

On Another Approach to the Definition of an L-fuzzy Valued Integral

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Abstract—We continue to develop a construction of an L-fuzzy valued measure extending a crisp measure defined on a σ -algebra of crisp sets to an L-fuzzy valued measure defined on a T_M -tribe. We describe two equivalent approaches to define an L-fuzzy valued integral of non-negative measurable functions.

Index Terms—L-set, L-fuzzy real number, L-fuzzy valued measure, L-fuzzy valued integral.

I. INTRODUCTION

One can find a lot of works regarding fuzzy approach to measure and integral. The biggest scope on this subject can be found in [1], [2], [3]. Our interest is in developing a theory where not only sets are fuzzy, but also measure and integral take fuzzy real values. In the series of our previous papers [4], [5], [6] the concepts of an L-fuzzy valued measure and integral were studied thoroughly. We continue to develop results obtained in the previous papers and show that L-fuzzy valued measure and integral preserve most of the properties of its crisp analogues (see e.g. [7]).

Inspired by the fact from the classical measure theory stating that an integral of a given non-negative function can be considered as a measure we obtain two different approaches to define an L-fuzzy valued integral.

The idea of the first approach is to extend a crisp measure ν to the L-fuzzy valued measure μ by using the construction described in [5] and then defining the L-fuzzy valued integral

$$\mu_f(E) = \int_E f \, d\mu,$$

where f is an integrable real-valued function and E is a measurable L-set [6].

The second approach shows that an L-fuzzy valued integral can be obtained by considering the measure ν_f using a crisp measure ν as following:

$$\nu_f(M) = \int_M f \, d\nu,$$

where f is an integrable non-negative function and M is a measurable set. And then extending it to the L-fuzzy valued measure $\tilde{\mu}_f$ according to the construction described in [5].

Finally we show that both approaches give us equivalent ways to define the L-fuzzy valued integral, i.e. $\mu_f = \tilde{\mu}_f$. This

result can be described by the following diagram:

$$\begin{array}{ccc} \nu & \longrightarrow & \mu \\ \downarrow & & \downarrow \\ \nu_f & \longrightarrow & \mu_f \end{array}$$

The second approach to the definition gives us an opportunity to simplify calculations of the L-fuzzy valued integral.

Our investigations of the L-fuzzy valued integral are motivated by our intention to develop results on approximation from [8], [9]. For the problems that can be solved only approximately the notion of the error of a method of approximation plays the fundamental role. In order to estimate the quality of approximation on an L-fuzzy set, we need an appropriate L-fuzzy analogue of a norm. Our intention is to use the L-fuzzy valued integral to define an L-fuzzy valued norm for investigation of the error of approximation on an L-set.

II. PRELIMINARIES

A. L-sets

Given a (crisp) universe X and a complete, completely distributive lattice $L(\wedge, \vee, 0_L, 1_L)$, an *L-subset* A of X (or, briefly, an *L-set* A) is a function $A : X \rightarrow L$. The class of all L-subsets of X is denoted L^X . The operations for L-sets A, B are defined by using the minimum triangular norm T_M , its corresponding conorm S_M and decreasing involution N :

$$\begin{aligned} (A \wedge B)(x) &= T_M(A(x), B(x)), \\ (A \vee B)(x) &= S_M(A(x), B(x)), \\ A^c(x) &= N(A(x)). \end{aligned}$$

A finite family of L-sets $A_1, A_2, \dots, A_n$ is said to be *T_M -disjoint* (see e.g. [1]) iff for each $k \in \{1, \dots, n\}$ we have

$$\left(\bigvee_{j=1, j \neq k}^n A_j \right) \wedge A_k = \emptyset.$$

A countable family of L-sets is said to be *T_M -disjoint* iff every finite subfamily of this family is *T_M -disjoint*.

We consider a class of L-sets called *T_M -tribe* (see e.g. [1]).

Definition II.1. A subclass $\Sigma \subset L^X$ is called a T_M -tribe on X if the following properties are satisfied:

- $\emptyset \in \Sigma$;
- for all $A \in \Sigma$ we have $A^c \in \Sigma$;
- for all sequences $(A_n)_{n \in \mathbb{N}} \subset \Sigma$ we have $\bigwedge_{n=1}^{\infty} A_n \in \Sigma$.

B. L-fuzzy real numbers

For our purposes we use the L-fuzzy real numbers as they were first defined by B. Hutton [10].

Definition II.2. An L-fuzzy real number is a function $z: \mathbb{R} \rightarrow L$ such that

- z is non-increasing;
- $\bigwedge_t z(t) = 0_L, \bigvee_t z(t) = 1_L$;
- z is left semi-continuous, i.e. for all $t_0 \in \mathbb{R}$ we have $\bigwedge_{t < t_0} z(t) = z(t_0)$.

In the original papers of this subject (see [10], [11], [12]) the L-fuzzy real numbers were defined not as order reversing functions, but as equivalence classes of such functions. However each class of equivalence has a unique left semi-continuous representative and therefore an L-fuzzy real number can be identified with this representative.

The set of all L-fuzzy real numbers is called the *L-fuzzy real line* and it is denoted by $\mathbb{R}(L)$. An L-fuzzy number z is called *non-negative* if $z(0) = 1_L$. We denote by $\mathbb{R}_+(L)$ the set of all non-negative L-fuzzy real numbers.

Operations with L-fuzzy real numbers such as addition $\oplus$ and multiplication by a real positive number are defined as following:

$$(z_1 \oplus z_2)(t) = \bigvee_{\tau} \{z_1(\tau) \wedge z_2(t - \tau)\}, \quad (zr)(t) = z\left(\frac{t}{r}\right).$$

The supremum and the infimum of a set of non-negative L-fuzzy numbers $F \subset \mathbb{R}_+(L)$ are defined by the formulas (see e.g. [8], [9]):

$$(\text{Inf } F)(t) = \bigwedge \{z(t) \mid z \in F\}, \quad t \in \mathbb{R},$$

$$\text{Sup } F = \text{Inf} \{z \mid z \in \mathbb{R}(L), z \geq z' \text{ for all } z' \in F\}.$$

Taking into account that F is bounded from below it is easy to see that $\text{Inf } F$ is an L-fuzzy real number. In case F is bounded from above (i.e. there exists $z_0 \in \mathbb{R}(L)$ such that $z \leq z_0$ for all $z \in F$), $\text{Sup } F$ is an L-fuzzy real number, otherwise the condition

$$\bigwedge_t \text{Sup } F(t) = 0_L$$

does not necessarily hold.

Going forward we will need also the countable addition of non-negative fuzzy real numbers. Given a sequence of non-negative fuzzy real numbers $(z_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+(L)$ we consider the

countable sum

$$\bigoplus_{n=1}^{\infty} z_n = \text{Sup} \{z_1 \oplus z_2 \oplus \dots \oplus z_n \mid n \in \mathbb{N}\}.$$

For $a \in \mathbb{R}_+$ and $\alpha \in L$ by $z(a, \alpha)$ we denote a special type of non-negative L-fuzzy real numbers

$$(z(a, \alpha))(t) = \begin{cases} 1, & t \leq 0, \\ \alpha, & 0 < t \leq a, \\ 0, & t > a, \end{cases}$$

that will play an important role in our work.

III. L-FUZZY VALUED MEASURE

A. Definition of an L-fuzzy valued measure

We consider a measure that is defined on a T_M -tribe and takes values in $\mathbb{R}_+(L)$.

Definition III.1. Let Σ be a T_M -tribe. A function $\mu: \Sigma \rightarrow \mathbb{R}_+(L)$ is called an L-fuzzy valued measure if it satisfies the following conditions:

- $\mu(\emptyset) = z(0, 1_L)$;
- μ is T_M -valuation, i.e. for all $A, B \in \Sigma$ it holds

$$\mu(A \wedge B) \oplus \mu(A \vee B) = \mu(A) \oplus \mu(B);$$

- μ is left T_M -continuous, i.e.

$$\text{Sup} \{\mu(A_n) \mid n \in \mathbb{N}\} = \mu(A),$$

$$\text{where } (A_n)_{n \in \mathbb{N}} \subset \Sigma, \bigvee_{n \in \mathbb{N}} A_n = A \in \Sigma.$$

B. Construction of an L-fuzzy valued measure

For a given σ -algebra of crisp sets $\Phi \subset 2^X$ and a finite measure $\nu: \Phi \rightarrow \mathbb{R}_+$ an L-fuzzy valued measure can be obtained by the following schema (see e.g. [4], [5]):

- For $M \in \Phi, \alpha \in L$ we define an L-fuzzy set

$$(A(M, \alpha))(x) = \begin{cases} \alpha, & x \in M, \\ 0, & x \notin M. \end{cases}$$

All these L-sets form a class of L-sets that we denote by $\wp$.

- Next we define an L-fuzzy valued function $m: \wp \rightarrow \mathbb{R}_+(L)$ by

$$m(A(M, \alpha)) = z(\nu(M), \alpha),$$

and we extend it to the L-fuzzy valued function $m^*: L^X \rightarrow \mathbb{R}_+(L)$ as following:

$$m^*(E) = \text{Inf} \left\{ \bigoplus_{n=1}^{\infty} m(E_n) \mid (E_n)_{n \in \mathbb{N}} \subset \wp : E \leq \bigvee_{n=1}^{\infty} E_n \right\}.$$

- We denote by Σ the T_M -tribe of all so called m^* -measurable L-sets $E \in L^X$ such that for all $B \in L^X$ it holds

$$m^*(E) \oplus m^*(B) = m^*(E \wedge B) \oplus m^*(E \vee B),$$

$$m^*(E^c) \oplus m^*(B) = m^*(E^c \wedge B) \oplus m^*(E^c \vee B).$$

- Finally, we define μ as the restriction of m^* to Σ we can state that μ is an L-fuzzy valued measure and

$$\mu(E) = m^*(E), \text{ for all } E \in \Sigma.$$

IV. AN L-FUZZY VALUED INTEGRAL

A. Definition of an L-fuzzy valued integral

In this section we define an L-fuzzy valued integral

$$\int_E f d\mu,$$

where $E \in \Sigma$ and $f : X \rightarrow \mathbb{R}$ is a non-negative measurable function with respect to σ -algebra Φ .

By analogy with the classical case ([7]) we define an L-fuzzy valued integral stepwise, first considering the case of simple non-negative measurable functions (for short SNMF):

$$\int_E \left(\sum_{i=1}^n c_i \chi_{C_i} \right) d\mu = \bigoplus_{i=1}^n (c_i \mu(C_i \wedge E)),$$

whenever

- $c_i \in \mathbb{R}_+, C_i \in \Phi$ for all $i = 1, \dots, n$,
- χ_{C_i} is the characteristic function of C_i , $i = 1, \dots, n$,
- $C_1, \dots, C_n$ are pairwise disjoint sets.

Then considering the case for non-negative measurable function f (for short NMF):

$$\int_E f d\mu = \text{Sup} \left\{ \int_E g d\mu \mid g \leq f \text{ and } g \text{ is SNMF} \right\}.$$

For $\mathbb{I}_f = \int_E f d\mu$ due to properties of the supremum of a set of L-fuzzy numbers, we have

- $\mathbb{I}_f$ is non-increasing,
- $\bigvee_t \mathbb{I}_f(t) = 1_L$,
- $\mathbb{I}_f$ is left semi-continuous, i.e. $\bigwedge_{t < t_0} \mathbb{I}_f(t) = \mathbb{I}_f(t_0)$.

Definition IV.1. We say that a non-negative measurable function f is L-fuzzy integrable over L-set E iff

$$\bigwedge_t \mathbb{I}_f(t) = 0_L.$$

Let us note that if f is non-negative integrable over X then it is also L-fuzzy integrable over every $E \in \Sigma$. Going forward we consider only such functions.

B. Properties of L-fuzzy valued integral

For L-fuzzy integrable non-negative functions $f, f_1, f_2, \dots, f_n, \dots$ and measurable L-sets $E, E_1, E_2, \dots, E_n, \dots \in \Sigma$ the following properties of L-fuzzy valued integral are true (see e.g. [6]).

- (I1) $\int_E d\mu = \mu(E)$
- (I2) $r \in \mathbb{R}_+ \Rightarrow \int_E r f d\mu = r \int_E f d\mu$
- (I3) $f_1 \leq f_2 \Rightarrow \int_E f_1 d\mu \leq \int_E f_2 d\mu$
- (I4) $E_1 \leq E_2 \Rightarrow \int_{E_1} f d\mu \leq \int_{E_2} f d\mu$
- (I5) $\int_E (f_1 + f_2) d\mu = \int_E f_1 d\mu \oplus \int_E f_2 d\mu$
- (I6) $E_1 \wedge E_2 = \emptyset \Rightarrow \int_{E_1 \vee E_2} f d\mu = \int_{E_1} f d\mu \oplus \int_{E_2} f d\mu$

$$(I7) (E_n)_{n \in \mathbb{N}} : E_n \leq E_{n+1} \text{ and } \bigvee_{n \in \mathbb{N}} E_n = E \Rightarrow$$

$$\int_E f d\mu = \text{Sup} \left\{ \int_{E_n} f d\mu \mid n \in \mathbb{N} \right\}$$

$$(I8) (f_n)_{n \in \mathbb{N}} : f_n \leq f_{n+1} \text{ and } \lim_{n \rightarrow \infty} f_n = f \Rightarrow$$

$$\int_E f d\mu = \text{Sup} \left\{ \int_E f_n d\mu \mid n \in \mathbb{N} \right\}$$

V. TWO APPROACHES TO DEFINE L-FUZZY VALUED MEASURE BASED ON A FUNCTION

A. The first approach

From the classical measure theory (see e.g [7]) it is known that having a measure ν and non-negative measurable function f we can get a new measure ν_f defined as following

$$\nu_f(M) = \int_M f d\nu.$$

By using the same logic in the fuzzy case we want to make sure that this statement will be preserved.

We define

$$\mu_f(E) = \int_E f d\mu,$$

where $\mu_f : \Sigma \rightarrow \mathbb{R}_+(L)$ and f is a real valued non-negative L-fuzzy integrable function.

Theorem V.1. μ_f is an L-fuzzy valued measure.

Proof: Due to integral property μ_f is left T_M -continuous. We need to prove that μ_f is T_M -valuation, i.e.

$$\mu_f(E_1 \vee E_2) \oplus \mu_f(E_1 \wedge E_2) = \mu_f(E_1) \oplus \mu_f(E_2) \text{ for all } E_1, E_2 \in \Sigma.$$

Considering $M_1 = \{x \in X \mid E_1(x) \geq E_2(x)\}$ and $M_2 = \{x \in X \mid E_1(x) < E_2(x)\}$ we can split L-sets E_1, E_2 into two T_M -disjoint L-sets as following:

$$E_1 = (E_1 \wedge M_1) \vee (E_1 \wedge M_2) \text{ and } E_2 = (E_2 \wedge M_1) \vee (E_2 \wedge M_2).$$

Obviously,

$$E_1 \vee E_2 = (E_1 \wedge M_1) \vee (E_2 \wedge M_2)$$

and

$$E_1 \wedge E_2 = (E_1 \wedge M_2) \vee (E_2 \wedge M_1).$$

Hence,

$$\begin{aligned} \int_{E_1} f d\mu \oplus \int_{E_2} f d\mu &= \int_{(E_1 \wedge M_1) \vee (E_1 \wedge M_2)} f d\mu \oplus \int_{(E_2 \wedge M_1) \vee (E_2 \wedge M_2)} f d\mu = \\ &= \int_{E_1 \wedge M_1} f d\mu \oplus \int_{E_1 \wedge M_2} f d\mu \oplus \int_{E_2 \wedge M_1} f d\mu \oplus \int_{E_2 \wedge M_2} f d\mu = \\ &= \int_{E_1 \vee E_2} f d\mu \oplus \int_{E_1 \wedge E_2} f d\mu. \end{aligned}$$

As we can see L-fuzzy valued measure μ_f was obtained by the following scheme:

- first extending crisp measure ν with construction described in section III-B;
- second defining a measure by using an integral with respect to L-fuzzy valued T_M -measure μ .

B. The second approach

Now if we change the order in scheme described above we get L-fuzzy valued measure $\tilde{\mu}_f$.

For a given σ -algebra $\Phi \subset 2^X$ and a finite measure $\nu : \Phi \rightarrow \mathbb{R}_+$ we define measure ν_f as following:

$$\nu_f(M) = \int_M f \, d\nu.$$

An L-fuzzy valued measure $\tilde{\mu}_f$ can be obtained by using the same scheme as in section III-B :

- We define an L-fuzzy valued function $m_f : \wp \rightarrow \mathbb{R}_+(L)$ by

$$m_f(A(M, \alpha)) = z(\nu_f(M), \alpha),$$

and we extend it to the L-fuzzy valued function $m_f^* : L^X \rightarrow \mathbb{R}_+(L)$ as following:

$$m_f^*(E) = \text{Inf} \left\{ \bigoplus_{n=1}^{\infty} m_f(E_n) \mid (E_n)_{n \in \mathbb{N}} \subset \wp : E \leq \bigvee_{n=1}^{\infty} E_n \right\}.$$

- We denote by Σ_f the class of all so called m_f^* -measurable L-sets $E \in L^X$ such that for all $B \in L^X$ it holds

$$m_f^*(B) \oplus m_f^*(E) = m_f^*(B \wedge E) \oplus m_f^*(B \vee E),$$

$$m_f^*(E^c) \oplus m_f^*(B) = m_f^*(E^c \wedge B) \oplus m_f^*(E^c \vee B).$$

- Finally, considering $\tilde{\mu}_f$ as restriction of m_f^* to Σ_f we obtain the L-fuzzy valued measure: $\tilde{\mu}_f = m_f^*(E)$.

VI. THE EQUIVALENCE OF TWO DEFINITIONS OF AN L-FUZZY VALUED INTEGRAL

In this section we show that two approaches to define an L-fuzzy valued integral described in the previous section give us the equivalent result.

Theorem VI.1. *For all integrable non-negative functions f it holds:*

(T1) $\Sigma \subset \Sigma_f$,

(T2) for all $E \in \Sigma$ we have $\mu_f = \tilde{\mu}_f$.

To prove the Theorem VI.1 we use the following Lemma.

Lemma VI.2. *For all $E \in L^X$ it holds:*

(L1) if f is SNMF (i.e. $f = \sum_{i=1}^n c_i \chi_{C_i}$), then

$$m_f^*(E) = \bigoplus_{i=1}^n c_i m^*(E \wedge C_i);$$

(L2) if $(f_n)_{n \in \mathbb{N}}$ is a sequence of SNMF such that $f_n \leq f_{n+1}$ and $\lim_{n \rightarrow \infty} f_n = f$, then

$$m_f^*(E) = \text{Sup} \{ m_{f_n}^*(E) \mid n \in \mathbb{N} \}.$$

Proof: (L1) First we show that equality holds for L-set $E = A(M, \alpha) \in \wp$. Taking into account the properties of L-fuzzy real numbers we get:

$$m_f(A(M, \alpha)) = z(\nu_f(M), \alpha) = z\left(\int_M f \, d\nu, \alpha\right) =$$

$$\begin{aligned} &= z\left(\int_M \sum_{i=1}^n c_i \chi_{C_i} \, d\nu, \alpha\right) = z\left(\sum_{i=1}^n c_i \nu(M \wedge C_i), \alpha\right) = \\ &= \bigoplus_{i=1}^n c_i z(\nu(M \wedge C_i), \alpha) = \bigoplus_{i=1}^n c_i m(A(M, \alpha) \wedge C_i). \end{aligned}$$

Now for the sequence of L-sets $(E_k)_{k \in \mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k$ we obtain:

$$\bigoplus_{k=1}^{\infty} m_f(E_k) = \bigoplus_{k=1}^{\infty} \bigoplus_{i=1}^n c_i m(E_k \wedge C_i) = \bigoplus_{i=1}^n c_i \bigoplus_{k=1}^{\infty} m(E_k \wedge C_i).$$

Finally, recalling the definition of m_f^* :

$$\begin{aligned} m_f^* &= \text{Inf} \left\{ \bigoplus_{k=1}^{\infty} m_f(E_k) \mid (E_k)_{k \in \mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k \right\} = \\ &= \bigoplus_{i=1}^n c_i \text{Inf} \left\{ \bigoplus_{k=1}^{\infty} m(E_k \wedge C_i) \mid (E_k)_{k \in \mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k \right\} = \\ &= \bigoplus_{i=1}^n c_i m^*(E \wedge C_i). \end{aligned}$$

(L2) We start with an L-set $E = A(M, \alpha) \in \wp$. Using the properties of L-fuzzy real numbers we get:

$$\begin{aligned} \text{Sup} \{ m_{f_n}(A(M, \alpha)) \mid n \in \mathbb{N} \} &= \text{Sup} \{ z(\nu_{f_n}(M), \alpha) \mid n \in \mathbb{N} \} = \\ &= z(\text{sup}_n(\nu_{f_n}(M)), \alpha) = z(\nu_f(M), \alpha) = m_f(A(M, \alpha)). \end{aligned}$$

Now taking any $E \in L^X$ and the fact that L is completely distributive lattice we obtain:

$$\begin{aligned} &\text{Sup} \{ m_{f_n}^*(E) \mid n \in \mathbb{N} \} = \\ &= \text{Sup} \left\{ \text{Inf} \left\{ \bigoplus_{k=1}^{\infty} m_{f_n}(E_k) \mid (E_k)_{k \in \mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k \right\} \mid n \in \mathbb{N} \right\} = \\ &= \text{Inf} \left\{ \text{Sup} \left\{ \bigoplus_{k=1}^{\infty} m_{f_n}(E_k) \mid n \in \mathbb{N} \right\} \mid (E_k)_{k \in \mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k \right\} = \\ &= \text{Inf} \left\{ \bigoplus_{k=1}^{\infty} m_f(E_k) \mid (E_k)_{k \in \mathbb{N}} \subset \wp : E \leq \bigvee_{k=1}^{\infty} E_k \right\} = m_f^*(E). \end{aligned}$$

Coming back to theorem VI.1.

Proof: (T1) We consider $E \in \Sigma$ and start with the case when f - SNMF or $f = \sum_{i=1}^n c_i \chi_{C_i}$. For every $i \in \{1, \dots, n\}$ if $E \in \Sigma$ then $E \wedge C_i \in \Sigma$ and following for all $B \in L^X$ we have

$$\begin{aligned} m^*(E \wedge C_i) \oplus m^*(B \wedge C_i) &= \\ &= m^*(E \wedge B \wedge C_i) \oplus m^*((E \wedge C_i) \vee (B \wedge C_i)). \end{aligned}$$

Now multiplying equality by c_i and summing up by i we obtain.

$$\sum_{i=1}^n c_i m^*(E \wedge C_i) \oplus \sum_{i=1}^n c_i m^*(B \wedge C_i) =$$

$$= \sum_{i=1}^{\infty} c_i m^*(E \wedge B \wedge C_i) \oplus \sum_{i=1}^{\infty} c_i m^*((E \wedge C_i) \vee (B \wedge C_i)).$$

And finally applying Lemma VI.2:

$$m_f^*(E) \oplus m_f^*(B) = m_f^*(E \wedge B) \oplus m_f^*(E \vee B).$$

By analogy we can prove the second equality

$$m_f^*(E^c) \oplus m_f^*(B) = m_f^*(E^c \wedge B) \oplus m_f^*(E^c \vee B).$$

In the case when f is NMF there exists a sequence of SNMF $(f_n)_{n \in \mathbb{N}}$ such that $f_n \leq f_{n+1}$ and $\lim_{n \rightarrow \infty} f_n = f$. For all f_n it holds

$$m_{f_n}^*(E) \oplus m_{f_n}^*(B) = m_{f_n}^*(E \wedge B) \oplus m_{f_n}^*(E \vee B).$$

Now taking Sup we obtain

$$\begin{aligned} & \text{Sup}\{m_{f_n}^*(E) | n \in \mathbb{N}\} \oplus \text{Sup}\{m_{f_n}^*(B) | n \in \mathbb{N}\} = \\ & = \text{Sup}\{m_{f_n}^*(E \wedge B) | n \in \mathbb{N}\} \oplus \text{Sup}\{m_{f_n}^*(E \vee B) | n \in \mathbb{N}\}. \end{aligned}$$

And due to Lemma VI.2

$$m^*(E) \oplus m^*(B) = m^*(E \wedge B) \oplus m^*(E \vee B).$$

The same logic can be used to prove the second equality.

(T2) We start again with SNMF f . Using Lemma VI.2 we see that

$$\begin{aligned} \mu_f(E) &= \int_E f d\mu = \bigoplus_{i=1}^n c_i \mu(E \wedge C_i) = \\ &= \bigoplus_{i=1}^n c_i m^*(E \wedge C_i) = m_f^*(E) = \tilde{\mu}_f(E). \end{aligned}$$

In the case when f is NMF we use a sequence $(f_n)_{n \in \mathbb{N}}$ such that $f_n \leq f_{n+1}$ and $\lim_{n \rightarrow \infty} f_n = f$. By using Lemma VI.2 and the fact that for the simple functions $\mu_{f_n} = \tilde{\mu}_{f_n}$ we get:

$$\begin{aligned} \mu_f(E) &= \int_E f d\mu = \text{Sup}\left\{\int_E f_n d\mu \mid n \in \mathbb{N}\right\} = \text{Sup}\{\mu_{f_n}(E) \mid n \in \mathbb{N}\} = \\ &= \text{Sup}\{\tilde{\mu}_{f_n}(E) \mid n \in \mathbb{N}\} = m_f^*(E) = \tilde{\mu}_f(E). \end{aligned}$$

■

This result gives us an opportunity to define an L-fuzzy valued integral by applying the following formula:

$$\int_E f d\mu = \tilde{\mu}_f(E).$$

Summarizing, we want to mention that both ways of defining the L-fuzzy valued integral have their own motivation. The first approach (see V-A) has a theoretical foundation, while the second approach (see V-B) gives us an opportunity to simplify calculations of the L-fuzzy valued integral.

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