

Relational Structure Analysis of Fuzzy Graph and its Application

For Analyzing Fuzzy Data of Human Relation

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Abstract— Generally, we could efficiently analyze the inexact information and investigate the fuzzy relation by applying the fuzzy graph theory^[1].

We would extend the fuzzy graph theory, and propose a fuzzy node fuzzy graph. Since a fuzzy node fuzzy graph is complicated to analyze, we would transform it to a simple fuzzy graph by using T-norm family. In addition, to investigate the relations between nodes, we would define the fuzzy contingency table.

In this paper, we would discuss about five subjects, (1) new T-norm "Uesu product", (2) fuzzy node fuzzy graph, (3) fuzzy contingency table, (4) decision analysis of the optimal fuzzy graph G_{λ_0} in the fuzzy graph sequence $\{G_{\lambda}\}$ and (5) its application to sociometry analysis.

By using the fuzzy node fuzzy graph theory, the new T-norm and the fuzzy contingency table, we could clarify the relational structure of fuzzy information. According to the decision method in section 2, we could find the optimal fuzzy graph G_{λ_0} in the fuzzy graph sequence $\{G_{\lambda}\}$, and clarify the structural feature of the fuzzy node fuzzy graph.

Moreover, we would illustrate its practical effectiveness with the case study concerning sociometry analysis.

Keywords-component; fuzzy node fuzzy graph; optimal fuzzy graph; sociometry analysis; T-norm; contingency table

I. T-NORM, FUZZY NODE FUZZY GRAPH AND FUZZY CONTINGENCY TABLE

A. T-norm and its Family

T-norm (Triangular Norm) is a binary operation that is defined by the following:

Definition 1 T-norm is a binary operation

$$p, q \in [0,1] \rightarrow T(p, q) \in [0,1]$$

satisfying the following properties:

(1) Commutativity :

$$T(p, q) = T(q, p)$$

(2) Associativity :

$$T(p, T(q, r)) = T(T(p, q), r)$$

(3) Monotonicity :

$$p \leq q, \quad r \leq s \Rightarrow T(p, r) \leq T(q, s)$$

(4) Boundary conditions :

$$T(p, 0) = 0, \quad T(p, 1) = p$$

The typical T-norms are the followings:

(1) Logical product :

$$T_L(p, q) = p \wedge q$$

(2) Algebraic product :

$$T_A(p, q) = pq$$

(3) Multi-valued product (Lukasiewicz product) :

$$T_M(p, q) = (p + q - 1) \vee 0$$

(4) Drastic product :

$$T_D(p, q) = \begin{cases} 0 & , p \vee q < 1 \\ p \wedge q, p \vee q = 1 \end{cases}$$

Here, “ $a \wedge b$ ” means “ $\min(a, b)$ ”, “ $a \vee b$ ” means “ $\max(a, b)$ ”.

Definition 2 Order of T-norms

For two T-norms T_α and T_β , if a relation

$$T_\alpha(p, q) \leq T_\beta(p, q), (p, q) \in [0, 1]^2$$

holds, we denote it by $T_\alpha \leq T_\beta$.

For any T-norm T , a relation $T_D \leq T \leq T_L$ always holds.

Definition 3 T-norm Family

For any $\lambda \in [a, b]$, when T_λ is T-norm, then we say that $\{T_\lambda\}$ is T-norm family that connects T_a with T_b .

The typical T-norm families are

(1) Dubois product :

$$T_\lambda(p, q) = \frac{pq}{p \vee q \vee \lambda}, \lambda \in [0, 1]$$

(2) Weber product :

$$T_\lambda(p, q) = 0 \vee \{(1 + \lambda)(p + q - 1) - \lambda pq\}, \lambda \geq -1$$

(3) Schweizer product :

$$T_\lambda(p, q) = \sqrt[\lambda]{0 \vee (p^\lambda + q^\lambda - 1)}, \lambda > 0$$

(4) Uesu product :

$$T_\lambda(p, q) = \begin{cases} 0 & , p \vee q < 1 - \lambda \\ p \wedge q, p \vee q \geq 1 - \lambda \end{cases}, \lambda \in [0, 1]$$

and so on.

The relation of typical T-norm families and Uesu product could be illustrated in Fig.1.

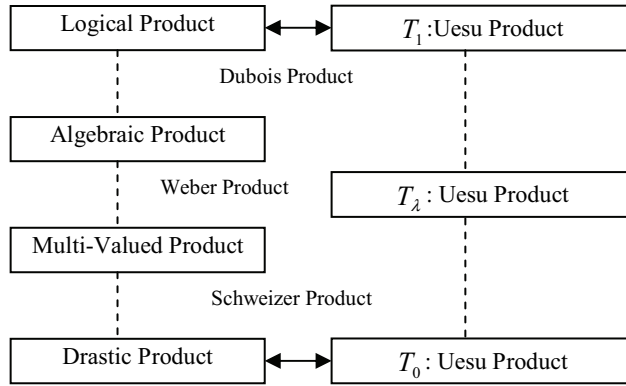


Figure 1. Relation of Typical T-norm Families

Here, in Fig.1, we could understand that Dubois product connects “logical product” with “algebraic product”, Weber product does “algebraic product” with “multi-valued product”, Schweizer product does “mulch-valued product” with “drastic product” and Uesu product does “logical product” with “drastic product”.

B. Fuzzy Node Fuzzy Graph

Definition 4 Crisp Node Fuzzy Graph

A crisp node fuzzy graph $G^{[6]}$ is defined by

$$G = (V, F) : V = \{v_i\}, F = (f_{ij}), 0 \leq f_{ij} \leq 1$$

where V is the set of the nodes and F is an $n \times n$ matrix. (i, j) component f_{ij} of F is a fuzziness of the arc from the node v_i to the node v_j .

Definition 5 Fuzzy Node Fuzzy Graph

A fuzzy node fuzzy graph $G^{[14]}$ is defined by

$$G = (V, F) : V = \{v_i / u_i\}, Y = (y_{ij}), 0 \leq u_i \leq 1, 0 \leq y_{ij} \leq 1$$

where V is the set of the nodes and the fuzziness u_i is a fuzziness of the node v_i . Y is a $n \times n$ matrix whose (i, j) component y_{ij} is a fuzziness of the arc from the node v_i to the node v_j .

A fuzzy node fuzzy graph is characterized by the fuzziness of the nodes and the fuzziness of the arcs. Therefore, the structure of a fuzzy node fuzzy graph is usually very complicated. Then, it should be interesting to transform a fuzzy node fuzzy graph

to a crisp node fuzzy graph and we present a method to transform a fuzzy node fuzzy graph to a crisp node fuzzy graph.

Definition 6 Transformation from Fuzzy Node Fuzzy Graph to Crisp Node Fuzzy Graph

A fuzzy node fuzzy graph $G = (V, Y)$ can be transformed to a crisp node fuzzy graph $G = (V, F)$ by the following method: Let

$$G = (V, F) : V = \{v_i\}, F = (f_{ij}), f_{ij} = T(u_i, y_{ij})$$

where the fuzziness f_{ij} of the arc from the node v_i to the node v_j could be defined by applying T-norms.

This fuzzy node fuzzy graph analysis would be applied to the sociometry analysis, the instruction structure analysis and so on.

C. Fuzzy Contingency Table

Definition 7 Fuzzy Contingency Table

Let x and y be two vectors as follows:

$$x = (x_i), y = (y_i) : 0 \leq x_i, y_i \leq 1, 1 \leq i \leq n.$$

Then the fuzzy contingency table of x and y defined accordingly:

TABLE I. FUZZY CONTINGENCY TABLE

$x \backslash y$	1	0	Sum
1	a_{xy}	b_{xy}	$\sum_{k=1}^n x_k$
0	c_{xy}	d_{xy}	$n - \sum_{k=1}^n x_k$
Sum	$\sum_{k=1}^n y_k$	$n - \sum_{k=1}^n y_k$	n

$$a_{xy} = \sum_{k=1}^n T(x_k, y_k), b_{xy} = \sum_{k=1}^n x_k - a_{xy},$$

$$c_{xy} = \sum_{k=1}^n y_k - a_{xy}, d_{xy} = n - \sum_{k=1}^n y_k - b_{xy}$$

From the fuzzy contingency table, we could computes any similarity coefficients. The typical similarity coefficients are

$$(1) \text{ Jaccard : } s_{xy} = \frac{a}{a+b+c}$$

$$(2) \text{ Sorensen-Dice : } s_{xy} = \frac{2a}{2a+b+c}$$

$$(3) \text{ Russell and Rao : } s_{xy} = \frac{a}{a+b+c+d}$$

$$(4) \text{ Simple Matching : } s_{xy} = \frac{a+d}{a+b+c+d}$$

$$(5) \text{ Rogers-Tanimoto : } s_{xy} = \frac{a+d}{a+2b+2c+d}$$

$$(6) \text{ Simpson : } s_{xy} = \frac{a}{(a+b) \wedge (a+c)}$$

$$(7) \text{ Ochiai : } s_{xy} = \frac{a}{\sqrt{(a+b)(a+c)}}$$

$$(8) \text{ Phi : } s_{xy} = \frac{ad - bc}{\sqrt{(a+b)(a+c)(b+d)(c+d)}}$$

$$(9) \text{ Yule : } s_{xy} = \frac{ad - bc}{ad + bc}$$

and so on.

II. FUZZY GRAPH AND ITS ANALYSIS

A. Cluster Analysis of Fuzzy Graph

A fuzzy graph G is defined by

$$G = (V, Y), V = \{v_i\}, F = (f_{ij}), 0 \leq f_{ij} \leq 1, f_{ii} = 1, 1 \leq i \leq n$$

where f_{ij} is a fuzziness of the arc from the node v_i to the node v_j .

In order to analyze the similarity structure among nodes of a fuzzy graph, we use the symmetric relation matrix $S = (s_{ij})$. A symmetric relation matrix S could be defined by using the fuzzy contingency table and any similarity coefficients. Here, we define the symmetric relation matrix S by using the simple matching similarity coefficient.

In order to analyze the clustering structure among nodes, we have its max-min transitive closure $\hat{S} = (\hat{s}_{ij})$ which is computed by $\hat{S} = S^n$.

After that, we define the c -cut matrix S_c of $\hat{S} = (\hat{s}_{ij})$ as follows:

$$S_c = (s_{ij}^c), \quad s_{ij}^c = \begin{cases} 1 & (\hat{s}_{ij} \geq c) \\ 0 & (\hat{s}_{ij} < c) \end{cases}, 0 \leq c \leq 1$$

From the matrix S_c , we define the cluster $CL_{S_c}(i)$.

$$J_c(i) = \{j \mid s_{ij}^c = 1, 1 \leq j \leq n\}$$

$$CL_{S_c}(i) = \{v_j \mid j \in J_c(i)\}$$

The cluster $CL_{S_c}(i)$ gives an equivalence relation among nodes.

Thence, we can construct the partition tree by changing the level c of the c -cut matrix which represents the clustering situation of nodes in a fuzzy graph.

B. Decision Analysis of Optimal Value λ_0

Concerning the transformation from a fuzzy node fuzzy graph $G = (V, Y)$ to a fuzzy graph $G_\lambda = (V, F_\lambda)$, we could analyze a fuzzy node fuzzy graph by using Uesu product T_λ .

By changing the parameter λ , a sequence $\{G_\lambda\}$ of fuzzy graphs is composed.

Here, we would choose the optimal fuzzy graph G_{λ_0} from this fuzzy graph sequence $\{G_\lambda\}$.

Then, we would discuss the decision method of the optimal parameter λ_0 .

In order to decide the optimal fuzzy graph G_{λ_0} , we would define two functions $d(\lambda)$ and $e(\lambda)$ as follows:

Definition 8 Distance Function $d(\lambda)$ and Connectivity Function $e(\lambda)$

$$d(\lambda) = d(F_\lambda, S_{c_0}) = \frac{1}{n^2 - n} \sum_{i=1}^n \sum_{j=1}^n |f_{ij}^\lambda - s_{ij}^{c_0}|$$

$$e(\lambda) = e(F_\lambda) = \frac{\gamma(F_\lambda)}{n^2 - n}$$

$$\text{where, } \gamma(F_\lambda) = \#(\Gamma_\lambda), \Gamma_\lambda = \{f_{ij}^\lambda \in F_\lambda \mid f_{ij}^\lambda > 0\}.$$

And F_λ, S_{c_0} are given as follows:

$$F_\lambda = (f_{ij}^\lambda),$$

$$f_{ij}^\lambda = \begin{cases} 0 & , u_i \vee y_{ij} < 1 - \lambda \\ u_i \wedge y_{ij}, u_i \vee y_{ij} \geq 1 - \lambda \end{cases}$$

$$S_{c_0} = (s_{ij}^{c_0}), \quad s_{ij}^{c_0} = \begin{cases} 1 & (\hat{s}_{ij} \geq c_0) \\ 0 & (\hat{s}_{ij} < c_0) \end{cases}$$

Here, $d(\lambda)$ evaluates the feature between the fuzzy node fuzzy graph $G_\lambda = (V, F_\lambda)$ and the optimal clustering cut level c_0 . If the value of $d(\lambda)$ is large, then G_λ reasonably shows the feature of the clustering level c_0 .

On the other hand, $e(\lambda)$ evaluates the connectivity information of G_λ . If the value of $e(\lambda)$ is small, then G_λ reasonably shows the feature of the connectivity information.

Here, we would normalize the values of $d(\lambda)$ and $e(\lambda)$ respectively, and define $f_d(\lambda)$ and $f_e(\lambda)$ as follows:

Definition 9 Fuzzy Distance Function $f_d(\lambda)$ and Fuzzy Connectivity Function $f_e(\lambda)$

$$f_d(\lambda) = \frac{d_M - d(\lambda)}{d_M - d_m}, \quad f_e(\lambda) = \frac{e(\lambda) - e_m}{e_M - e_m}$$

where,

$$d_M = \bigvee_{\lambda \in [0,1]} \{d(\lambda)\}, d_m = \bigwedge_{\lambda \in [0,1]} \{d(\lambda)\}, e_M = \bigvee_{\lambda \in [0,1]} \{e(\lambda)\} \text{ and } e_m = \bigwedge_{\lambda \in [0,1]} \{e(\lambda)\}.$$

By applying the maximal decision of the fuzzy decision, we could reasonably find the optimal value λ_0 concerning the sequence $\{G_\lambda\}$.

Definition 10 Decision of Optimal Value λ_0

$$f_m(\lambda) = f_d(\lambda) \wedge f_e(\lambda)$$

$$\lambda_0 = \bigwedge \{\lambda : f_m(\lambda) = \bigvee_{x \in [0,1]} f_m(x)\}$$

III. APPLICATION TO SOCIOGRAM ANALYSIS

We would present a practical case study of the fuzzy sociogram analysis^{[5],[11]} that has been experimented in high

school.

If we executed the questionnaires in Fig.2 to 10 students, we could obtain the response table L .

From the table L , we obtained the response matrix $K^{(1)}$, $K^{(2)}$, $K^{(3)}$ and the evaluation matrix R .

By analyzing the matrix R , we have the preferring matrix $Y=(y_{ij})$ in Fig.3 and the amicable matrix $S=(s_{ij})$ in Fig.4. From the amicable matrix S , we obtained the fuzziness u_i of member v_i . We illustrated the fuzzy node fuzzy graph $G=(V, Y)$.

We could transform the fuzzy node fuzzy graph $G=(V, Y)$ to the fuzzy graph $G=(V, F)$ by applying Uesu product . By changing parameter λ , we obtained fuzzy graph sequence $\{G_\lambda\}$.

Next, we could obtain the partition tree P in Fig.5 if we execute the cluster analysis concerning the amicability matrix S .

Here, we could decide the optimal cut level $c_0=0.78$.

By using c_0 -cut matrix Sc_0 and the fuzzy graph sequence $\{G_\lambda\}$, we could decide the optimal fuzzy graph G_{λ_0} . Here, by calculating the distance function $d(\lambda)$ and the connectivity function $e(\lambda)$.

$$d(\lambda)=\frac{1}{90}\sum_{i=1}^{10}\sum_{j=1}^{10}\left|f_{ij}^\lambda-s_{ij}^{c_0}\right|$$

$$e(\lambda)=\frac{\gamma(F_\lambda)}{90}$$

where, $\gamma(F_\lambda)=\#(\Gamma_\lambda)$, $\Gamma_\lambda=\{f_{ij}^\lambda\in F_\lambda\left|f_{ij}^\lambda>0\right.\}$.

Then, we have normalized the value of $d(\lambda)$ and $e(\lambda)$, so we have calculated $f_d(\lambda)$ and $f_e(\lambda)$ as follows:

$$f_d(\lambda)=\frac{0.23-d(\lambda)}{0.23-0.15}, f_e(\lambda)=\frac{e(\lambda)-0.21}{0.64-0.21}.$$

Select your friends and write them in order for each group activities.

Q₁ : Study in Group

Q₂ : Take Lunch Together

Q₃ : Do Sport Together

	1	2	3	4	...
Q ₁					
Q ₂					
Q ₃					

Figure 2. Questionnaires

	1	2	3	4	5	6	7	8	9	10
1	1	0.50		0.42		0.58	1.00			
2	0.17	1	0.33	0.92			0.58	0.08	0.42	
3			0.92	1	0.58		0.58			0.33
4	1.00	0.33		1			0.33		0.75	
5		0.33	0.42	0.33	1		0.17	0.83	0.42	
6	0.08	0.75		0.33		1	0.67	0.08		
7	0.58	0.58				0.17	1		0.58	
8		0.33	0.08		1.00	0.08	0.42	1	0.42	0.17
9	0.58	0.42		0.83			0.33		1	
10	0.17	0.83					0.92	0.33		1

Figure 3. Preferring Matrix Y

	1	2	3	4	5	6	7	8	9	10
1	1	0.49	0.49	0.65	0.44	0.63	0.57	0.48	0.70	0.53
2	0.49	1	0.55	0.63	0.33	0.52	0.78	0.37	0.56	0.52
3	0.49	0.55	1	0.63	0.72	0.65	0.45	0.73	0.59	0.75
4	0.65	0.63	0.63	1	0.43	0.53	0.51	0.46	0.63	0.48
5	0.44	0.33	0.72	0.43	1	0.63	0.32	0.97	0.58	0.68
6	0.63	0.52	0.65	0.53	0.63	1	0.52	0.65	0.48	0.68
7	0.57	0.78	0.45	0.51	0.32	0.52	1	0.35	0.52	0.53
8	0.48	0.37	0.73	0.46	0.97	0.65	0.35	1	0.59	0.68
9	0.70	0.56	0.59	0.63	0.58	0.48	0.52	0.59	1	0.56
10	0.53	0.52	0.75	0.48	0.68	0.68	0.53	0.68	0.56	1

Figure 4. Amicable Matrix S

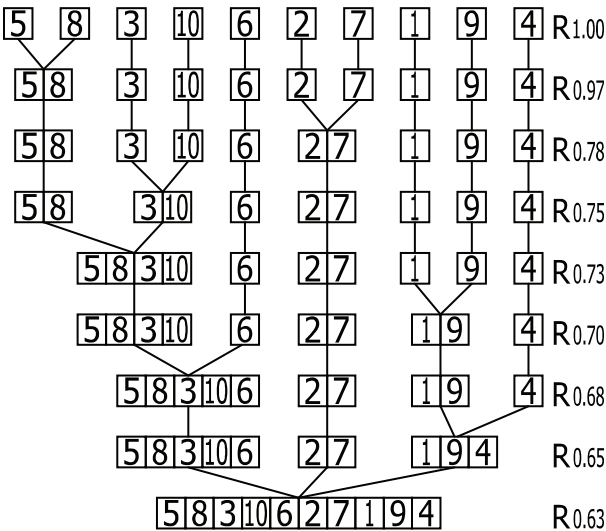


Figure 5. Partition Tree P

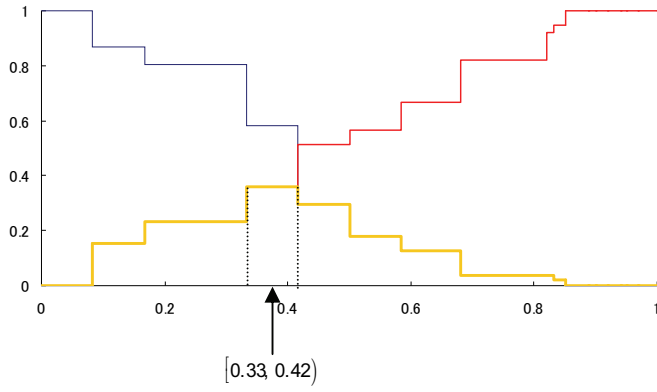


Figure 6. Fuzzy Decision $f_d(F) \wedge f_e(F)$

By applying the fuzzy decision, we could obtain the optimal value $\lambda_0 = 0.33$ as shown in Fig.6.

We would illustrate the fuzzy sociogram U^F , $z = c_0 = 0.58$ in Fig.7 which is based on the fuzzy graph $G_{\lambda_0} = (V, F_{\lambda_0})$, $\lambda_0 = 0.33$ in Fig.7. In this graph, we indicated the students in the same cluster at the optimal cut level $z = c_0 = 0.58$ with the same color and the opinion leaders are boldly marked.

This fuzzy sociogram U^F effectively represents the following results:

- (1) The opinion leaders are 2 and 7 because these nodes are connected with many other nodes.
- (2) The relatively isolated member is 10 because this node has not been connected with the others.
- (3) The subgroups are $\{5, 8, 3, 10\}$ and $\{2, 7\}$.

In this way, we could effectively analyze and clarify the interpersonal relation among members by applying the fuzzy node fuzzy graph and our analysis method.

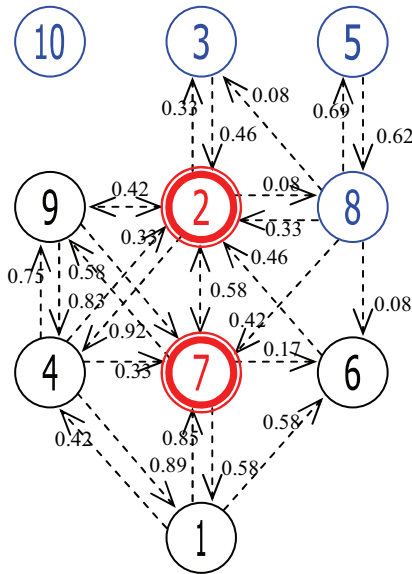


Figure 7. Fuzzy Sociogram U^F , $\lambda_0 = 0.33$, $z = 0.73$

IV. CONCLUSION

In this paper, we have mainly discussed the followings;

- (1) fuzzy node fuzzy graph
- (2) new T-norm "Uesu product"
- (3) decision analysis of the optimal fuzzy graph G_{λ_0} in the fuzzy graph sequence $\{G_{\lambda}\}$
- (4) fuzzy contingency table
- (5) practical application and its effectiveness concerning the sociometry analysis

We have illustrated the practical effectiveness of the analysis method with the case study concerning the sociometry analysis. This method is also available for the instruction/cognition analysis in education[10], the opinion poll in psychology and so on.

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