

A Theoretical Approach to Liu's Generalized Lambda-Fuzzy Measure

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Abstract— λ -measure, P-measure, additive measure and B-measure are four well known fuzzy measure, all of them are univalent fuzzy measure with only one formulaic fuzzy measure solution. In this paper, first of all, it is proved that both λ -measure and additive measure are larger than P-measure and less than B-measure, in other words, both λ -measure and additive measure are never equal to the smallest fuzzy measure, P-measure and the largest fuzzy measure, B-measure. Furthermore, a novel multivalent fuzzy measure with infinitely many fuzzy measure solutions based on λ -measure, called a generalized λ -measure, is proposed. It is proved that the measure function of this new measure is continuous and increasing on λ , each different λ value corresponding different fuzzy measure, from the smallest fuzzy measure, P-measure to the largest fuzzy measure, B-measure, and containing λ -measure itself, in other words, all of above four univalent fuzzy measures are special cases of this new fuzzy measure. Moreover, this new generalized λ -measure is also satisfying the monotone condition rather than the old generalized λ -measure proposed by L.-Z. Yang, M.-H. Ha, X.-J. Wang and Z.-R. Zhang. However, the old one is just the λ -measure without the monotone condition. It is a generalized measure but no more a fuzzy measure. Hence, this new and true generalized fuzzy measure is called Liu's generalized λ -measure. Some properties about this new measure are also discussed.

Keywords- λ -measure; P-measure; B-measure; L-measure; Liu's generalized λ -measure

I. INTRODUCTION

When interactions among independent variables exist in forecasting problems, the performances of the multiple linear regression models and ridge regression models [1] are not good enough. For improving this situation, G. Choquet proposed his fuzzy integral, called Choquet integral, in 1953 [2], M. Sugeno rediscovered the concepts of fuzzy integral and fuzzy measure and also proposed his λ -measure in 1974 [3], L. A. Zadeh's simple sub-additive fuzzy measure with closed form, P-measure, was proposed in 1978 [4], the author, H.-C. Liu, proved that the P-measure is the smallest fuzzy measure and proposed the largest fuzzy measure, B-measure in 2009 [6], all of above measures and traditional additive measure [5] are univalent fuzzy measure with only one formulaic fuzzy measure solution, the author, H.-C. Liu, proposed some multivalent fuzzy measures with infinitely many fuzzy measure

solutions, called L-measure, completed L-measure, maximized L-measure, etc, [6-9]. It is showed that the interactions problem can be improved by using Choquet integral regression models with λ -measure and above multivalent fuzzy measures [6-9].

However, although all of L-measure, completed L-measure and maximized L-measure are multivalent fuzzy measures with infinitely many fuzzy measure solutions, but they do not contain λ -measure, in other words, λ -measure is not a special case of above multivalent fuzzy measures and all of the above multivalent fuzzy measures are not a generalized λ -measure.

L.-Z. Yang, M.-H. Ha, X.-J. Wang, and Z.-R. Zhang proposed a generalized λ -measure by deleting the monotone condition of λ -measure in 2008 [10], and then λ -measure is a special case of their generalized λ -measure, but if the monotone condition is not satisfied, then this generalized λ -measure is not a fuzzy measure, in other words, this generalized λ -measure is a generalized measure of λ -measure, but not a generalized fuzzy measure of λ -measure, since a fuzzy measure must be a monotone measure.

In this paper, a novel multivalent fuzzy measure satisfying the monotone condition with infinitely many fuzzy measure solutions based on λ -measure also containing λ -measure itself and rather than the generalized λ -measure proposed by L.-Z. Yang, M.-H. Ha, X.-J. Wang, and Z.-R. Zhang, is considered. Some relative properties about this new generalized λ -measure are also considered.

II. FUZZY MEASURES

A. Fuzzy measures

Definition 1. [3,5] A measure μ on a finite set X is a fuzzy measure, if its measure function $g_\mu : 2^X \rightarrow [0, \infty)$ satisfying

$$(i) \ g_\mu(\emptyset) = 0 \quad (\text{vanishing at the empty set}) \quad (1)$$

$$(ii) \ A \subseteq B \Rightarrow g_\mu(A) \leq g_\mu(B) \quad (\text{monotonicity}) \quad (2)$$

B. Fuzzy support function, support

Definition 2. [6] A fuzzy support function of a fuzzy measure μ on a finite set X is a function $s : X \rightarrow [0, \infty)$ satisfying:

$$s(x) = \mu(\{x\}), x \in X \quad (3)$$

$s(x)$ is called the support value of singleton x .

C. Additive measure

Definition 3. [5] For a given fuzzy support function, $s(x)$ on a finite set X , a measure σ on X is an additive measure, if its measure function $g_\sigma : 2^X \rightarrow [0, \infty)$ satisfying :

$$(i) \quad g_\sigma(\emptyset) = 0 \quad (\text{vanishing at the empty set}) \quad (4)$$

$$(ii) \quad \forall A \in 2^X \Rightarrow g_\sigma(A) = \sum_{x \in A} s(x) = \sum_{x \in A} g_\sigma(\{x\}) \quad (5)$$

Theorem 1. [5] An additive measure on a finite set X is a fuzzy measure on X satisfying

$$(i) \quad A, B \in 2^X, A \cap B = \emptyset \Rightarrow g_\sigma(A \cup B) = g_\sigma(A) + g_\sigma(B) \quad (6)$$

$$(ii) \quad A, B \in 2^X, A \subseteq B \Rightarrow g_\sigma(A) \leq g_\sigma(B) \quad (7)$$

Proof:

$$(i) \quad A, B \in 2^X, A \cap B = \emptyset \\ \Rightarrow g_\sigma(A \cup B) = \sum_{x \in A \cup B} g_\sigma(\{x\}) = \sum_{x \in A} g_\sigma(\{x\}) + \sum_{x \in B} g_\sigma(\{x\}) \\ = g_\sigma(A) + g_\sigma(B) \quad (8)$$

$$(ii) \quad A, B \in 2^X, A \subseteq B \Rightarrow g_\sigma(A) = \sum_{x \in A} g_\sigma(\{x\}) \leq \sum_{x \in B} g_\sigma(\{x\}) = g_\sigma(B) \quad (9)$$

D. λ -measure [3]

Definition 4. [3] For a given fuzzy support function, $s(x)$, on a finite set X , a measure on X is a λ -measure, if its measure function $g_\lambda : 2^X \rightarrow [0, \infty)$ satisfying

$$(i) \quad g_\lambda(\emptyset) = 0 \quad (\text{Vanishing at the empty set}) \quad (10)$$

$$(ii) \quad \lambda \in \left[-\frac{1}{\max_{A \subset X} \{g_\lambda(A)\}}, \infty \right) \quad (11)$$

$$(iii) \quad A, B \in 2^X, A \cap B = \emptyset, A \cup B \neq X \\ \Rightarrow g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B) \quad (12)$$

$$(iv) \quad \forall A \subset X \Rightarrow g_\lambda(A) = \begin{cases} \frac{1}{\lambda} \left(-1 + \prod_{x \in A} [1 + \lambda s(x)] \right) & \lambda \neq 0 \\ \sum_{x \in A} s(x) & \lambda = 0 \end{cases} \quad (13)$$

Note that

(i) Eq. (12) and Eq. (13) are called σ - λ -rule of λ -measure.

(ii) Eq. (11) is the monotone condition of λ -measure.

E. P-measure

Definition 5. [4] For a given fuzzy support function, $s(x)$, on a finite set X , a measure is called P-measure, g_P , if its measure function $g_P : 2^X \rightarrow [0, \infty)$ satisfying

$$(i) \quad g_P(\emptyset) = 0 \quad (\text{vanishing at the empty set})$$

$$(ii) \quad \forall A \in 2^X \Rightarrow g_P(A) = \max_{x \in A} \{s(x)\} = \max_{x \in A} g_P(\{x\}) \quad (14)$$

III. NORMALIZED FUZZY MEASURES

A. Normalized fuzzy measures [2,3]

Definition 6. [5,6] A fuzzy measure μ on a finite set X is a normalized fuzzy measure, if its measure function $g_\mu : 2^X \rightarrow [0, 1]$ satisfying the following axioms:

$$(i) \quad g_\mu(\emptyset) = 0, g_\mu(X) = 1 \quad (\text{normalized conditions}) \quad (15)$$

$$(ii) \quad A \subseteq B \Rightarrow g_\mu(A) \leq g_\mu(B) \quad (\text{monotonicity}) \quad (16)$$

B. Fuzzy density function

Definition 7. A fuzzy density function of a normalized fuzzy measure μ on a finite set X is a function $d : X \rightarrow [0, 1]$ satisfying:

$$s(x) = g_\mu(\{x\}), x \in X \quad (17)$$

$s(x)$ is called the density of singleton x .

C. Normalized additive measure

Definition 8. [5] For a given fuzzy density function, $d(x)$ on a finite set X , a measure σ on X is an normalized additive measure, if its measure function $g_\sigma : 2^X \rightarrow [0, 1]$ satisfying :

$$(i) \quad g_\sigma(\emptyset) = 0, g_\sigma(X) = 1 \quad (\text{Normalized conditions}) \quad (18)$$

$$(ii) \quad \forall A \in 2^X \Rightarrow g_\sigma(A) = \sum_{x \in A} s(x) = \sum_{x \in A} g_\sigma(\{x\}) \quad (19)$$

Theorem 2. [5] An normalized additive measure on a finite set X is a fuzzy measure on X satisfying

$$(i) \quad A, B \in 2^X, A \cap B = \emptyset \Rightarrow g_\sigma(A \cup B) = g_\sigma(A) + g_\sigma(B) \quad (20)$$

$$(ii) \quad A, B \in 2^X, A \subseteq B \Rightarrow g_\sigma(A) \leq g_\sigma(B) \quad (21)$$

$$(iii) \quad \sum_{x \in X} g_\sigma(\{x\}) = 1 \quad (22)$$

D. Normalized λ -measure

Definition 9. [3] For a given fuzzy density function, $d(x)$, on a finite set x , a normalized fuzzy measure is called a normalized λ -measure, if its measure function $g_\lambda : 2^X \rightarrow [0, 1]$ satisfying the following axioms:

$$(i) \quad g_\lambda(\emptyset) = 0, g_\lambda(X) = 1 \quad (\text{normalized conditions}) \quad (23)$$

$$(ii) \quad \lambda \in [-1, \infty) \quad (24)$$

$$(iii) \quad A, B \in 2^X, A \cap B = \emptyset, A \cup B \neq X \\ \Rightarrow g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B) \quad (25)$$

$$(iv) \quad \forall A \subset X \Rightarrow g_\lambda(A) = \begin{cases} \frac{1}{\lambda} \left(-1 + \prod_{x \in A} [1 + \lambda d(x)] \right) & \lambda \neq 0 \\ \sum_{x \in A} d(x) & \lambda = 0 \end{cases} \quad (26)$$

$$(v) \quad \prod_{x \in X} [1 + \lambda d(x)] = 1 + \lambda > 0, d(x) = g_\lambda(\{x\}) \quad (27)$$

Note that Eq. (27) is the upper bounded condition of Definition 9.

Theorem 3. [3] If $d(x)$ is the fuzzy density function of a normalized λ -measure on a finite set X , then normalized λ -measure is a normalized fuzzy measure on X with exact one solution of the value of λ satisfying

$$(i) \quad \sum_{x \in X} d(x) > 1 \Rightarrow -1 \leq \lambda < 0, \quad (28)$$

$$(ii) \quad \sum_{x \in X} d(x) = 1 \Rightarrow \lambda = 0 \quad (29)$$

$$(iii) \quad \sum_{x \in X} d(x) < 1 \Rightarrow \lambda > 0, \quad (30)$$

Definition 10. [3] Let μ be a normalized fuzzy measure,

$g_\mu : 2^X \rightarrow [0, 1]$ be its measure function

(i) If the formula (31) is satisfied, then μ is called a normalized sub-additive measure,

$$\forall A, B \in 2^X, A \cap B = \emptyset \Rightarrow g_\mu(A \cup B) < g_\mu(A) + g_\mu(B) \quad (31)$$

(ii) If the formula (32) is satisfied, then μ is called a normalized additive measure,

$$\forall A, B \in 2^X, A \cap B = \emptyset \Rightarrow g_\mu(A \cup B) = g_\mu(A) + g_\mu(B) \quad (32)$$

(iii) If the formula (33) is satisfied, then μ is called a normalized supper-additive measure,

$$\forall A, B \in 2^X, A \cap B = \emptyset \Rightarrow g_\mu(A \cup B) > g_\mu(A) + g_\mu(B) \quad (33)$$

Theorem 4. [3] Let $d(x)$ be the fuzzy density function of a normalized λ -measure on a finite set X ,

(i) If $\sum_{x \in X} s(x) > 1$, then normalized λ -measure is a

normalized sub-additive measure

(ii) If $\sum_{x \in X} s(x) = 1$, then normalized λ -measure is a

normalized additive measure

(iii) If $\sum_{x \in X} s(x) < 1$, then normalized λ -measure is a

normalized supper-additive measure

E. Normalized P-measure

Definition 11. [4] For a given fuzzy density function $d(x)$ on a finite set X , a normalized fuzzy measure is called a normalized P-measure, if its measure function $g_P : 2^X \rightarrow [0, 1]$ satisfying

$$(i) \quad g_P(\emptyset) = 0, g_P(X) = 1 \text{ (normalized conditions)} \quad (34)$$

$$(ii) \quad \forall A \in 2^X \Rightarrow g_P(A) = \max_{x \in X} \{s(x)\} = \max_{x \in X} \{g_P(\{x\})\} \quad (35)$$

IV. REGULAR FUZZY MEASURES

Definition 12. A fuzzy density function $d(x)$ of a normalized fuzzy measure μ on a finite set X is called a regular fuzzy density function, if it satisfies

$$\forall x \in X \Rightarrow d(x) = g_\mu(\{x\}) > 0 \quad (36)$$

Definition 13. Any normalized fuzzy measure is called a regular fuzzy measure, if its fuzzy density function is a regular fuzzy density function

Definition 14.

(i) A normalized additive measure is called a regular additive measure, if its fuzzy density function is a regular fuzzy density function

(ii) A normalized sub-additive measure is called a regular sub-additive measure, if its fuzzy density function is a regular fuzzy density function.

(iii) A normalized supper-additive measure is called a regular supper-additive measure, if its fuzzy density function is a regular fuzzy density function

(iv) A normalized λ -measure is called a regular λ -measure, if its fuzzy density function is a regular fuzzy density function

(v) A normalized P-measure is called a regular P-measure, if its fuzzy density function is a regular fuzzy density function.

V. COMPARISON BETWEEN TWO MEASURES

A. B-measure

Definition 15. [6] For any given fuzzy density function, $d(x)$, on a finite set, X , a normalized fuzzy measure is called a B-measure, if its measure function, $g_B : 2^X \rightarrow [0, 1]$, satisfies

$$g_B(A) = \begin{cases} 0 & A = \emptyset \\ d(x) & A = \{x\}, x \in X \\ 1 & |A| > 1, A \subset X \end{cases} \quad (37)$$

Definition 16. A B-measure is called a regular B-measure, if its fuzzy density function is a regular fuzzy density function.

B. Definition of Comparisons of fuzzy measures

Definition 17. [6] Comparison of two fuzzy measure

For any given fuzzy density function, $d(x)$, on a finite set, X , let μ_1 and μ_2 be two normalized fuzzy measures on X ,

(i) If $g_{\mu_1}(A) = g_{\mu_2}(A)$, $\forall A \subset X$, then we say that μ_1 -measure is equal to μ_2 -measure, denoted as

$$\mu_1\text{-measure} = \mu_2\text{-measure}. \quad (38)$$

(ii) If $g_{\mu_1}(A) < g_{\mu_2}(A)$, $\forall A \subset X, 1 < |A| < |X|$ then we say that μ_1 -measure is less than μ_2 -measure, or μ_2 -measure is larger than μ_1 -measure, denoted as

$$\mu_1\text{-measure} < \mu_2\text{-measure}. \quad (39)$$

(iii) If $g_{\mu_1}(A) \leq g_{\mu_2}(A)$, $\forall A \subset X, 1 < |A| < |X|$, then we say that μ_1 -measure is not larger than μ_2 -measure, or μ_2 -measure is not smaller than μ_1 -measure, denoted as

$$\mu_1 - \text{measure} \leq \mu_2 - \text{measure}. \quad (40)$$

C. Important properties for comparisons of fuzzy measures

Theorem 5. For any given fuzzy density function $d(x)$ on a finite set X , P -measure is not larger than any other fuzzy measure μ , that is

$$P - \text{measure} \leq \mu - \text{measure} \quad (41)$$

Proof.

Let $\forall A \subset X$ and $1 < |A| < |X|$,

From the definition of monotonicity, we have

$$g_\mu(A) \geq g_\mu(\{x\}), \forall x \in A \quad (42)$$

And then

$$g_\mu(A) \geq \max_{x \in A} \{g_\mu(\{x\})\} \quad (43)$$

$$\text{Since } g_\mu(\{x\}) = g_p(\{x\}) = d(x), \forall x \in A \quad (44)$$

We can obtain

$$g_\mu(A) \geq \max_{x \in A} \{g_\mu(\{x\})\} = \max_{x \in A} \{g_p(\{x\})\} = g_p(A) \quad (45)$$

That is $g_p(A) \leq g_\mu(A)$, $\forall A \subset X$, the proof is completed.

Theorem 6. For any given fuzzy density function $d(x)$ on a finite set X , B -measure is not less than any other fuzzy measure μ , that is

$$\mu - \text{measure} \leq B - \text{measure} \quad (46)$$

Proof.

Let $\forall A \subset X$ and $1 < |A| < |X|$,

From the definition of B -measure, we have

$$g_\mu(A) \leq g_B(A) = 1, \forall x \in A \quad (47)$$

That is $g_\mu(A) \leq g_B(A)$, $\forall A \subset X$, the proof is completed

Theorem 7.

$$\text{regular } P - \text{measure} < \text{regular } \lambda - \text{measure}$$

Proof:

$$\text{i) Let } \forall A \subset X \text{ and } 1 < |A| < |X| \quad (48)$$

$$\text{Satisfy } A = B \cup C, B \neq \emptyset \neq C, B \cap C = \emptyset \quad (49)$$

From the definition of regular fuzzy measure, we know

$$\text{that } d(x) > 0, \forall x \in X \quad (50)$$

$$\text{Since } B \neq \emptyset \neq C, \text{ then } g_\lambda(B), g_\lambda(C), g_p(B), g_p(C) > 0 \quad (51)$$

$$\text{ii) From the definition of regular } \lambda - \text{measure, we know that } \lambda > -1 \quad (52)$$

$$\text{We can get } g_\lambda(C)[1 - \lambda g_\lambda(B)] > 0 \quad (53)$$

Then

$$\begin{aligned} g_\lambda(A) &= g_\lambda(B \cup C) = g_\lambda(B) + g_\lambda(C) - \lambda g_\lambda(B) g_\lambda(C) \\ &= g_\lambda(B) + g_\lambda(C)[1 - \lambda g_\lambda(B)] > g_\lambda(B), \end{aligned} \quad (54)$$

$$\text{That is } g_\lambda(A) = g_\lambda(B \cup C) > g_\lambda(B), \quad (55)$$

$$\text{Similarly we have } g_\lambda(A) = g_\lambda(B \cup C) > g_\lambda(C), \quad (56)$$

$$\text{Then } g_\lambda(A) = g_\lambda(B \cup C) > \max[g_\lambda(B), g_\lambda(C)] \quad (57)$$

In general, we can obtain

$$g_\lambda(A) > \max_{x \in A} \{g_\lambda(\{x\})\} = \max_{x \in A} \{g_p(\{x\})\} = g_p(A) \quad (58)$$

$$\Rightarrow g_\lambda(A) > g_p(A), \forall A \subset X, 1 < |A| < |X|$$

Then *regular* P -measure < *regular* λ -measure.

Theorem 8.

$$\text{regular } \lambda - \text{measure} < B - \text{measure}$$

Proof:

(i) Suppose that *regular* λ -measure < B -measure is not true, then there exist $A \subset X$ such that $1 < |A| < |X|$ satisfies

$$g_\lambda(A) = 1 \quad (59)$$

$$\text{(ii) Since } 1 < |A| < |X|, \text{ then } A' = X - A \neq \emptyset \quad (60)$$

$$\text{and } A \neq \emptyset \neq A' \quad (61)$$

From the definition of regular λ -measure,

we know that $\lambda > -1$ and $g_\lambda(A') > 0$

$$\text{then } g_\lambda(A')[1 + \lambda] > 0 \quad (62)$$

(iii) Since

$$\begin{aligned} g_\lambda(X) &= g_\lambda(A \cup A') \\ &= g_\lambda(A) + g_\lambda(A') + \lambda g_\lambda(A) g_\lambda(A') \\ &= g_\lambda(A) + g_\lambda(A')[1 + \lambda g_\lambda(A)] \\ &= 1 + g_\lambda(A')[1 + \lambda] > 1 \end{aligned} \quad (63)$$

But $g_\lambda(X) = 1$, it leads a contradiction,

hence λ -measure < B -measure.

Theorem 9.

$$\text{regular } P - \text{measure} < \text{regular } \lambda - \text{measure} < B - \text{measure} \quad (64)$$

Proof: It is trivial!

Theorem 10.

$$\text{regular } P - \text{measure} < \text{regular additive measure}$$

Proof:

$$\text{(i) Let } \forall A \subset X \text{ and } 1 < |A| < |X| \quad (65)$$

$$\text{Satisfy } A = B \cup C, B \neq \emptyset \neq C, B \cap C = \emptyset \quad (66)$$

From the definition of regular fuzzy measure, we know

$$\text{that } d(x) > 0, \forall x \in X \quad (67)$$

$$\text{Since } B \neq \emptyset \neq C, \text{ then } g_\sigma(B), g_\sigma(C) > 0 \quad (68)$$

$$\text{ii) From the definition of regular additive measure, we know that}$$

$$g_\sigma(A) = g_\sigma(B \cup C) = g_\sigma(B) + g_\sigma(C) \quad (69)$$

$$\text{That is } g_\lambda(A) = g_\lambda(B \cup C) > g_\lambda(B), \quad (70)$$

$$\text{and } g_\lambda(A) = g_\lambda(B \cup C) > g_\lambda(C), \quad (71)$$

$$\text{Then } g_\lambda(A) = g_\lambda(B \cup C) > \max[g_\lambda(B), g_\lambda(C)] \quad (72)$$

In general, we can obtain

$$g_{\sigma}(A) > \max_{x \in A} \{g_{\sigma}(\{x\})\} = \max_{x \in A} \{g_P(\{x\})\} = g_P(A) \quad (73)$$

$$\Rightarrow g_{\sigma}(A) > g_P(A), \forall A \subset X, 1 < |A| < |X|$$

Then *regular P-measure* < *regular additive measure*.

Theorem 11.

regular additive measure < *B-measure*

Proof:

(i) Suppose that *regular additive measure* < *B-measure* is not true,

then there exist $A \subset X$ such that $1 < |A| < |X|$ satisfies

$$g_{\sigma}(A) = 1 \quad (74)$$

(ii) Since $1 < |A| < |X|$, then $A' = X - A \neq \emptyset$ (75)

$$\text{and } A \neq \emptyset \neq A' \quad (76)$$

From the definition of regular additive measure,

$$\text{we know that } g_{\sigma}(A') > 0 \quad (77)$$

(iii) Since

$$\begin{aligned} g_{\sigma}(X) &= g_{\sigma}(A \cup A') = g_{\sigma}(A) + g_{\sigma}(A') \\ &> g_{\sigma}(A) = 1 \\ \Rightarrow g_{\sigma}(X) &> 1 \end{aligned} \quad (78)$$

But $g_{\lambda}(X) = 1$, it leads a contradiction,

Hence *regular additive measure* < *B-measure*.

Theorem 12.

regular P-measure < *regular additive measure* < *B-measure* (79)

Proof: It is trivial!.

Note that

- (i) If there is no confusion, for brief, the regular fuzzy measure is also called fuzzy measure.
- (ii) Both λ -measure and additive measure are larger than the smallest fuzzy measure, P-measure and less than the largest fuzzy measure, B-measure.

VI. YANG'S GENERALIZED LAMBDA MEASURE

Yang's generalized λ -measure was proposed by L.-Z. Yang, M.-H. Ha, X.-J. Wang and Z.-R. Zhang in 2008 [10], its formal definition is list as below'

Definition 18. [10] For a given fuzzy support function, $s(x)$, on a finite set X , a measure on X is a Yang's generalized λ -measure, if its measure function $g_{\lambda} : 2^X \rightarrow [0, \infty)$ satisfying

$$(i) \quad g_{\lambda}(\emptyset) = 0 \quad (\text{Vanishing at the empty set}) \quad (80)$$

$$(ii) \quad \begin{aligned} A, B \in 2^X, A \cap B = \emptyset, A \cup B \neq X \\ \Rightarrow g_{\lambda}(A \cup B) = g_{\lambda}(A) + g_{\lambda}(B) + \lambda g_{\lambda}(A) g_{\lambda}(B) \end{aligned} \quad (81)$$

$$(iii) \quad \forall A \subset X \Rightarrow g_{\lambda}(A) = \begin{cases} \frac{1}{\lambda} \left(-1 + \prod_{x \in A} [1 + \lambda s(x)] \right) & \lambda \neq 0 \\ \sum_{x \in A} s(x) & \lambda = 0 \end{cases} \quad (82)$$

Note that

- (i) Eq. (84) and Eq. (85) are called σ - λ -rule of λ -measure.
- (ii) To compare the definition of Yang's generalized λ -measure with the definition of λ -measure, we know that Yang's generalized λ -measure is just the λ -measure without the monotone condition $\lambda \in \left[-\frac{1}{\max_{A \subset X} \{g_{\lambda}(A)\}}, \infty \right)$, in other words, Yang's generalized λ -measure is a generalization of λ -measure not necessarily monotone,
- (iii) Since a fuzzy measure must be a monotone measure. If a Yang's generalized λ -measure does not satisfy the monotone condition, then it is not a fuzzy measure, hence Yang's generalized λ -measure is only a generalized measure of λ -measure, but not a generalized fuzzy measure of λ -measure

VII. LIU'S GENERALIZED LAMBDA FUZZY MEASURE

A. Definition of multivalent fuzzy measures

Definition 19. [6] Multivalent fuzzy measures

- (i) A fuzzy measure is called a univalent fuzzy measure, if it has only one fuzzy measure solution.
- (ii) A fuzzy measure is called a multivalent fuzzy measure, if its fuzzy measure solutions are more than one
- (iii) A normalized fuzzy measure is called a univalent normalized fuzzy measure, if it has only one normalized fuzzy measure solution.
- (iv) A normalized fuzzy measure is called a multivalent normalized fuzzy measure, if its normalized fuzzy measure solutions are more than one
- (v) A regular fuzzy measure is called a univalent regular fuzzy measure, if it has only one regular fuzzy measure solution.
- (vi) A regular fuzzy measure is called a multivalent regular fuzzy measure, if its regular fuzzy measure solutions are more than one

B. Definition of multivalent λ -measures

Definition 20. Multivalent λ -measures

$$\text{For any } \lambda \in \left[-\frac{1}{\max_{A \subset X} \{g_{\lambda}(A)\}}, \infty \right) \quad (83)$$

and a given fuzzy support function, $s(x)$ on a finite set X , a measure on X is a multivalent λ -measure, if its measure function $g_{\lambda} : 2^X \rightarrow [0, \infty)$ satisfying

$$(i) \quad g_{\lambda}(\emptyset) = 0 \quad (\text{Vanishing at the empty set}) \quad (84)$$

$$(ii) \quad \begin{aligned} \forall A, B \in 2^X, A \cap B = \emptyset \\ \Rightarrow g_{\lambda}(A \cup B) = g_{\lambda}(A) + g_{\lambda}(B) + \lambda g_{\lambda}(A) g_{\lambda}(B) \end{aligned} \quad (85)$$

$$(iii) \quad \forall A \subset X \Rightarrow g_{\lambda}(A) = \begin{cases} \frac{1}{\lambda} \left(-1 + \prod_{x \in A} [1 + \lambda s(x)] \right) & \lambda \neq 0 \\ \sum_{x \in A} s(x) & \lambda = 0 \end{cases} \quad (86)$$

Note that multivalent λ -measure do not need the condition;

$$\prod_{x \in X} [1 + \lambda s(x)] = 1 + \lambda > 0, s(x) = g_\lambda(\{x\})$$

Theorem 13. Multivalent λ - measure is a multivalent fuzzy measure

Proof;

(I) For any $\lambda \in \left[-\frac{1}{\max_{E \subset X} \{g_\lambda(E)\}}, \infty\right)$, let $\forall A, B \in 2^X, A \subseteq B$,

$$\text{to prove that } g_\lambda(A) \leq g_\lambda(B) \quad (87)$$

$$\text{Since } \lambda \in \left[-\frac{1}{\max_{E \subset X} \{g_\lambda(E)\}}, \infty\right),$$

$$\text{then } \lambda > -\frac{1}{\max_{E \subset X} \{g_\lambda(E)\}} \Rightarrow 1 + \lambda \max_{E \subset X} \{g_\lambda(E)\} > 0 \quad (88)$$

$$\text{(i) if } -\frac{1}{\max_{E \subset X} \{g_\lambda(E)\}} < \lambda < 0 \quad (89)$$

$$\text{Then } 1 + \lambda g_\lambda(A) \geq 1 + \lambda \max_{E \subset X} \{g_\lambda(E)\} > 0 \quad (90)$$

$$\begin{aligned} g_\lambda(B) &= g_\lambda(A \cup (B-A)) \\ &= g_\lambda(A) + g_\lambda(B-A) + \lambda g_\lambda(A) g_\lambda(B-A) \\ &= g_\lambda(A) + g_\lambda(B-A) [1 + \lambda g_\lambda(A)] \\ &\geq g_\lambda(A) + g_\lambda(B-A) \left[1 - \frac{g_\lambda(A)}{\max_{E \subset X} \{g_\lambda(E)\}}\right] \\ &\geq g_\lambda(A) \end{aligned} \quad (91)$$

$$\text{(ii) if } \lambda \geq 0 \quad (92)$$

$$\begin{aligned} g_\lambda(B) &= g_\lambda(A \cup (B-A)) \\ &= g_\lambda(A) + g_\lambda(B-A) + \lambda g_\lambda(A) g_\lambda(B-A) \\ &\geq g_\lambda(A) \end{aligned} \quad (93)$$

(II) For any $\forall A \subset X$ to prove $g_\lambda(\emptyset) = 0$

$$\begin{aligned} g_\lambda(\emptyset) &= g_\lambda(\emptyset \cup \emptyset) = g_\lambda(\emptyset) + g_\lambda(\emptyset) + \lambda g_\lambda(\emptyset) g_\lambda(\emptyset) \\ &= 2g_\lambda(\emptyset) + \lambda [g_\lambda(\emptyset)]^2 \\ &\Rightarrow g_\lambda(\emptyset) + \lambda [g_\lambda(\emptyset)]^2 = g_\lambda(\emptyset) [1 + \lambda g_\lambda(\emptyset)] = 0 \end{aligned} \quad (94)$$

$$\text{Since } [1 + \lambda g_\lambda(\emptyset)] > 0$$

$$\Rightarrow g_\lambda(\emptyset) = 0$$

The proof is completed.

Definition 21. Multivalent normalized λ - measures

$$\text{For any } \lambda \in [-1, \infty) \quad (95)$$

and a given fuzzy density function $d(x)$ on a finite set X , a measure on X is a multivalent normalized λ - measure, if its measure function $g_\lambda : 2^X \rightarrow [0, 1]$ satisfying

$$\text{(i) } g_\lambda(\emptyset) = 0, g_\lambda(X) = 1 \text{ (normalized conditions)} \quad (96)$$

$$\text{(ii) } \forall A, B \in 2^X, A \cap B = \emptyset$$

$$\Rightarrow g_\lambda(A \cup B) = [g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)] \wedge 1 \quad (97)$$

$$\text{(iii) } \forall A \subset X \Rightarrow g_\lambda(A) = \begin{cases} \left[\frac{1}{\lambda} \left(-1 + \prod_{x \in A} [1 + \lambda d(x)] \right) \right] \wedge 1 & \lambda \neq 0 \\ \sum_{x \in A} d(x) & \lambda = 0 \end{cases} \quad (98)$$

Note that multivalent normalized λ - measure also do not need the condition; $\prod_{x \in X} [1 + \lambda d(x)] = 1 + \lambda > 0, d(x) = g_\lambda(\{x\})$.

Theorem 14. Multivalent normalized λ - measure is a multivalent normalized fuzzy measure

Proof;

(I) For any $\lambda \in [-1, \infty)$, let $\forall A, B \in 2^X, A \subseteq B$,

$$\text{to prove that } g_\lambda(A) \leq g_\lambda(B)$$

$$\text{(i) if } -1 \leq \lambda \leq 0 \text{ then } 1 + \lambda g_\lambda(A) \geq 1 - g_\lambda(A) \geq 0 \quad (99)$$

$$\begin{aligned} g_\lambda(B) &= g_\lambda(A \cup (B-A)) \\ &= g_\lambda(A) + g_\lambda(B-A) + \lambda g_\lambda(A) g_\lambda(B-A) \\ &= g_\lambda(A) + g_\lambda(B-A) [1 + \lambda g_\lambda(A)] \geq g_\lambda(A) \end{aligned} \quad (100)$$

(ii) if $\lambda > 0$ then

$$\begin{aligned} g_\lambda(B) &= g_\lambda(A \cup (B-A)) \\ &= g_\lambda(A) + g_\lambda(B-A) + \lambda g_\lambda(A) g_\lambda(B-A) \\ &= g_\lambda(A) + g_\lambda(B-A) [1 + \lambda g_\lambda(A)] \geq g_\lambda(A) \end{aligned} \quad (101)$$

(II) For any $\lambda \in [-1, \infty)$, and any $A \subset X$ to prove

$$0 \leq g_\lambda(A) \leq 1 \quad (102)$$

$$\text{From (I), we know that } 0 = g_\lambda(\emptyset) \leq g_\lambda(A) \leq g_\lambda(X) = 1$$

The proof is completed.

Definition 22. Multivalent regular λ - measures

$$\text{For any } \lambda \in [-1, \infty) \quad (103)$$

and a given fuzzy regular density function $d(x)$ on a finite set X , a measure on X is a multivalent regular λ - measure, if its measure function $g_\lambda : 2^X \rightarrow [0, 1]$ satisfying

$$\text{(i) } g_\lambda(\emptyset) = 0, g_\lambda(X) = 1 \text{ (Normalized conditions)} \quad (104)$$

$$\text{(ii) } \forall A, B \in 2^X, A \cap B = \emptyset$$

$$\Rightarrow g_\lambda(A \cup B) = [g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B)] \wedge 1 \quad (105)$$

$$\text{(iii) } \forall A \subset X \Rightarrow g_\lambda(A) = \begin{cases} \left[\frac{1}{\lambda} \left(-1 + \prod_{x \in A} [1 + \lambda d(x)] \right) \right] \wedge 1 & \lambda \neq 0 \\ \sum_{x \in A} d(x) & \lambda = 0 \end{cases} \quad (106)$$

Theorem 15. Multivalent regular λ - measure is a multivalent regular fuzzy measure

Proof: it is trivial.

C. Completed measure

Definition 23 [6] Completed measure

For any given fuzzy density function, $d(x)$, on a finite set, X , a multivalent fuzzy measure, μ -measure, with determined coefficient, $\mu \in R$ it is called a completed fuzzy measure, if it satisfies

$$(i) \quad \forall \mu \in [a, b] \ni \mu\text{-measure is a fuzzy measure}$$

$$(ii) \quad \mu = a \Rightarrow g_\mu(A) = g_p(A), \forall A \subset X \quad (107)$$

$$(iii) \quad \mu = b \Rightarrow g_\mu(A) = g_p(A), \forall A \subset X \quad (108)$$

Note that all of regular λ -measure, multivalent λ -measure and multivalent normalized λ -measure are not completed fuzzy measure.

Definition 23 Liu's generalized λ -measure

For any given fuzzy density function $d(x)$ on a finite set X ,

$$\text{And } \forall \lambda \in [a, b] = \left[\frac{-2}{\min_{0 \neq x \in X} \{d(x)\}}, \frac{1}{\left[\min_{0 \neq x \in X} \{d(x)\} \right]^2} \right] \quad (109)$$

a measure on X is called a Liu's generalized λ -measure, if its measure function $g_\lambda : 2^X \rightarrow [0, 1]$ satisfying

$$(i) \quad g_\lambda(\emptyset) = 0, g_\lambda(X) = 1 \quad (\text{Normalized conditions}) \quad (110)$$

$$(ii) \quad \forall A, B \in 2^X, A \cap B = \emptyset$$

$$\Rightarrow g_\lambda(A \cup B)$$

$$\begin{cases} \frac{\lambda+1}{a+1} \max_{x \in A \cup B} \{d(x)\} + \frac{a-\lambda}{a+1} [g_\lambda(A) + g_\lambda(B) - g_\lambda(A)g_\lambda(B)] & a \leq \lambda < -1 \\ [g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A)g_\lambda(B)] & -1 \leq \lambda \leq 0 \\ [g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A)g_\lambda(B)] \wedge 1 & 0 < \lambda \leq b \end{cases} \quad (111)$$

$$a = \frac{-2}{\min_{0 \neq x \in X} \{d(x)\}} < -1, b = \frac{1}{\left[\min_{0 \neq x \in X} \{d(x)\} \right]^2} > 1 \quad (112)$$

$$(iii) \quad \forall A \subset X \Rightarrow g_\lambda(A) = \begin{cases} \frac{\lambda+1}{a+1} \max_{x \in A} \{d(x)\} + \frac{a-\lambda}{a+1} f_{-1}(A) & a \leq \lambda < -1 \\ f_\lambda(A) & -1 \leq \lambda < 0 \\ \sum_{x \in A} d(x) & \lambda = 0 \\ [f_\lambda(A)] \wedge 1 & 0 < \lambda \leq b \end{cases} \quad (113)$$

$$\text{where } f_\lambda(A) = \left[\frac{1}{\lambda} \left(-1 + \prod_{x \in A} [1 + \lambda d(x)] \right) \right] \quad (114)$$

Theorem 16. Liu's generalized λ -measure is a Completed, continuous and increasing normalized fuzzy measure

Proof:

(I) Let $\forall A \in 2^X$ to prove that $0 \leq g_\lambda(A) \leq 1$,

From (113) and the definition of fuzzy density function we know that

$$\text{If } A = \emptyset, \text{ or } A = X, \text{ or } |A| = 1 \Rightarrow 0 \leq g_\lambda(A) \leq 1 \quad (115)$$

$$\text{If } |A| > 1, A = B \cup C, B \cap C = \emptyset, B \neq \emptyset \neq C \quad (116)$$

From (113), we know that

$$0 \leq \max_{x \in A} \{d(x)\} = \max_{x \in B \cup C} \{d(x)\} \leq g_\lambda(B \cup C) = g_\lambda(A) \leq 1 \quad (117)$$

(II) Let $\forall A, B \in 2^X, A \subseteq B$ to prove that $g_\lambda(A) \leq g_\lambda(B)$

From (102) and (106), we can obtain that

$$\forall A \in 2^X \Rightarrow g_\lambda(A) \geq \max_{x \in A} \{d(x)\} = g_p(A) \quad (118)$$

Since $B = A \cup (B - A)$, then we have

$$\begin{aligned} g_\lambda(B) &= g_\lambda(A \cup (B - A)) \\ &= \begin{cases} \frac{\lambda+1}{a+1} \max_{x \in B} \{d(x)\} + \frac{a-\lambda}{a+1} [g_\lambda(A) + g_\lambda(B - A) - g_\lambda(A)g_\lambda(B - A)] & a \leq \lambda < -1 \\ [g_\lambda(A) + g_\lambda(B - A) + \lambda g_\lambda(A)g_\lambda(B - A)] & -1 \leq \lambda \leq 0 \\ [g_\lambda(A) + g_\lambda(B - A) + \lambda g_\lambda(A)g_\lambda(B - A)] \wedge 1 & 0 < \lambda \leq b \end{cases} \quad (119) \end{aligned}$$

(i) if $0 < \lambda \leq b$, then $g_\lambda(A) \leq g_\lambda(B)$ is trivial.

(ii) if $-1 \leq \lambda \leq 0$, since $0 \leq g_\lambda(A) \leq 1$ then

$$\begin{aligned} \lambda g_\lambda(A) &\geq \lambda \geq -1 \text{ and } [1 + \lambda g_\lambda(A)] \geq 0 \text{ we can obtain} \\ g_\lambda(B) &= g_\lambda(A) + g_\lambda(B - A) + \lambda g_\lambda(A)g_\lambda(B - A) \\ &= g_\lambda(A) + g_\lambda(B - A) [1 + \lambda g_\lambda(A)] \geq g_\lambda(A) \end{aligned} \quad (120)$$

That is $g_\lambda(A) \leq g_\lambda(B)$

$$(iii) \text{ if } a \leq \lambda < -1 \text{ then } g_\lambda(B) = \frac{\lambda+1}{a+1} g_p(B) + \frac{a-\lambda}{a+1} f_{-1}(B) \quad (121)$$

Where $g_p(A) \leq g_p(B), f_{-1}(A) \leq f_{-1}(B)$

We can obtain

$$g_\lambda(A) = \frac{\lambda+1}{a+1} g_p(A) + \frac{a-\lambda}{a+1} f_{-1}(A) \leq \frac{\lambda+1}{a+1} g_p(B) + \frac{a-\lambda}{a+1} f_{-1}(B) = g_\lambda(B) \quad (122)$$

And we know that Liu's generalized λ -measure is a fuzzy measure.

$$(II) \quad \text{Let } \lambda = a = \frac{-2}{\min_{0 \neq x \in X} \{d(x)\}} \text{ to prove}$$

$$g_\lambda(A) = g_a(A) = \max_{x \in A} \{d(x)\} = g_p(A) \quad (123)$$

$$\text{Let } \forall A, B \subset X, A \cap B = \emptyset, \text{ and } \lambda = a = \frac{-2}{\min_{0 \neq x \in X} \{d(x)\}}$$

then

$$\begin{aligned} a &\leq \left(\frac{1}{\min_{0 \neq x \in A} \{d(x)\}} + \frac{1}{\min_{0 \neq x \in B} \{d(x)\}} \right) \leq \left(\frac{1}{g_a(A)} + \frac{1}{g_a(B)} \right) \\ &\leq \frac{-g_a(A) - g_a(B)}{g_a(A)g_a(B)} \leq \frac{\max_{x \in A \cup B} \{d(x)\} - g_a(A) - g_a(B)}{g_a(A)g_a(B)} \\ &\Rightarrow \alpha g_a(A)g_a(B) \leq \max_{x \in A \cup B} \{d(x)\} - g_a(A) - g_a(B), \forall A, B \subset X \quad (124) \\ &\Rightarrow [g_a(A) + g_a(B) + \alpha g_a(A)g_a(B)] \leq \max_{x \in A \cup B} \{d(x)\} \end{aligned}$$

Then we can obtain

$$\begin{aligned}
& g_a(A \cup B) \\
&= [g_a(A) + g_a(B) + a g_a(A) g_a(B)] \vee \max_{x \in A \cup B} \{d(x)\} = \max_{x \in A \cup B} \{d(x)\} \quad (125) \\
&\Rightarrow g_a(A \cup B) = \max_{x \in A \cup B} \{d(x)\}
\end{aligned}$$

That is

$$g_\lambda(A) = g_a(A) = \max_{x \in A} \{d(x)\} = g_p(A), \forall A \subseteq X$$

$$(III) \quad \text{Let } \lambda = b = \frac{1}{\left[\min_{0 \neq x \in X} \{d(x)\} \right]^2} \quad (126)$$

$$\text{to prove } g_\lambda(A) = g_b(A) = g_B(A), \forall A \subseteq X, |A| > 1 \quad (127)$$

let

$$\forall A, B \subseteq X, A \cap B = \emptyset, g_\lambda(A) g_\lambda(B) > 0,$$

$$|A \cup B| > 1, \lambda = b = \frac{1}{\left[\min_{0 \neq x \in X} \{d(x)\} \right]^2} \quad (128)$$

$$\text{then } \lambda = b = \frac{1}{\left[\min_{0 \neq x \in X} \{d(x)\} \right]^2} \geq \frac{1}{[g_b(A) \wedge g_b(B)]^2} \quad (129)$$

$$\geq \frac{1}{g_b(A) g_b(B)} \geq \frac{1 - g_b(A) - g_b(B)}{g_b(A) g_b(B)}$$

$$\begin{aligned}
&\Rightarrow g_b(A) + g_b(B) + b g_b(A) g_b(B) \geq 1, \forall A, B \subseteq X \\
&\Rightarrow g_b(A \cup B) = [g_b(A) + g_b(B) + b g_b(A) g_b(B)] \wedge 1 = 1
\end{aligned} \quad (130)$$

In other words, $\forall A \subseteq X, |A| > 1 \Rightarrow g_b(A) = 1$

Then $g_\lambda(A) = g_b(A) = g_B(A), \forall A \subseteq X, |A| > 1$

- (IV) From above we know that Liu's generalized λ -measure is a Completed normalized fuzzy measure
- (V) It is trivial that the measure function of Liu's generalized λ -measure is a continuous and increasing function on λ value.

VIII. CONCLUSIONS AND FUTURE WORKS

In this paper, first of all, it is proved that both λ -measure and additive measure are never equal to the smallest fuzzy measure, P-measure and the largest fuzzy measure, B-measure.

Second, this paper pointed out that the generalized λ -measure proposed by L.-Z. Yang, M.-H. Ha, X.-J. Wang, and Z.-R. Zhang, is a generalized measure of λ -measure, but not a generalized fuzzy measure of λ -measure, since a fuzzy measure must be a monotone measure.

Furthermore, a novel multivalent fuzzy measure with infinitely many fuzzy measure solutions based on λ -measure,

called Liu's generalized λ -measure, is proposed. It is proved that the measure function of this new measure is continuous and increasing on λ , each different λ value corresponding different fuzzy measure, from the smallest fuzzy measure, P-measure to the largest fuzzy measure, B-measure, and containing additive measure and λ -measure itself. Moreover, it is a generalize fuzzy measure of λ -measure satisfying the monotone condition, and rather than the old generalized λ -measure proposed by L.-Z. Yang, M.-H. Ha, X.-J. Wang, and Z.-R. Zhang. Some properties about this new measure are also discussed.

In the future, we can use Choquet integral with respect to this new fuzzy measure to analysis forecasting problem and multiple criterion decision making problem.

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