

Multiple Criteria Group Decision Making with Triangular Interval Type-2 Fuzzy Sets

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Abstract—In this paper, the triangular interval type-2 fuzzy set extension of the Analytic Hierarchy Process is introduced. The priority vectors are elicited from the pairwise comparison matrix with triangular interval type-2 fuzzy sets which represent the linguistic relative importance judged by experts. First, the triangular interval type-2 fuzzy set logarithmic regression function is defined with respect to the pairwise comparison matrix of the criteria. Then, the sets of linear systems corresponding to the end points of the upper and lower membership functions are formulated by minimizing the logarithmic regression function. The non-normalized weights are solved. It is shown in this paper that the sets of linear systems are all dependent. Accordingly, the final normalized weights can be obtained. A quadratic programming model for finding the optimum normalized weights is constructed in the sense of minimizing the fuzziness of the interval type-2 fuzzy weights. Analogously, if the pairwise comparisons are for the alternatives, then the final normalized triangular interval type-2 fuzzy rates are also obtained. The aggregation process of the triangular interval type-2 fuzzy sets is proposed to obtain the overall rate of each alternative. The interval type-2 fuzzy set ranking method is proposed and used to find the final preference of the alternatives. A numerical example about selecting the best professor candidate is examined.

Keywords: *Analytic Hierarchy Process, Interval Type-2 Fuzzy Sets, Multiple Criteria Decision Making, Pairwise Comparison Matrix*

I. INTRODUCTION

Analytic hierarchy process (AHP) is developed by Saaty[1] to find the priority of a set of alternatives under the given relative importance opinions to the pairs of criteria. In real life, some judgments might not be given as an exact relative scale. The linguistic scales are used instead. The fuzzy version of the extensions of Saaty's priority theory was proposed by de Graan[2] and Loostma[3]. Based on their fuzzy extension, van Laarhoven and Pedrycz[4] formally presented a method to find the fuzzy priority vector from a fuzzy pairwise comparison matrix(FPCM) with the triangular fuzzy numbers(TFN) as the matrix elements. By proposing a normalization modification of the fuzzy priority vector from Loostma's LSMR algorithm, Boender, Graan and Lootsma[5] pointed out that the normalized solution violates the first-order conditions, i.e. the applied normalization destroys optimality. Another approach to find the priority vector was based on the row and column geometric means(RCGM) which was studied by Uppuluri[6],

Aczel[7] and Koczkodaj[8]. Buckley[9] was the first one to develop the fuzzy geometric mean method to find the fuzzy priority vector(weights). Actually, Buckley developed the interval geometric mean method for obtaining the interval priority vector for interval comparison matrix. His algorithm can be applied to any general fuzzy numbers and not just for TFN. However, the numerical evaluation process is tedious. The original Saaty's AHP model is of crisp opinions, which is too restrictive for experts to judge. Almost all the studies dealing with the extension to fuzzy criteria are using type-1 fuzzy sets. The uncertainty of the relative importance among criteria can be further given by words which are modelled by interval type-2 fuzzy sets(IT2FSs)([10]).

In this paper, the triangular IT2FS extension of the classical AHP is introduced and the IT2FS priority vector(weighting vector) are elicited from the pairwise comparison matrix with triangular IT2FS elements which represent the linguistic relative importance judged by experts. First, the triangular IT2FS logarithmic regression function is defined with respect to the triangular IT2FS pairwise comparison matrix of the criteria. Secondly, five sets of linear systems corresponding to the end points of the upper and lower membership functions are formulated by minimizing the logarithmic regression function. Finally, the non-normalized triangular IT2FS weight of the criteria are solved. It is shown in this paper that the sets of linear systems are all dependent. Accordingly, the final normalized weights can be obtained. A linear programming model for finding the optimum normalized weights is constructed in the sense of minimizing the fuzziness of the interval type-2 fuzzy weights. The final normalized weights can be obtained. Analogously, if the pairwise comparison are given for the alternatives, then the final normalized triangular IT2FS rates are also obtained by the same mathematical analysis. The aggregation process of the triangular IT2FS is also proposed to obtain the overall rate of each alternative. After obtaining the triangular IT2FS overall rates of the alternatives, the IT2FS ranking method is introduced and used to find the preference of the alternatives.

The rest of this paper is organized as follows: Section 2 presents definitions of the triangular IT2FS and some arithmetic and transcendental operations. Section 3 formulates the mathematical model by proposing the triangular IT2FS logarithmic regression function. The non-normalized solutions are obtained. Section 4 proposes the normalized solutions by applying the dependent conditions of the linear systems. It is shown that the sets of linear systems are all dependent. Accordingly, the final normalized weights can be obtained. A quadratic programming model for finding the optimum

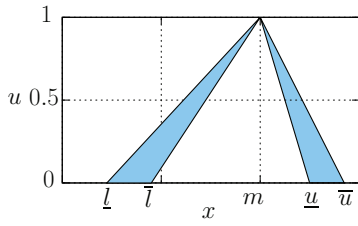


Figure 1. Footprint of uncertainty(FOU) of the triangular interval type-2 fuzzy number.

normalized weights is constructed in the sense of minimizing the fuzziness of the interval type-2 fuzzy weights. Section 5 proposes the aggregation process of the triangular IT2FS weights of the criteria and the rates of the decision alternatives. Section 6 proposes the IT2FS ranking method. Section 7 is a numerical example about the decision of selecting the best candidate of a college professor in Operations Research.

II. TRIANGULAR INTERVAL TYPE-2 FUZZY SETS

A. Some Definitions of Expressions

We need to define some terms and approximation operations for the model.

The triangular interval type-2 fuzzy number is defined as in the following.

Let $\tilde{a} = ([\underline{l}, \bar{l}], m, [\underline{u}, \bar{u}])$ be a triangular interval type-2 fuzzy number. The upper and lower membership functions of $\tilde{a}$ is

$$\begin{aligned} \text{UMF}_{\tilde{a}}(x) &= \begin{cases} \frac{x - \underline{l}}{m - \underline{l}}, & \underline{l} \leq x \leq m \\ \frac{x - \bar{u}}{m - \bar{u}}, & m \leq x \leq \bar{u} \end{cases}, \\ \text{LMF}_{\tilde{a}}(x) &= \begin{cases} \frac{x - \bar{l}}{m - \bar{l}}, & \bar{l} \leq x \leq m \\ \frac{x - \underline{u}}{m - \underline{u}}, & m \leq x \leq \underline{u} \end{cases}. \end{aligned}$$

The footprint of uncertainty(FOU) of the membership function is depicted on Fig. 1.

Some approximation operations are defined as follows.

Definition 1. The inverse of a triangular interval type-2 fuzzy number is defined as

$$\frac{1}{\tilde{a}} = \left(\left[\frac{1}{\bar{u}}, \frac{1}{\underline{u}} \right], \frac{1}{m}, \left[\frac{1}{\bar{l}}, \frac{1}{\underline{l}} \right] \right).$$

Definition 2. The logarithm of a triangular interval type-2 fuzzy number is defined as

$$\ln \tilde{a} = ([\ln \underline{l}, \ln \bar{l}], \ln m, [\ln \underline{u}, \ln \bar{u}]).$$

Definition 3. The exponential of a triangular interval type-2 fuzzy number is defined as

$$\exp \tilde{a} = ([\exp \underline{l}, \exp \bar{l}], \exp m, [\exp \underline{u}, \exp \bar{u}]).$$

Definition 4. The power to the positive integer n of a triangular interval type-2 fuzzy number is defined as

$$\tilde{a}^n = ([\underline{l}^n, \bar{l}^n], m^n, [\underline{u}^n, \bar{u}^n]).$$

Definition 5. The negative of a triangular interval type-2 fuzzy number is defined as

$$-\tilde{a} = ([-\bar{u}, -\underline{u}], -m, [-\bar{l}, -\underline{l}]).$$

Based on the definitions above, we can define the binary operations as follows.

Definition 6. Let $\tilde{a} = ([\underline{a}^l, \bar{a}^l], a^m, [\underline{a}^u, \bar{a}^u])$ and $\tilde{b} = ([\underline{b}^l, \bar{b}^l], b^m, [\underline{b}^u, \bar{b}^u])$ be two triangular interval type-2 fuzzy numbers. The sum of $\tilde{a}$ and $\tilde{b}$ is

$$\tilde{a} + \tilde{b} = ([\underline{a}^l + \underline{b}^l, \bar{a}^l + \bar{b}^l], a^m + b^m, [\underline{a}^u + \underline{b}^u, \bar{a}^u + \bar{b}^u]).$$

Definition 7. Let $\tilde{a} = ([\underline{a}^l, \bar{a}^l], a^m, [\underline{a}^u, \bar{a}^u])$ and $\tilde{b} = ([\underline{b}^l, \bar{b}^l], b^m, [\underline{b}^u, \bar{b}^u])$ be two triangular interval type-2 fuzzy numbers. The difference of $\tilde{a}$ and $\tilde{b}$ is defined thru the negation as $\tilde{a} - \tilde{b} = \tilde{a} + (-\tilde{b})$.

Definition 8. Let $\tilde{a} = ([\underline{a}^l, \bar{a}^l], a^m, [\underline{a}^u, \bar{a}^u])$ and $\tilde{b} = ([\underline{b}^l, \bar{b}^l], b^m, [\underline{b}^u, \bar{b}^u])$ be two triangular interval type-2 fuzzy numbers. The product of $\tilde{a}$ and $\tilde{b}$ is

$$\begin{aligned} \tilde{a} \cdot \tilde{b} &= \left(\left[\min \left\{ \underline{a}^l \cdot \underline{b}^l, \underline{a}^l \cdot \bar{b}^l, \bar{a}^l \cdot \underline{b}^l, \bar{a}^l \cdot \bar{b}^l \right\}, \right. \right. \\ &\quad \left. \max \left\{ \underline{a}^l \cdot \underline{b}^l, \underline{a}^l \cdot \bar{b}^l, \bar{a}^l \cdot \underline{b}^l, \bar{a}^l \cdot \bar{b}^l \right\} \right], \\ &\quad a^m \cdot b^m, \left[\min \left\{ \underline{a}^u \cdot \underline{b}^u, \underline{a}^u \cdot \bar{b}^u, \bar{a}^u \cdot \underline{b}^u, \bar{a}^u \cdot \bar{b}^u \right\}, \right. \\ &\quad \left. \max \left\{ \underline{a}^u \cdot \underline{b}^u, \underline{a}^u \cdot \bar{b}^u, \bar{a}^u \cdot \underline{b}^u, \bar{a}^u \cdot \bar{b}^u \right\} \right] \right). \end{aligned}$$

Definition 9. Let $\tilde{a} = ([\underline{a}^l, \bar{a}^l], a^m, [\underline{a}^u, \bar{a}^u])$ and $\tilde{b} = ([\underline{b}^l, \bar{b}^l], b^m, [\underline{b}^u, \bar{b}^u])$ be two triangular interval type-2 fuzzy numbers, where $\underline{b}^l > 0$ or $\bar{b}^u < 0$. The quotient $\tilde{a}$ and $\tilde{b}$ is defined thru the inverse and multiplication as $\tilde{a}/\tilde{b} = \tilde{a} \cdot (1/\tilde{b})$.

All the definitions above are well-defined, because if the type-2 fuzzy numbers are reduced to the type-1 fuzzy numbers or even further to the crisp numbers, then the operations would result in the ordinary traditional definitions of the operations of real numbers.

III. MATHEMATICAL MODEL OF LOGARITHMIC REGRESSION FUNCTION

The interval type-2 fuzzy pairwise comparison matrix (IT2FPCM) is like

$$\begin{pmatrix} 1 & \tilde{r}_{12} & \cdots & \tilde{r}_{1n} \\ \tilde{r}_{21} & 1 & \cdots & \tilde{r}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{r}_{n1} & \tilde{r}_{n2} & \cdots & 1 \end{pmatrix}$$

where $\tilde{r}_{ij} = [\tilde{r}_{ij1}, \tilde{r}_{ij2}, \dots, \tilde{r}_{ij\delta_{ij}}]^T$ and δ_{ij} is the number of judgments given by experts for the preference of the criterion i to the criterion j . The triangular interval type-2 fuzzy number $\tilde{r}_{ijk}$ is the relative comparison judgment of the criterion i to the criterion j given by the k^{th} expert. Note that $\delta_{ij} = \delta_{ji}$ and some δ_{ij} could be 0 but not all $\delta_{ij} = 0$ for $i = 1, 2, \dots, n; i \neq j$. In other words, it is not reasonable that no any judgments of criterion i to other criteria are given.

Let $\tilde{w}_i$ be the absolute weight for the criterion i . Based on the assumption that $\tilde{r}_{ijk} \approx \tilde{w}_i/\tilde{w}_j$, and to obtain the optimal solutions for $\tilde{w}_i$, it is required that

$$\text{Min} J = \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} \left(\ln \tilde{r}_{ijk} - \ln \frac{\tilde{w}_i}{\tilde{w}_j} \right)^2. \quad (1)$$

Let $\tilde{r}_{ijk} = \left([r_{ijk}^l, r_{ijk}^u], r^m, [r_{ijk}^u, r_{ijk}^l] \right)$ and $\tilde{w}_i = ([w_i^l, w_i^u], w_i^m, [w_i^u, w_i^l])$. Using the definitions, the logarithmic regression equation (2) can be expressed as

$$\begin{aligned} \text{Min } J &= \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} \left(\ln \tilde{r}_{ijk} - \ln \frac{\tilde{w}_i}{\tilde{w}_j} \right)^2 \\ &= \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln \tilde{r}_{ijk} - \ln \tilde{w}_i + \ln \tilde{w}_j)^2. \end{aligned} \quad (2)$$

Extending the type-1 fuzzy models from [4], [5], [11] and using the definitions 2, 5, 9, the approximation of (2) can be expressed as

$$\text{Min } J = \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^l - \ln w_i^l + \ln w_j^l)^2 \quad (3)$$

$$+ \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^l - \ln w_i^l + \ln w_j^l)^2 \quad (4)$$

$$+ \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^m - \ln w_i^m + \ln w_j^m)^2$$

$$+ \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^u - \ln w_i^u + \ln w_j^u)^2 \quad (5)$$

$$+ \sum_{i=1}^n \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^u - \ln w_i^u + \ln w_j^l)^2. \quad (6)$$

To get the minimum of J , it is required that

$$\frac{\partial J}{\partial w_i^l} = \frac{-2}{w_i^l} \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^l - \ln w_i^l + \ln w_j^l) = 0, \quad (7)$$

$$\frac{\partial J}{\partial w_i^l} = \frac{-2}{w_i^l} \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^l - \ln w_i^l + \ln w_j^l) = 0, \quad (8)$$

$$\frac{\partial J}{\partial w_i^m} = \frac{-2}{w_i^m} \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^m - \ln w_i^m + \ln w_j^m) = 0, \quad (9)$$

$$\frac{\partial J}{\partial w_i^u} = \frac{-2}{w_i^u} \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^u - \ln w_i^u + \ln w_j^l) = 0, \quad (10)$$

$$\frac{\partial J}{\partial w_i^u} = \frac{-2}{w_i^u} \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} (\ln r_{ijk}^u - \ln w_i^u + \ln w_j^l) = 0, \quad (11)$$

for $i = 1, 2, \dots, n$. The equations (7)~(11) can be divided into three uncoupled linear systems,

$$\begin{bmatrix} D & E \\ E & D \end{bmatrix} \begin{bmatrix} \underline{\mathbf{1}} \\ \underline{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{b}}^l \\ \underline{\mathbf{b}}^u \end{bmatrix}, \quad (12)$$

$$[D + E] \underline{\mathbf{m}} = \underline{\mathbf{b}}^m, \quad (13)$$

$$\begin{bmatrix} D & E \\ E & D \end{bmatrix} \begin{bmatrix} \underline{\mathbf{1}} \\ \underline{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{b}}^l \\ \underline{\mathbf{b}}^u \end{bmatrix}, \quad (14)$$

where $D = \text{diag}(\sum_{j=2}^n \delta_{1j}, \sum_{j=1, j \neq 2}^n \delta_{2j}, \dots, \sum_{j=1}^{n-1} \delta_{nj})$, and $E =$

$[e_{ij}]$ with $e_{ij} = \begin{cases} 0, & i = j \\ -\delta_{ij}, & i \neq j \end{cases}$, and the unknowns are

$\mathbf{t} = [\ln w_i^t]$, $t = l$ or u , the knowns are $\mathbf{b}^t = [b_i^t]$, $b_i^t = \sum_{j=1, j \neq i}^n \sum_{k=1}^{\delta_{ij}} \ln r_{ijk}^t$, $t = l$ or u or m . Both $\mathbf{t}$ and $\mathbf{b}$ can be appended the underline or overline. Furthermore, let $A = \begin{bmatrix} D & E \\ E & D \end{bmatrix}$, $A^m = D + E$, the linear systems (12)~(14) can be expressed as

$$A \mathbf{w}_1 = \mathbf{b}_1, \quad (15)$$

$$A^m \mathbf{w}^m = \mathbf{b}^m, \quad (16)$$

$$A \mathbf{w}_2 = \mathbf{b}_2. \quad (17)$$

The linear systems (15)~(17) which are uncoupled can be solved independently. If each $\delta_{ij} \neq 0$, then it can be easily shown that A and A^m are not of full rank. Each of the linear system (15)~(17) has infinite many solutions. The solutions exist for systems (15)~(17).

A. Theorems

Since the judgment matrix is reciprocal, we have $\tilde{r}_{jik} = 1/\tilde{r}_{ijk}$ and $\ln \tilde{r}_{jik} = -\ln \tilde{r}_{ijk}$. It follows that

$$\begin{aligned} & \ln \left(([\underline{r}_{jik}^l, \bar{r}_{jik}^l], r_{jik}^m, [\underline{r}_{jik}^u, \bar{r}_{jik}^u]) \right) \\ &= -\ln \left(([\underline{r}_{ijk}^l, \bar{r}_{ijk}^l], r_{ijk}^m, [\underline{r}_{ijk}^u, \bar{r}_{ijk}^u]) \right). \end{aligned}$$

The rank of A and the rank of the augmented matrices $[A \ \mathbf{b}_1]$ and $[A \ \mathbf{b}_2]$ are the same and the dimension of the null space of A is 1. Also, the rank of A^m and the augmented matrix $[A^m \ \mathbf{b}^m]$ are the same and the dimension of the null space of A^m is equal to 1. Therefore, all the three linear systems (15)~(17) have infinite many solutions.

It is suffice to assume that each of δ_{ij} , $i = 1, 2, \dots, n; i \neq j$ is not equal to 0 on the following theorems and corollaries. Let $\mathbf{1} = [1, 1, \dots, 1]^t$ in the sequel.

Theorem 1. *The nullity of the coefficient matrix A of the linear system (15) is 1 and the null space of A is $\text{span}\{\mathbf{1}\}$.*

Proof: The proof can be easily followed by the row reduced echelon form of A . ■

Theorem 2. *The rank of A is the same as the augmented matrix $[A \ \mathbf{b}_1]$ of the linear system (15).*

Proof: The proof can be easily followed by the row reduced echelon form of the augmented matrix $[A \ \mathbf{b}_1]$. ■

Theorem 3. *The rank of A is the same as the augmented matrix $[A \ \mathbf{b}_2]$ of the linear system (17).*

Proof: The proof is the same as in Theorem 2. ■

Corollary 1. *The linear system (15) has infinite solutions with two degrees of freedom.*

Proof: The solution space has two degrees of freedom, because the nullity of the coefficient matrix is 1. ■

Corollary 2. *The linear system (17) has infinite solutions with two degrees of freedom.*

Proof: The proof is the same as in 1. ■

Let A^+ be the Moore-Penrose pseudo-inverse of A [12, p.432], then the particular solutions of (15) and (17) are

$$\mathbf{w}_1^p = A^+ \mathbf{b}_1, \quad (18)$$

$$\mathbf{w}_2^p = A^+ \mathbf{b}_2. \quad (19)$$

The general solution can be expressed as

$$\mathbf{w}_1 = A^+ \mathbf{b}_1 + p_1 \mathbf{1}, \quad p_1 \in \mathbb{R}. \quad (20)$$

$$\mathbf{w}_2 = A^+ \mathbf{b}_2 + p_2 \mathbf{1}, \quad p_2 \in \mathbb{R}. \quad (21)$$

Theorem 4. *The nullity of the coefficient matrix A^m of the linear system (16) is 1 and the null space of A^m is $\text{span}\{\mathbf{1}\}$.*

Proof: The proof can be easily followed by the row reduced echelon form of A^m . The nullity of A^m is 1, and the null space is $\text{span}\{\mathbf{1}\}$. ■

Theorem 5. *The rank of A^m is the same as the augmented matrix $[A^m \ \mathbf{b}^m]$ of the linear system (16).*

Proof: The proof can be easily followed by the row reduced echelon form of the augmented matrix $[A^m \ \mathbf{b}^m]$. The rank of the augmented matrix $[A^m \ \mathbf{b}^m]$ is $(n-1)$ which is equal to the rank of A^m . ■

Corollary 3. *The linear system (16) has infinite solutions with one degrees of freedom.*

Proof: The solution space has one degrees of freedom, because the nullity of the coefficient is 1. ■

Let A^{m+} be the Moore-Penrose pseudo-inverse of A^m [12, p.432], then the particular solution $\mathbf{w}^{m,p}$ is $A^{m+} \mathbf{b}^m$. The general solution can be expressed as

$$\mathbf{w}^m = A^{m+} \mathbf{b}^m + p_m \mathbf{1}, \quad p_m \in \mathbb{R}. \quad (22)$$

IV. NORMALIZATION

The weight $\tilde{w}_i = ([\underline{l}_i, \bar{l}_i], m_i, [\underline{u}_i, \bar{u}_i])$ is a triangular interval type-2 fuzzy weights for the criteria. It should be well-defined and satisfied the inequalities

$$\underline{l}_i \leq \bar{l}_i \leq m_i \leq \underline{u}_i \leq \bar{u}_i. \quad (23)$$

From Corollaries 1 and 3, the final solution of the weight of a certain criterion has three degrees of freedom. The suitable or the optimal parameters can be selected such that the final solution of the weight is a well-defined triangular interval type-2 fuzzy number.

From (20) and (22), we have

$$\ln \tilde{w}_i = ([\underline{l}_i + p_1, \bar{l}_i + p_2], m_i + p_m, [\underline{u}_i + p_2, \bar{u}_i + p_1]),$$

where s, t, v are three arbitrary parameters. The parameter v can be independently determined by normalizing modal vector in the sense of unit vector sum. Namely, $\sum_{i=1}^n \exp(m_i + p_m) = 1$, then we have

$$p_m = -\ln \left(\sum_{i=1}^n \exp m_i \right). \quad (24)$$

The other two parameters s, t can be determined by minimizing the spans of the supports of the FOU. A quadratic programming (QP) model can be constructed to find the optimal

solution for p_1 and p_2 .

$$\begin{aligned}
& \min J \\
& = \sum_{i=1}^n \left(((\bar{l}_i + p_2) - (\underline{l}_i + p_1))^2 \right. \\
& \quad + ((m_i + p_m) - (\bar{l}_i + p_2))^2 \\
& \quad + ((\underline{u}_i + p_2) - (m_i + p_m))^2 \\
& \quad \left. + ((\bar{u}_i + p_1) - (\underline{u}_i + p_2))^2 \right) \\
& \underline{l}_i + p_1 \leq \bar{l}_i + p_2, i = 1, 2, \dots, n \\
& \underline{u}_i + p_2 \leq \bar{u}_i + p_1, i = 1, 2, \dots, n \\
& \bar{l}_i + p_2 \leq m_i + p_m, i = 1, 2, \dots, n \\
& m_i + p_m \leq \underline{u}_i + p_2, i = 1, 2, \dots, n \\
& p_1, p_2 \text{ unbounded.}
\end{aligned}$$

Let

$$\begin{aligned}
J' = J - \sum_{i=1}^n & (l_i^2 + 2\bar{l}_i^2 + 2\underline{u}_i^2 + \bar{u}_i^2 \\
& + 2m_i^2 + 2p_m^2 - 2p_m\bar{l}_i \\
& - 2\bar{l}_i\bar{l}_i - 2m_i\underline{u}_i - 2p_m\underline{u}_i \\
& - 2\bar{u}_i\underline{u}_i + 4p_m m_i - 2m_i\bar{l}_i)
\end{aligned}$$

and introduce the following matrices and vectors,

$$\begin{aligned}
\mathbf{x} &= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, Q = \begin{bmatrix} 4n & -4n \\ -4n & 8n \end{bmatrix}, \\
\mathbf{c} &= \begin{bmatrix} \sum_{i=1}^n (-2\bar{l}_i + 2\underline{l}_i + 2\bar{u}_i - 2\underline{u}_i) \\ \sum_{i=1}^n (4\bar{l}_i - 2\underline{l}_i - 2\bar{u}_i + 4\underline{u}_i - 4m_i - 4v) \end{bmatrix},
\end{aligned}$$

the objective function of the QP model can be expressed as

$$\text{Min } J' = \frac{1}{2} \mathbf{x}^t Q \mathbf{x} + \mathbf{x}^t \mathbf{c}.$$

Since Q is a positive definite matrix, the minimizer exists and is unique. Suppose the optimal solution exists at $p_1 = p_1^*$ and $p_2 = p_2^*$, the optimal normalized i th IT2 triangular fuzzy weight can be expressed as

$$\begin{aligned}
\tilde{w}_i &= \frac{(\exp(p_1^*) \cdot \exp \underline{l}_i, \exp(p_2^*) \cdot \exp \bar{l}_i)}{\exp m_i} \\
&= \frac{\sum_{j=1}^n \exp m_j}{(\exp(p_2^*) \cdot \exp \underline{u}_i, \exp(p_1^*) \cdot \exp \bar{u}_i)}.
\end{aligned}$$

V. AGGREGATION

Let $\tilde{r}_i = ([\underline{r}_i^l, \bar{r}_i^l], r_i^m, [\underline{r}_i^u, \bar{r}_i^u])$ be the rate for a certain alternative corresponding to the i^{th} criterion whose weight is $\tilde{w}_i = ([\underline{w}_i^l, \bar{w}_i^l], w_i^m, [\underline{w}_i^u, \bar{w}_i^u])$. We propose two approaches to find the IT2 fuzzy weighted average(FWA) for IT2 TFNs. One is using the linear programming models to find the uncertainty supports of the lower membership function (LMF) and upper membership function(UMF). The other is a direct extension from the interval weighted average to the IT2 FWA of the IT2 TFNs.

A. Aggregation by Linear Programming

Model-I(lower bound of UMF)

$$\begin{aligned}
\text{Min } \underline{r}_{\text{ave}}^l &= \sum_{i=1}^n \underline{r}_i^l \cdot w_i, \\
\frac{\underline{w}_i^l}{\sum_{j=1}^n \underline{w}_j^u} &\leq w_i \leq \frac{\bar{w}_i^u}{\sum_{j=1}^n \underline{w}_j^l}, i = 1, 2, \dots, n, \\
\sum_{i=1}^n w_i &= 1.
\end{aligned}$$

Model-II(lower bound of LMF)

$$\begin{aligned}
\text{Min } \bar{r}_{\text{ave}}^l &= \sum_{i=1}^n \bar{r}_i^l \cdot w_i, \\
\frac{\bar{w}_i^l}{\sum_{j=1}^n \underline{w}_j^u} &\leq w_i \leq \frac{\underline{w}_i^u}{\sum_{j=1}^n \bar{w}_j^l}, i = 1, 2, \dots, n, \\
\sum_{i=1}^n w_i &= 1.
\end{aligned}$$

Model-III(upper bound of LMF)

$$\begin{aligned}
\text{Max } \underline{r}_{\text{ave}}^u &= \sum_{i=1}^n \underline{r}_i^u \cdot w_i, \\
\frac{\bar{w}_i^l}{\sum_{j=1}^n \bar{w}_j^u} &\leq w_i \leq \frac{\underline{w}_i^u}{\sum_{j=1}^n \bar{w}_j^l}, i = 1, 2, \dots, n, \\
\sum_{i=1}^n w_i &= 1.
\end{aligned}$$

Model-IV(upper bound of UMF)

$$\begin{aligned}
\text{Max } \bar{r}_{\text{ave}}^u &= \sum_{i=1}^n \bar{r}_i^u \cdot w_i, \\
\frac{\underline{w}_i^l}{\sum_{j=1}^n \bar{w}_j^u} &\leq w_i \leq \frac{\bar{w}_i^u}{\sum_{j=1}^n \underline{w}_j^l}, i = 1, 2, \dots, n, \\
\sum_{i=1}^n w_i &= 1.
\end{aligned}$$

Let $\underline{r}_{\text{ave}}^{l*}, \bar{r}_{\text{ave}}^{l*}, \underline{r}_{\text{ave}}^{u*}, \bar{r}_{\text{ave}}^{u*}$ be the optimal solutions of the above four linear programming models, respectively, and let the modal weighted average be $r_{\text{ave}}^m = \frac{\sum_{i=1}^n r_i^m \cdot w_i^m}{\sum_{j=1}^n w_i^m}$, then interval type-2 fuzzy weighted average is

$$\tilde{r}_{\text{ave}} = ([\underline{r}_{\text{ave}}^{l*}, \bar{r}_{\text{ave}}^{l*}], r_{\text{ave}}^m, [\underline{r}_{\text{ave}}^{u*}, \bar{r}_{\text{ave}}^{u*}]).$$

B. Aggregation from Extension of Type-1 FWA

By extending the type-1 fuzzy weighted average, [13] introduced a set of formulas which can be used to find the supports of the LMF and UMF of the IT2 FWA as follows,

$$\underline{r}_{ave}^l = \frac{\sum_{i=1}^{k^l} \underline{r}_i^l \cdot \bar{w}_i^u + \sum_{i=k^l+1}^n \underline{r}_i^l \cdot \underline{w}_i^l}{\sum_{i=1}^{k^l} \bar{w}_i^u + \sum_{i=k^l+1}^n \underline{w}_i^l}, \quad (25)$$

$$\bar{r}_{ave}^l = \frac{\sum_{i=1}^{\bar{k}^l} \bar{r}_i^l \cdot \underline{w}_i^u + \sum_{i=\bar{k}^l+1}^n \bar{r}_i^l \cdot \bar{w}_i^l}{\sum_{i=1}^{\bar{k}^l} \underline{w}_i^u + \sum_{i=\bar{k}^l+1}^n \bar{w}_i^l}, \quad (26)$$

$$\underline{r}_{ave}^u = \frac{\sum_{i=1}^{k^u} \underline{r}_i^u \cdot \underline{w}_i^l + \sum_{i=k^u+1}^n \underline{r}_i^u \cdot \bar{w}_i^u}{\sum_{i=1}^{k^u} \underline{w}_i^l + \sum_{i=k^u+1}^n \bar{w}_i^u}, \quad (27)$$

$$\bar{r}_{ave}^u = \frac{\sum_{i=1}^{\bar{k}^u} \bar{r}_i^u \cdot \bar{w}_i^l + \sum_{i=\bar{k}^u+1}^n \bar{r}_i^u \cdot \underline{w}_i^u}{\sum_{i=1}^{\bar{k}^u} \bar{w}_i^l + \sum_{i=\bar{k}^u+1}^n \underline{w}_i^u}, \quad (28)$$

where $\underline{k}^l, \bar{k}^l, k^u, \bar{k}^u$ are the indexes satisfying the following inequalities respectively,

$$\begin{aligned} \underline{r}_{\underline{k}^l}^l &\leq \underline{r}_{ave}^l \leq \underline{r}_{\underline{k}^l+1}^l, \\ \bar{r}_{\bar{k}^l}^l &\leq \bar{r}_{ave}^l \leq \bar{r}_{\bar{k}^l+1}^l, \\ \underline{r}_{k^u}^u &\leq \underline{r}_{ave}^u \leq \underline{r}_{k^u+1}^u, \\ \bar{r}_{\bar{k}^u}^u &\leq \bar{r}_{ave}^u \leq \bar{r}_{\bar{k}^u+1}^u. \end{aligned}$$

These indexes are commonly referred to as the thresholds for the supports of the LMFs and UMFs of the IT2 fuzzy weights and have been proved to be unique [14], [13]. Some algorithms [15], [14], [16] were developed to determine these thresholds. We adopt the algorithm from [16]. The algorithm for finding $\underline{r}_{ave}^l$ is as follows:

Stage1: Sort $\underline{r}_i^l$, for $i = 1, 2, \dots, n$ in non-decreasing order and rearrange $[\underline{w}_i^l, \bar{w}_i^u]$ with respect to the reordered $\underline{r}_i^l$.

Stage2: Construct the function of index

$$\tau(i) = \frac{\sum_{j=1}^i (\underline{r}_j^l - \underline{r}_i^l) \cdot \bar{w}_j^u + \sum_{j=i+1}^n (\underline{r}_j^l - \underline{r}_i^l) \cdot \underline{w}_j^l}{\sum_{j=1}^i \bar{w}_j^u + \sum_{j=i+1}^n \underline{w}_j^l}.$$

Stage3: The threshold $\underline{k}^l$ exists at the index where $\tau(\underline{k}^l) > 0$ and $\tau(\underline{k}^l + 1) \leq 0$. ([16] proved that such index exists and is unique.)

Stage4: Substitute the threshold $\underline{k}^l$ into (25) to get $\underline{r}_{ave}^l$.

The algorithms for determining the thresholds for other supports can be constructed similarly by defining the corresponding specific index functions.

VI. RANKING OF IT2FSS

The final stage of the decision making process is to rank the overall ratings of the alternatives. It is not trivial to rank a set of IT2FSSs. Most of the ranking algorithms are for type-1 FSSs. So far there are two mature ranking algorithms proposed in literature. [17] adopted a statistical viewpoint and interpret each type-2 fuzzy number as an ensemble of ordinary fuzzy numbers. By ranking the 32 IT2FSSs which represents words for computation, [18] pointed out that Mitchell's method is highly impractical and prohibitive. Wu and Mendel proposed an extension of the type-1 FSSs centroid method to obtain

the centroids of a set of IT2FSSs, which are used for ranking these IT2FSSs. The centroid of an IT2FSS is approximated by discretizing the primary domain. Let x_i be the i th sample point of the primary variable x in ascending order with index i , then the IT2FSS centroid can be obtained from the following formulas

$$\begin{aligned} c_l(j) &= \frac{\sum_{i=1}^j x_i \bar{\mu}(x_i) + \sum_{i=j+1}^n x_i \underline{\mu}(x_i)}{\sum_{i=1}^j \bar{\mu}(x_i) + \sum_{i=j+1}^n \underline{\mu}(x_i)}, \\ c_u(j) &= \frac{\sum_{i=1}^j x_i \underline{\mu}(x_i) + \sum_{i=j+1}^n x_i \bar{\mu}(x_i)}{\sum_{i=1}^j \underline{\mu}(x_i) + \sum_{i=j+1}^n \bar{\mu}(x_i)}, \end{aligned}$$

for a specific switching indexes is $c = [c_l, c_u]$ where indexes k_l and k_u are the switching index such that

VII. NUMERICAL EXAMPLE

The fuzzy logarithmic least squares for fuzzy pairwise comparison matrix was proposed by [4] and modified by [5]. We investigate the same numerical example in both papers in which the type-1 fuzzy judgments are all extending to the IT2 triangular fuzzy numbers.

One of the three candidates (A, B and C) seeking for a post of professor in Operations Research is to be determined by a committee of three members. The following four criteria are identified by the committee for selecting the best candidates.

- C1: Mathematical creativity,
- C2: Implementation creativity,
- C3: Administrative capability,
- C4: Human maturity.

Fig. 2 shows the hierarchy structure of the decision making problem.

The committee members need to give the pairwise comparison judgments. All the linguistic judgments from committee members are transformed to the corresponding IT2 triangular fuzzy numbers. The set of IT2 triangular fuzzy scales that is the type-2 fuzzy extension of Saaty's scale ([19]) defined as

$$\begin{aligned} \tilde{1} &= ([1, 1], 1, [3/2, 2]), \\ \tilde{2} &= ([1, 3/2], 2, [5/2, 3]), \\ \tilde{3} &= ([2, 5/2], 3, [7/2, 4]). \end{aligned}$$

Additionally, the reciprocals are defined by Definition 1.

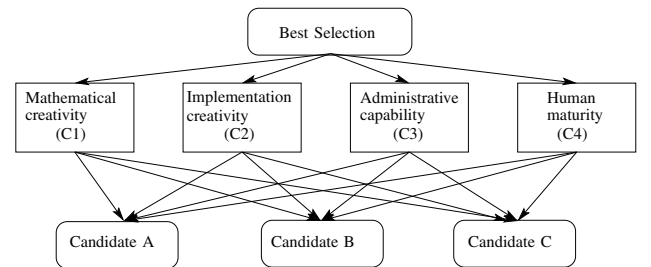


Figure 2. Hierarchy structure for the best candidate decision problem

Table I
THE PAIRWISE COMPARISON JUDGMENTS OF THE FOUR CRITERIA

$C_1 - C_2$	$C_1 - C_3$	$C_1 - C_4$	$C_2 - C_3$	$C_2 - C_4$	$C_3 - C_4$
$\{\tilde{1}, \tilde{\frac{1}{2}}, \tilde{\frac{1}{2}}\}$	$\{\tilde{1}\}$	$\{\tilde{\frac{1}{3}}, \tilde{\frac{1}{3}}, \tilde{\frac{1}{2}}\}$	$\{\tilde{\frac{2}{3}}, \tilde{\frac{2}{3}}\}$	$\{\tilde{1}, \tilde{\frac{1}{2}}, \tilde{\frac{1}{2}}\}$	$\{\tilde{\frac{1}{2}}\}$

Table II
PAIRWISE COMPARISONS OF THE THREE CANDIDATES WITH RESPECT TO THE FOUR CRITERIA

	A-B	A-C	B-C
C1	$\{\tilde{1}, \tilde{1}\}$	$\{\tilde{1}, \tilde{\frac{1}{2}}\}$	$\{\tilde{\frac{1}{2}}\}$
C2	$\{\tilde{\frac{2}{3}}\}$	$\{\tilde{\frac{2}{3}}\}$	
C3	$\{\tilde{\frac{2}{3}}, \tilde{\frac{2}{3}}, \tilde{\frac{1}{2}}\}$	$\{\tilde{\frac{2}{3}}\}$	$\{\tilde{1}\}$
C4		$\{\tilde{\frac{2}{3}}, \tilde{\frac{1}{2}}\}$	$\{\tilde{\frac{1}{2}}\}$

Table I shows the pairwise comparison judgments of the four criteria. Note that, not each of the committee members is required to give judgment for pairs of criteria. For example, only one member gives the judgment $\tilde{\frac{1}{2}}$ for the pair (C_3, C_4) .

Table II contains the pairwise comparison judgments of the three candidates with respect to the four criteria.

Using the IT2 FLLSM and the optimal normalization process developed in this paper, the IT2 TFN weights for the four criteria are obtained and is shown in Table III and Fig. 3.

Table IV shows the IT2 fuzzy rates for three candidates with respective to the four criteria.

We apply two approaches from Section V-A and Section V-B to finding the overall aggregated IT2 fuzzy rates of the three candidates as in Tables V and VI.

Fig. 4 and Fig. 5 show the overall aggregated IT2 fuzzy rates for the three candidates. It is clear that candidate A is the best candidate.

Table III
THE IT2 FUZZY WEIGHTS FOR FOUR CRITERIA NORMALIZED BY MINIMIZING THE QUADRATIC SUM OF FOUR SPANS.

C1	$([0.1718, 0.1836], 0.1945, [0.2283, 0.2680])$
C2	$([0.2609, 0.2904], 0.3192, [0.3617, 0.3977])$
C3	$([0.1149, 0.1246], 0.1398, [0.1490, 0.1649])$
C4	$([0.2531, 0.2982], 0.3466, [0.3708, 0.3946])$

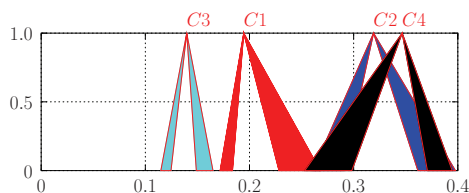


Figure 3. The IT2 fuzzy weights for four criteria normalized by minimizing the quadratic sum of four spans.

Table IV
THE IT2 FUZZY RATES FOR THREE CANDIDATES WITH RESPECT TO THE FOUR CRITERIA

	candidate A
C1	$([0.2891, 0.2891], 0.2891, [0.3448, 0.3694])$
C2	$([0.3782, 0.4664], 0.5455, [0.6186, 0.6872])$
C3	$([0.4620, 0.5375], 0.5789, [0.6326, 0.6991])$
C4	$([0.2184, 0.2321], 0.2500, [0.2752, 0.3150])$
	candidate B
C1	$([0.1801, 0.2171], 0.2651, [0.2730, 0.2818])$
C2	$([0.1718, 0.1767], 0.1818, [0.1865, 0.1891])$
C3	$([0.1975, 0.2040], 0.2167, [0.2572, 0.2988])$
C4	$([0.3783, 0.4494], 0.5000, [0.5328, 0.5456])$
	candidate C
C1	$([0.2679, 0.3652], 0.4458, [0.4840, 0.5136])$
C2	$([0.2291, 0.2474], 0.2727, [0.3109, 0.3782])$
C3	$([0.1616, 0.1760], 0.2045, [0.2094, 0.2136])$
C4	$([0.1819, 0.2131], 0.2500, [0.2996, 0.3783])$

Table V
THE IT2 FUZZY RATES FOR THE OVERALL GLOBAL FUZZY RATES USING APPROACH IN SECTION V-A

	Linear programming model(V-A)
A	$([0.2894, 0.3398], 0.3979, [0.4833, 0.5682])$
B	$([0.2185, 0.2611], 0.3132, [0.3560, 0.3947])$
C	$([0.1998, 0.2411], 0.2890, [0.3406, 0.4081])$

Table VII provides the comparative results from [4] and [5], in which the triangular fuzzy sets represented by (l, m, r) are used. It shows that candidate A is also the best candidate. Note that the middle points of the results, 0.40 for candidate A, 0.31 for candidate B and 0.29 for candidate C, are exactly the same (despite of the decimal digits) as our approach in Tables V and VI, namely 0.3979 for candidate A, 0.3131 for candidate B and 0.2890 for candidate C.

VIII. CONCLUSION

In this article, we have introduced an overall process for implementing interval type-2 fuzzy sets on group decision

Table VI
THE IT2 FUZZY RATES FOR THE OVERALL GLOBAL FUZZY RATES USING APPROACH IN SECTION V-B.

	Fuzzy Weighted Average(V-B)
A	$([0.3037, 0.3495], 0.3979, [0.4668, 0.5385])$
B	$([0.2260, 0.2679], 0.3132, [0.3445, 0.3687])$
C	$([0.2058, 0.2457], 0.2890, [0.3359, 0.3976])$

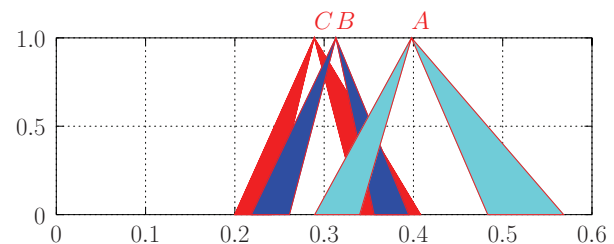


Figure 4. The global fuzzy rates of three candidates using linear programming model

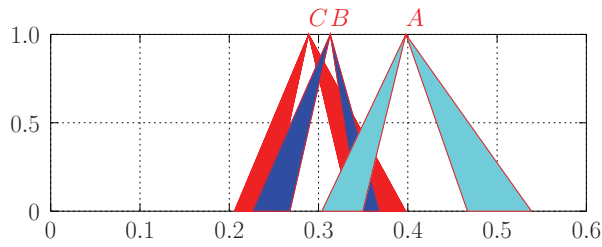


Figure 5. The global fuzzy rates of three candidates using IT2 fuzzy weighted average

Table VII
RESULTS OBTAINED IN [4] AND [5]

	Laarhoven and Pedrycz[4]	Boender, Graan and Lootsma[5]
A	(0.23, 0.40, 0.71)	(0.30, 0.40, 0.54)
B	(0.17, 0.31, 0.58)	(0.22, 0.31, 0.42)
C	(0.19, 0.29, 0.51)	(0.23, 0.29, 0.36)

making problems with uncertain judgments. Since most decision making problems exist in real world. The judgments are always made by human beings. Not all the experts are familiar with the mathematical properties of the decision making process. If the type of judgments given by experts can be allowed to as flexible as possible, then the collected decision data would be much more valuable. Type-2 fuzzy sets are the better mathematical quantities than type-1 fuzzy sets and even better than the crisp scales. Using type-2 fuzzy sets on decision making problems can lead to more realistic and accurate results. This paper proposes a very important approach on decision making problem with type-2 fuzzy sets specifically on the very popular decision tool, Analytic Hierarchy Process.

We extend the logarithmic least squares method proposed by [4] and formulate a quadratic model for the optimal solutions of the interval type-2 fuzzy priority weights. We also give a detail discussion about the simultaneous linear equations of the logarithmic decision variables. After obtaining these interval type-2 fuzzy weights, we construct a linear programming model to get the overall rate of interval type-2 fuzzy rates. Finally, the individual overall type-2 fuzzy rates are sorted using the centroids of the type-2 fuzzy sets.

The proposed model is for triangular type interval type-2 fuzzy sets. For different purposes, other kinds lower and upper membership functions can be used. For example, one can use triangular membership function for lower membership function and trapezoidal membership function for upper membership function or both trapezoidal membership functions for lower and upper membership function. In addition to the logarithmic least square method, other algorithms (for example, [9]) can be used to obtain the optimal type-2 fuzzy priority weights and rates. We need further studies on these topics.

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