

On Spline Methods of Approximation under L-fuzzy Information

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Abstract—This work is closely related to our previous papers on algorithms of approximation under L -fuzzy information. In the classical theory of approximation central algorithms were worked out on the basis of usual, that is crisp splines. We describe central methods for solution of linear problems with balanced L -fuzzy information and develop the concept of L -fuzzy splines.

Index Terms—Error of approximation, central algorithm of approximation, spline, L -fuzzy real number, L -fuzzy information, L -fuzzy spline.

I. INTRODUCTION

In the literature one can find two different approaches to the approximation problem when the conditions which must be satisfied under the process of approximation are not strictly defined. One of them, we call it *the statistical approach*, is applicable in cases when the approximation relies on information given with some error and is based on the use of distribution functions. The second, so called *interval method* considers approximation when imprecision of the used information is implied by inevitable interval estimation. As different from these two approaches we propose here an alternative approach to approximation problem which could be helpful in cases when the imprecision of information used in the approximation process has fuzzy, or vague nature.

We consider the problem of approximation of an operator B defined on a set X and taking values in a normed space Y . To find an approximation of the exact value Bx , we must know something about the problem element $x \in X$. We denote the information of x by Ax and suppose that this information is given by an information vector z . Usually the information is imprecise, thus there may exist many different information vectors z , corresponding to one problem element $x \in X$. Under such interpretation it seems natural to consider information Ax as a fuzzy, rather than as a crisp one, i.e. to realize it as an L -fuzzy set Ax by using a lattice L .

In order to estimate the "quality" of *approximation under fuzzy data*, we need an appropriate L -fuzzy analogue of a real number. Notice that the error of a method of approximation in this case is characterized by the supremum of the corresponding L -fuzzy set of real numbers. The principal aim of our work (see also [1], [2], [3]) is to generalize for the case

of L -fuzzy information the idea of an *optimal error method* of approximation (the method whose error is the infimum of the errors of all methods for a given problem using the same information) and the concept of a *central algorithm*, which is always an optimal error algorithm and in the crisp case is useful in practice as well as in the general theory (see e.g. [4]).

In the classical approximation theory central methods for solution of linear problems with balanced information usually are based on splines. In the crisp case (see e.g. [5], [6], [7], [8]) one can distinguish three types of splines used in the study of approximation problems: they are respectively *the interpolation splines*, *the smoothing splines* and, the most general, *splines on the basis of convex sets*. Generalizing the last type of crisp splines, here we introduce the concept of an *L -fuzzy spline* on the basis of L -fuzzy sets and with the help of such splines initiate to develop an approach to study of the approximation problem in case when L -fuzzy information is taken as the basis for this approximation.

We suppose that $L = (L, \leq, \wedge, \vee)$ is a complete, completely distributive lattice, and denote its bottom and top elements by 0 and 1 respectively.

II. PRELIMINARIES

A. L -fuzzy real numbers

In the literature on fuzzy mathematics one can find several different schemes for defining fuzzy real numbers. For our purposes the most appropriate is the approach originating in B. Hutton's paper [9] and further developed by several authors (see e.g. [10], [11], [12], [13], [14], [15]). However, we had to modify it a little.

Definition II.1. An L -fuzzy real number is a function $z: \mathbb{R} \rightarrow L$ such that

- z is non-increasing;
- $\bigwedge_t z(t) = 0$;
- z is left semi-continuous, i.e. for all $t_0 \in \mathbb{R}$ we have $\bigwedge_{t < t_0} z(t) = z(t_0)$.

Thus we omit the condition $\bigvee_t z(t) = 1$, which is requested by B.Hutton and his followers.

In the original papers on this subject L -fuzzy real numbers were defined not as order reversing functions, but as equivalence classes of such functions. However, as noticed in [13], each class of equivalence has a unique left semi-continuous representative and therefore an L -fuzzy real number can be identified with this representative.

The set $\mathbb{R}(L)$ of all L -fuzzy real numbers is called the L -fuzzy real line.

Given $z_1, z_2 \in \mathbb{R}(L)$ let

$$(z_1 \oplus z_2)(t) = \bigvee_{\tau} (z_1(\tau) \wedge z_2(t - \tau)).$$

The operation of fuzzy addition $\oplus$ is a jointly continuous extension of addition $+$ from the real line $\mathbb{R}$ to the fuzzy real line $\mathbb{R}(L)$.

The operation $\oplus$ can be naturally extended to the case of L -fuzzy subsets M, N of a linear space X (that is mappings $M, N : X \rightarrow L$) as follows:

$$M \oplus N : X \rightarrow L : (M \oplus N)(x) := \bigvee_{y \in X} (M(y) \wedge N(x - y)).$$

The definition of the product $\odot$ of L -fuzzy numbers is more complicated if compared with the definition of their sum. However, for our purposes we need only to multiply an L -fuzzy number by a real number $\gamma \in \mathbb{R}$. In this case we define the multiplication by the following simple formula:

$$(\gamma \odot z)(t) = z\left(\frac{t}{\gamma}\right) \text{ if } \gamma \neq 0,$$

and

$$(0 \odot z)(t) = \begin{cases} \bigvee_{\tau \in \mathbb{R}} z(\tau) & \text{for } t \leq 0, \\ 0 & \text{for } t > 0. \end{cases}$$

In case $\gamma > 0$ this definition is equivalent to the standard one (see e.g. [11]). The operation of product is in accordance with the addition and is jointly continuous.

The operation $\odot$ can be naturally extended to the case of an L -fuzzy subset M of a linear space X as follows:

$$(\gamma \odot M)(x) = M\left(\frac{x}{\gamma}\right), \quad x \in X, \text{ for any real number } \gamma \neq 0.$$

B. Infimum and supremum

Here we recall some concepts and results from [1], [2].

Definition II.2. The infimum and the supremum of a set of L -fuzzy numbers $F \subset \mathbb{R}(L)$ are defined by the formulas :

$$(\text{Inf } F)(t) = \bigwedge \{z(t) \mid z \in F\}, \quad t \in \mathbb{R},$$

$$\text{Sup } F = \text{Inf} \{z \mid z \in \mathbb{R}(L), z \geq z' \text{ for all } z' \in F\}.$$

It is easy to see that $\text{Inf } F$ is an L -fuzzy real number. In case F is bounded from above (i.e. there exists $z_0 \in \mathbb{R}(L)$ such that $z \leq z_0$ for all $z \in F$), $\text{Sup } F$ is an L -fuzzy real number, otherwise the condition

$$\bigwedge_t \text{Sup } F(t) = 0$$

does not necessarily hold.

Definition II.3. By the supremum of a bounded L -fuzzy set $M : \mathbb{R} \rightarrow L$ we call the L -fuzzy real number

$$\text{Sup}(M) : \mathbb{R} \rightarrow L$$

defined by the equality

$$\text{Sup}(M)(t) = \bigwedge_{u < t} \left(\bigvee_{\tau > u} M(\tau) \right).$$

One can prove that $\text{Sup}(M)$ is an L -fuzzy real number. Let us note that we define the supremum only for the case of a bounded L -fuzzy set. Recall that in case X is metric, an L -fuzzy set $M : X \rightarrow L$ is called *bounded* if $M^{-1}[\alpha, 1]$ is a bounded subset of X for any $\alpha \neq 0$. Similarly the supremum for an unbounded L -fuzzy set could be defined, however in this case it wouldn't be a fuzzy number: the condition $\bigwedge_t z(t) = 0$ will be violated.

Obviously, in case when M is crisp, this definition reduces to the classical definition of the supremum.

In case $M = \varphi(\mathcal{V})$, where $\mathcal{V} : X \rightarrow L$ and $\varphi : X \rightarrow \mathbb{R}$, we write

$$\text{Sup}_{x \in \mathcal{V}} \varphi(x) \quad \text{or} \quad \text{Sup}\{\varphi(x) : x \in \mathcal{V}\}$$

instead of

$$\text{Sup}(\varphi(\mathcal{V})).$$

Recall that the image $\varphi(\mathcal{V})$ of a fuzzy set $\mathcal{V} \in L^X$ under a mapping $\varphi : X \rightarrow \mathbb{R}$ is defined by the equality

$$\varphi(\mathcal{V})(x) = \bigvee \{\mathcal{V}(y) : \varphi(y) = x\}.$$

Given an L -fuzzy set $M : X \rightarrow L$, where X is a normed space, we define its L -valued norm as an L -fuzzy number

$$\|M\|^L : \mathbb{R} \rightarrow L$$

given by the equality

$$\|M\|^L(t) = \text{Sup}\{M(x) \mid \|x\|_X = t\}.$$

One can prove that

$$\|\gamma \odot M\|^L = |\gamma| \odot \|M\|^L$$

and

$$\|M \oplus N\|^L \leq \|M\|^L \oplus \|N\|^L$$

for any L -fuzzy sets $M, N : X \rightarrow L$, and hence

$$\|\cdot\|^L : L^X \rightarrow L^{\mathbb{R}}$$

inherits the standard properties of a norm.

C. Radius, diameter and centre of an L -fuzzy set

Here we recall some concepts and results from [3]. Given an L -fuzzy set $V : X \rightarrow L$, where X is a normed space.

Definition II.4. The radius and the diameter of a bounded L -fuzzy set V are defined, respectively by

$$\text{rad}(V) = \inf_{x \in X} \sup_{y \in V} \|x - y\|_X$$

and

$$\text{diam}(V) = \sup_{x, y \in V} \|x - y\|_X.$$

Definition II.5. If for a bounded L -fuzzy set V there exists $c \in X$ such that

$$\text{rad}(V) = \sup_{x \in V} \|c - x\|_X$$

then c is called a centre of V and it is denoted by $c(V)$.

Note that $c(V)$ can be an element "outside" V and need not be unique.

We say that an L -fuzzy set V is symmetric with respect to an element $x_0 \in X$, if for any element $x \in X$ it holds

$$V(x_0 + x) = V(x_0 - x).$$

We call an L -fuzzy set symmetric in case it is symmetric with respect to some element.

Proposition II.6. If a bounded L -fuzzy set V is symmetric with respect to an element $c \in Y$, then c is the centre of V .

III. ON METHODS OF APPROXIMATION UNDER L -FUZZY INFORMATION

A. Approximation under L -fuzzy information

In this section, following [2], [3] we generalize the well known notions of "classic" or "crisp" approximation theory (see, e.g. [4]) to the L -fuzzy case. We consider the problem (A, B) of approximation of an operator $B : X \rightarrow Y$ defined in a set X and taking values in a normed space Y under information given by an operator $A : X \rightarrow L^{\mathbb{R}^n}$.

By a method (an algorithm) of solution of the problem (A, B) we mean any operator $\varphi : \mathbb{R}^n \rightarrow Y$. For an information vector $z \in \mathbb{R}^n$ the value $\varphi(z)$ must approximate the exact value of solution Bx for the problem element x corresponding to the information vector z :

$$x \in X \implies z \tilde{A} x \in L^{\mathbb{R}^n} \implies \varphi(z) \approx Bx \in Y.$$

For most problems the information operator is not one-to-one and a given information vector z does not determine the problem element x (the solution element Bx) in the unique way. Thus there may exist many different problem elements (solution elements) with the same information. Such problems also cannot be solved exactly. For problems which can be solved only approximately the notion of the error of a method of approximation plays the fundamental role.

B. Error of a method of approximation under L -fuzzy information

We investigate the error of approximation on a non-empty subset X_0 of the set X . Let $\varphi : \mathbb{R}^n \rightarrow Y$ be a method for solving the problem (A, B) . Our aim is to define the error of the method for an element $x \in X_0$. There may exist many different elements $u \in X_0$ corresponding to the same information Ax . Since each of them has the correspondness degree characterized by the value $Ax(z)$, such elements form an L -fuzzy subset of X_0 . Let us denote it by U_z :

$$U_z(x) = Ax(z) \quad \text{for all } x \in X_0; \quad U_z(x) = 0 \quad \text{for all } x \notin X_0.$$

Thus knowing only z it is impossible to recognize which problem element $x \in U_z$ is being actually approximated and we consider all of them.

Definition III.1. The error of approximation an element $x \in X_0$ by a method φ is given by the value

$$e(\varphi, z) = \sup_{x \in U_z} \|Bx - \varphi(z)\|_Y.$$

It is easy to see that the radius of the L -fuzzy set $V_z = B(U_z)$ plays the essential role for the value $e(\varphi, z)$.

Proposition III.2. $\text{rad}(V_z) \leq e(\varphi, z)$.

Definition III.3. The error of approximation by a method φ is given by the value

$$e(\varphi) = \sup_{z \in \mathbb{R}^n} e(\varphi, z).$$

Definition III.4. A method φ^o is called an optimal error method iff

$$e(\varphi^o) = \inf_{\varphi} e(\varphi).$$

C. Central methods of approximation under L -fuzzy information

In [2] it is proved that for any method φ

$$e(\varphi) = \sup_{x \in X_0} \sup_{z \in Ax} \|Bx - \varphi(z)\|_Y$$

and

$$r(A, B) \leq e(\varphi),$$

where $r(A, B)$ is the radius of information for a problem (A, B) defined by

$$r(A, B) = \sup_z \text{rad}(V_z).$$

In the sequel we assume that the radius $r(A, B)$ of information is finite.

Definition III.5. A method φ^c is called a central method on $\mathbb{R}^n$ iff

$$\varphi^c(z) = c(V_z) \quad \text{for all } z \in \mathbb{R}^n.$$

Proposition III.6. For any central algorithm φ^c

$$e(\varphi^c) = r(A, B).$$

This proposition implies that:

- 1) any central algorithm φ^c is an optimal error algorithm;
- 2) if for problem (A, B) there exists a central algorithm, then

$$r(A, B) = e(\varphi^o).$$

IV. SPLINE METHODS FOR SOLVING LINEAR PROBLEMS OF APPROXIMATION

A. Linear problems with balanced information

In this work we consider algorithms of approximation of a linear operator under balanced L -fuzzy information, i.e. we investigate approximation methods under assumptions that

$$A : X \rightarrow L^{\mathbb{R}^n}$$

is a linear operator defined on a vector space X , i.e.

$$A(\gamma_1 x_1 + \gamma_2 x_2) = \gamma_1 A x_1 \oplus \gamma_2 A x_2 \text{ for all } x_1, x_2 \in X, \gamma_1, \gamma_2 \in \mathbb{R},$$

and an information A is balanced in the following sense: for any $x \in X$ the set $Ax \in L^{\mathbb{R}^n}$ is symmetric, i.e. there exists an element $z_x \in \mathbb{R}^n$ such that

$$Ax(z_x + z) = Ax(z_x - z) \text{ for all } z \in \mathbb{R}^n$$

and

$$\bigvee_{z \in \mathbb{R}^n} Ax(z) = Ax(z_x).$$

Under these assumptions (A, B) is called a linear problem with balanced information.

We assume also that the set X_0 is given by a linear operator T defined on X and taking values in a Hilbert space W as follows

$$X_0 = \{x \in X : \|Tx\|_W \leq 1\}.$$

Note that central methods for solution of linear problems with balanced information worked out on the basis of *usual*, that is *crisp* splines. To define them we use the defuzzification

$$a : X \rightarrow \mathbb{R}^n$$

of an operator A :

$$a(x) = z_x \text{ for all } x \in X.$$

B. Spline methods

In the sequel assume that $T(Ker(a))$ is closed and $a(X) = \mathbb{R}^n$. Under this assumption for any $z \in \mathbb{R}^n$ there exists an element $s_z \in X$ such as

$$\|Ts_z\|_W = \inf_{x \in X, z_x = z} \|Tx\|_W,$$

called a *spline interpolating* z . The space of all interpolating splines $S(T, a)$ is described by the formula

$$S(T, a) = \{s \in X : \langle Ts, Tx \rangle_W = 0 \text{ for all } x \in Ker(a)\},$$

where $Ker(a)$ is the kernel of an operator a .

For the notions and results on splines see e.g. [5], [6], [7], [8].

The spline algorithm may be derived as follows. Let

$$e^i = (0, 0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$$

denote the i -th unit vector, $i = 1, 2, \dots, n$. Find a spline s_i interpolating e^i . The spline algorithm φ^s is of the form:

$$\varphi^s(z) = Bs(z), \text{ where } s(z) = \sum_{i=1}^n z_i s_i \text{ for all } z \in \mathbb{R}^n.$$

Proposition IV.1. *The method φ^s is a central method.*

This proposition implies that:

- 1) the spline algorithm φ^s is an optimal error algorithm;
- 2) for linear problems (A, B) with balanced information:

$$r(A, B) = e(\varphi^s).$$

C. L -fuzzy splines

In this section we generalize the concept of a spline. For a given vector $z \in \mathbb{R}^n$ we consider the following problem of minimization

$$\|TS_z\|^L = \inf_{x \in A^{-1}(z)} \|Tx\|_W,$$

where S_z is an L -fuzzy set in X . A solution of this problem $S_z : X \rightarrow L$ is called an *L -fuzzy spline interpolating* z . We denote by $S(T, A, L)$ the set of all L -fuzzy splines.

Proposition IV.2. *If an information operator $A : X \rightarrow L^{\mathbb{R}^n}$ is given by a usual operator $a : X \rightarrow \mathbb{R}^n$ as follows*

$$Ax(z) = \begin{cases} 1, & \text{if } z = a(x), \\ 0, & \text{if } z \neq a(x), \end{cases}$$

then for any vector $z \in \mathbb{R}^n$ an interpolating L -fuzzy spline $S_z \in S(T, A, L)$ is the crisp set of all splines $s_z \in S(T, a)$ interpolating z , i.e.

$$S_z(X) = \{0, 1\}, \text{ and } S_z(s_z) = 1$$

iff

$$\|Ts_z\|_W = \inf_{x \in X, a(x)=z} \|Tx\|_W.$$

Proposition IV.3. *If an information operator $A : X \rightarrow L^{\mathbb{R}^n}$ is given by a usual operator $a : X \rightarrow \mathbb{R}^n$ and a positive number ε as follows*

$$Ax(z) = \begin{cases} 1, & \text{if } \|a(x) - z\| \leq \varepsilon, \\ 0, & \text{if } \|a(x) - z\| > \varepsilon, \end{cases}$$

then for any vector $z \in \mathbb{R}^n$ an interpolating L -fuzzy spline $S_z \in S(T, A, L)$ is the crisp set of all splines $s_z \in S(T, a)$ smoothing z with parameter ε , i.e.

$$S_z(X) = \{0, 1\}, \text{ and } S_z(s_z) = 1$$

iff

$$\|Ts_z\|_W = \inf_{x \in X, \|a(x)-z\| \leq \varepsilon} \|Tx\|_W.$$

V. CONCLUSION

Central algorithms play fundamental role in classical approximation theory. In our previous work [3] we have introduced the generalized concept of a central algorithm of approximation of an operator under L -fuzzy information. The central methods for solution of linear problems with balanced information rely on splines. In [3] we studied central algorithm of approximation of an operator under L -fuzzy information which is based on usual, that is crisp splines. As different from that research the aim of this work is to start further development of the approximation theory in the context of fuzzy information which would be based on the use of L -fuzzy splines - the concept introduced and studied here. The research developed here embraces the case when an L -fuzzy spline $S_z \in S(T, A, L)$ interpolating vector z , is the crisp set of all splines $s_z \in S(T, a)$ interpolating or smoothing it. The general case of an L -fuzzy interpolating spline will be the subject of a forthcoming research.

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