13)$$

where $\varphi_\mu(x_i) = |\mu_A - \mu_B|/2$, $\varphi_\nu(x_i) = |\nu_A - \nu_B|/2$ and p is a positive integer.

IV. INTUITIONISTIC FUZZY SIMILARITY MEASURES PROPOSED IN LITERATURE

In this section, we present similarity measures between two IFSs A and B from literature, we recall that similarity measures $s(A, B)$ and distance measures $d(A, B)$ are dual concepts, so, if $s(A, B)$ is a similarity measure between IFSs A and B , then $d(A, B) = 1 - s(A, B)$ and inversely $s(A, B) = 1 - d(A, B)$.

A. Definition 1

In their paper, Hong and Kim [10] proposed the following similarity measure:

$$S_H(A, B) = 1 - \frac{\sum_{i=1}^n |\mu_A - \mu_B| + |\nu_A - \nu_B|}{2n} \quad (14)$$

S_H can not distinguish positive difference from negative difference, so some counter cases exist for S_H [12]. For example, if $A = \{(x, 0, 0)\}$ and $B = \{(x, 0.3, 0.5)\}$ then $S_H(A, B) = 0.6$ which is not reasonable. An example of another type of counter intuitive is given in [12]: If $A = \{(x, 1, 0)\}$, $B = \{(x, 0, 0)\}$ and $C = \{(x, 0.5, 0.5)\}$ then, $S_H(A, B) = S_H(C, B) = 0.5$.

B. Definition 2

In [11] the following measure are presented:

$$S_L(A, B) = 1 - \frac{\sum_{i=1}^n |S_A(x_i) - S_B(x_i)|}{4n} - \frac{\sum_{i=1}^n |\mu_A - \mu_B| + |\nu_A - \nu_B|}{4n} \quad (15)$$

where $S_A(x_i) = \mu_A - \nu_A$ and $S_B(x_i) = \mu_B - \nu_B$.

C. Definition 3

Dengfeng and Chuntian [13] proposed the following similarity measure between IFSs.

$$S_d^p(A, B) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^n |\varphi_A(x_i) - \varphi_B(x_i)|^p} \quad (16)$$

here:

$$\varphi_A(x_i) = \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2}$$

and

$$\varphi_B(x_i) = \frac{\mu_B(x_i) + 1 - \nu_B(x_i)}{2}$$

with $1 \leq p < +\infty$

Mitchell [14], Liang and Shi [15] present some counter-intuitive cases for S_d^p which are existing because S_d^p does not verify the property condition (P2) of similarity measures defined as:

(P2) $S(A, B) = 1$ if and only if $A = B$

As an example [12], the computed similarity S_d^p (16) using the sets $A = \{(x, 0.1, 0.1)\}$ and $B = \{(x, 0.5, 0.5)\}$ with $p = 1$, is $S_d^1(A, B) = 1$.

D. Definition 4

In [15] some similarity measures are given, we present below one of them.

$$S_e^p(A, B) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^n (\varphi_{\mu_{AB}}(x_i) + \varphi_{\nu_{AB}}(x_i))^p} \quad (17)$$

where

$$\varphi_{\mu_{AB}}(x_i) = \frac{|\mu_A - \mu_B|}{2}$$

and

$$\varphi_{\nu_{AB}}(x_i) = \left| \frac{1 - \nu_A}{2} - \frac{1 - \nu_B}{2} \right|$$

make the use of two end points of the subintervals in IFSs.

E. Definition 5

Hung and Yang [20] defined the L_p distance between two intervals $I_A(x_i) = [\mu_A, 1 - \nu_A]$ and $I_B(x_i) = [\mu_B, 1 - \nu_B]$, $i = 1, 2, \dots, n$ as:

$$d_p(I_A(x_i), I_B(x_i)) = (|\mu_A - \mu_B|^p + |\nu_A - \nu_B|^p)^{1/p}, \quad p \geq 1$$

In consequence, the distance $L_p(A, B)$ between A and B is defined as:

$$L_p(A, B) = \frac{1}{n} \sum_{i=1}^n d_p(I_A, I_B) \quad (18)$$

Then, the authors proposed some similarity measures based on L_p metric, we present the following measure from them:

$$S_l^p(A, B) = \frac{2^{\frac{1}{p}} - L_p(A, B)}{2^{\frac{1}{p}}} \quad (19)$$

F. Definition 6

In [24] the authors gave a generalization of measures between fuzzy sets proposed in [25] and presented the following measures between IFSs.

$$S_{pk2}(A, B) = 1 - \frac{1}{2} \left(\max_i |\mu_A - \mu_B| + \max_i |\nu_A - \nu_B| \right) \quad (20)$$

$$S_{pk3}(A, B) = 1 - \frac{\sum_{i=1}^n |\mu_A - \mu_B| + |\nu_A - \nu_B|}{\sum_{i=1}^n (|\mu_A + \mu_B| + |\nu_A + \nu_B|)} \quad (21)$$

V. RELATIONSHIP BETWEEN THE PRESENTED MEASURES

- The Hausdorff distance (9) is the same of the Hamming distance (10). So we have:

$$d_{Hd}(A, B) = l_h(A, B) \quad (22)$$

We mention that this measure should be the Hausdorff distance because it is the generalization of the Hausdorff distance between crisp sets.

- If we replace $\pi_A(x_i)$ and $\pi_B(x_i)$ with their formulas in $l_{IFS}(A, B)$ (6) we find that $H(A, B) \leq l_{IFS}(A, B) \leq 2H(A, B)$ where $H(A, B)$ has the equation (1). It is obvious that $H(A, B) \leq l_{IFS}(A, B)$, thus we show below that $l_{IFS}(A, B) \leq 2H(A, B)$

Proof:

$$\begin{aligned} l_{IFS}(A, B) &= \frac{1}{2n} \sum_{i=1}^n |\mu_A - \mu_B| + |\nu_A - \nu_B| \quad (23) \\ &+ |1 - \mu_A - \nu_A - 1 + \mu_B + \nu_B| \\ &\leq \frac{1}{2n} \sum_{i=1}^n |\mu_A - \mu_B| + |\nu_A - \nu_B| \\ &+ |\mu_B - \mu_A| + |\nu_B - \nu_A| \\ &\leq \frac{1}{2n} \sum_{i=1}^n 2|\mu_A - \mu_B| + 2|\nu_A - \nu_B| \\ &\leq \frac{1}{n} \sum_{i=1}^n |\mu_A - \mu_B| + |\nu_A - \nu_B| \end{aligned} \quad (24)$$

Then

$$H(A, B) \leq l_{IFS}(A, B) \leq 2H(A, B) \quad (25)$$

If we examine the formulas (23, 25) we can remark that the measure definition l_{IFS} is very close to that of $H(A, B)$.

- In the sections (III-A, III-B, III-D), we presented three different definitions of Hamming and euclidean distance measures from literature. If we examine measures S_H (14), S_L (15) and d_1 (12) we remark that they are related on intuitionistic fuzzy Hamming distance measure H (1), d_1 is on relation with Hamming distance l_h (10) in addition of H . In the same way, formulas d_2^p (13) and S_l^p (19) depend on the power p , if $p=1$, all these formulas can be written using Hamming distances H and l_h . So, we can write formulas S_H (14), S_L (15) as follows:
 - $S_H(A, B)$ (14) is a dual concept of $H(A, B)$ (1), so it can be written as:

$$S_H(A, B) = 1 - H(A, B) \quad (26)$$

- $S_L(A, B)$ (15) is on relation with $H(A, B)$ (1), we have:

$$S_L(A, B) = 1 - \frac{\sum_{i=1}^n |S_A(x_i) - S_B(x_i)|}{4n} - \frac{1}{2}H(A, B) \quad (27)$$

where $S_A(x_i)$ and $S_B(x_i)$ are defined in section IV-B.

- d_1 is on relation with Hamming distances H and l_h as:

$$d_1(A, B) = \frac{1}{2} (H(A, B) + l_h(A, B)) \quad (28)$$

- if $p=1$ and from (22, 28), the similarity measures d_2^p (13), S_l^p (19) are written as:

$$* \quad d_2^1(A, B) = \sqrt{H(A, B)} \quad (29)$$

$$* \quad S_l^1(A, B) = 1 - H(A, B) \quad (30)$$

- The IFSM S_e^p (17) is the dual concept of the distance measure d_2^p (13). It is easy to find in (17) that:

$$\varphi_{\nu_{AB}}(x_i) = \left| \frac{1 - \nu_A}{2} - \frac{1 - \nu_B}{2} \right|$$

$$\varphi_{\nu_{AB}}(x_i) = \left| \frac{\nu_A - \nu_B}{2} \right|$$

Thus:

$$S_e^p(A, B) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^n \left(\frac{|\mu_A - \mu_B|}{2} + \frac{|\nu_A - \nu_B|}{2} \right)^p}$$

So we have the equality:

$$S_e^p = 1 - d_2^p \quad (31)$$

Despite the different interpretations of the measures (17), (13), they lead to the same formula.

When $p=1$, from (31, 29) we have:

$$S_e^1(A, B) = 1 - \sqrt{H(A, B)} \quad (32)$$

- The IFSM S_d^p (16) can be written as:

$$S_d^p(A, B) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^n \frac{|\mu_A - \mu_B + \nu_B - \nu_A|}{2}}$$

$$\leq 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^n \frac{|\mu_A - \mu_B| + |\nu_B - \nu_A|}{2}}$$

$$\leq S_e^p$$

$$\leq 1 - d_2^p$$

- The measure S_{pk2} (20) presented in section IV-F is the dual concept of the distance J (5), Thus:

$$S_{pk2}(A, B) = 1 - J(A, B) \quad (33)$$

- The measure S_{pk3} (21) presented in section IV-F is written as:

$$S_{pk3}(A, B) = 1 - \frac{2nH(A, B)}{\sum_{i=1}^n |\mu_A + \mu_B| + |\nu_A + \nu_B|} \quad (34)$$

VI. CONCLUDING REMARKS

We showed that some measures are on relationship with the Hamming distance. Despite the different interpretations of distance measures or similarity measures, some of them are equal such as S_e^p (17) and d_2^p (13). In addition, the similarity measure $S_H(A, B)$ (14) is demonstrated to have some counter intuitive cases. So, from the equality (26), we deduce that all the following formula measures having close relation with the Hamming distance $H(A, B)$ (1) have the same counter intuitive cases of S_H :

d_1 (12), d_2^p (13), S_e^p (17), S_d^p (16), l_{IFS} (6) and S_{pk3} (21).

ACKNOWLEDGMENT

The authors would like to acknowledge the financial support of this work by grants from General Direction of Scientific Research (DGRST), Tunisia, under the ARUB program.

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