

On Monotonicity Of Type $\langle 1, 1 \rangle$ Fuzzy Quantifiers Determined By Fuzzy Measures

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Abstract—In this contribution, we study a very important semantic property of generalized quantifiers called the monotonicity for fuzzy quantifiers of type $\langle 1, 1 \rangle$ defined using fuzzy measures and Sugeno type of fuzzy integrals. We show that fuzzy integrals can ensure under some natural conditions the monotonicity of fuzzy quantifiers. Finally, we propose the concept of concave fuzzy quantifiers and prove that each concave fuzzy quantifier is expressible as the conjunction of a non-increasing and a non-decreasing fuzzy quantifier.

Index Terms—fuzzy quantifier; fuzzy measure; residuated lattice operation; monotonicity of fuzzy quantifiers

I. INTRODUCTION

This contribution continues our study of semantic properties of type $\langle 1, 1 \rangle$ fuzzy quantifiers defined by fuzzy integrals published in [7] (see also [4]) and presents some results on the monotonicity which belongs among the basic semantic properties as, e.g., the isomorphism invariance, extension, conservativity, investigated in the theory of generalized quantifiers (Boolean theory) as well as fuzzy quantifiers theory (many valued theory). For more information, we refer to [8], [13], [26].

Generalized quantifiers evolved, from pioneering works of Mostowski [20], Lindström [17], Barwise and Cooper [1], into quite large research field with deep results. For overview as well as new results see the recent monograph [26].

Research in the field of fuzzy quantifiers started with works of Zadeh [31], Thiele [29], Ralescu [28] and others, see also [18], [19], [22]. Important contribution was made by Hájek in [12], where he considers especially quantifier “many” using relative frequencies. He also points out interconnections between generalized quantifiers and modalities. A comprehensive study of fuzzy quantifiers was undertaken by Glöckner [8] (see also [9]). In paper [24], Novák studies so-called intermediate quantifiers, mainly from syntactic point of view in the frame of fuzzy type theory [23]. This research has been continued in [21], where all of generalized Aristotelian syllogisms (from Peterson’s book [27]) has been syntactically proved using slightly different definition of fuzzy quantifiers than ours.

The semantic interpretation of many generalized quantifiers is often connected to measurement of “size” of sets in concern. Consider e.g. the quantifier “many”. The truth value of proposition “many books have a red cover” clearly depends on the “size” of the set of red books. Therefore, it is natural to consider measures (and integrals) of (fuzzy) sets as natural

tools for the modeling of important classes of generalized quantifiers. This motivated us to introduce fuzzy quantifiers of type $\langle 1 \rangle$, which are denotations of important noun phrases of natural language, e.g. “something” in “Something is broken.”, “everyone” in “Everyone likes Bob.”, “nobody” in “Nobody knows everything.”, using tools of fuzzy measure theory ([30], see also [10], [15]). More precisely, we used a Sugeno type of fuzzy integrals and defined fuzzy integrals of type $\langle 1 \rangle$ in [5].

A natural extension of this research was to study quantifiers of type $\langle 1, 1 \rangle$, e.g. “every” in “Every book has leaves.”, “most” in “Most birds fly.”, which take two arguments. These type $\langle 1, 1 \rangle$ quantifiers are claimed to be the most important from the point of view of natural language semantics [26]. To introduce this type of fuzzy quantifiers by fuzzy integrals, we had to work with slightly different definition of fuzzy measures and integrals than in [5]. Note that the first argument of $\langle 1, 1 \rangle$ quantifier is called *restriction* and the second one is *scope* (i.e., in “Every book has leaves”, “to be a book” is the restriction and “to have leaves” is the scope). It is natural to think of the restriction as of a *new universe* for the quantifier (in our example, for the determination of truth value of this sentence, only these objects fulfilling the restriction condition are important, i.e., books). Because we are working with fuzzy subsets of some universe M , we in effect should be able to define quantifiers on *fuzzy universes*. Therefore, we defined a new type of fuzzy measure spaces on algebras of subsets of a fuzzy set A and a fuzzy integral on these fuzzy measure spaces (see [3], [6], [7]).

In [5], we have showed that type $\langle 1 \rangle$ fuzzy quantifiers determined by fuzzy measures are monotone fuzzy quantifiers. The type of monotonicity, i.e., whether a fuzzy quantifier is monotone increasing (non-decreasing) or monotone decreasing (non-increasing), depends on the type of the fuzzy integral,¹ e.g., the fuzzy quantifier “many” defined using a fuzzy measure is monotone increasing, but the fuzzy quantifier “few” defined using complementary fuzzy measure is monotone decreasing. For type $\langle 1, 1 \rangle$ fuzzy quantifiers, the situation is more complicated, since operations between restriction and scope are used in our modeling.

¹In [5], we proposed two types of fuzzy integrals defined on fuzzy and complementary fuzzy measure spaces to model, e.g., “many” and “few”. Note that under some conditions both types can be interconnected by the operation of negation.

The paper is structured as follows. In the following section, we recall basic notions from the fuzzy set theory and fuzzy measures and integrals. The third section is devoted to the residuated lattice operations using them operations between fuzzy sets are defined. Such operations between fuzzy sets are subsequently used as a means for a combination of two arguments of type $\langle 1, 1 \rangle$ fuzzy quantifiers. Fuzzy quantifiers defined using fuzzy integrals are introduced in the fourth section. The monotonicity of fuzzy quantifiers is studied in the fifth section. The last section is a conclusion.

II. PRELIMINARIES

A. Structures of truth values

In this paper, we suppose that the structure of truth values is a *complete residuated lattice* (see e.g. [2]), i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, \rightarrow, \otimes, \perp, \top \rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, \perp, \top \rangle$ is a complete lattice, where $\perp$ is the least element and $\top$ is the greatest element of L , respectively, $\langle L, \otimes, \top \rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative and the identity $a \otimes \top = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \quad (1)$$

holds for each $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering. A residuated lattice is *divisible*, if $a \otimes (a \rightarrow b) = a \wedge b$ holds for arbitrary $a, b \in L$, and satisfies the *law of double negation*, if $(a \rightarrow \perp) \rightarrow \perp = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called an *MV-algebra*. For more information about residuated lattices we refer to [2], [25].

EXAMPLE 1 One can easily prove (see also [12]) that the algebra

$$\mathbf{L}_T = \langle [0, 1], \min, \max, T, \rightarrow_T, 0, 1 \rangle,$$

where T is a left continuous t -norm and $a \rightarrow_T b = \bigvee \{c \in [0, 1] \mid T(a, c) \leq b\}$ defines the residuum, is a complete residuated lattice. Moreover, if T is the Łukasiewicz t -norm, i.e., $T(a, b) = \max(a + b - 1, 0)$ for all $a, b \in [0, 1]$, then $\mathbf{L}_T$ is a complete MV-algebra called a *Łukasiewicz algebra* (on $[0, 1]$).

Let us define two additional operations for all $a, b \in L$:

$$\begin{aligned} a \leftrightarrow b &= (a \rightarrow b) \wedge (b \rightarrow a), & (\text{biresiduum}) \\ \neg a &= a \rightarrow \perp. & (\text{negation}) \end{aligned}$$

B. Fuzzy sets

Let $\mathbf{L}$ be a complete residuated lattice and M be a universe of discourse (possibly empty). A mapping $A : M \rightarrow L$ is called a *fuzzy set on M* . A value $A(m)$ is called a *membership degree of m in the fuzzy set A* . The set of all fuzzy sets on M is denoted by $\mathcal{F}(M)$. Obviously, if $M = \emptyset$, then the empty mapping $\emptyset$ is the unique fuzzy set on $\emptyset$ and thus $\mathcal{F}(\emptyset) = \{\emptyset\}$. A fuzzy set A on M is called *crisp*, if there is a subset X of M such that $A = 1_X$, where 1_X denotes the characteristic function of X . Particularly, $1_\emptyset$ denotes the empty fuzzy set on

M , i.e., $1_\emptyset(m) = \perp$ for any $m \in M$. This convention will be also kept for $M = \emptyset$. The set of all crisp fuzzy sets on M is denoted by $\mathcal{P}(M)$. A fuzzy set A is *constant*, if there is $c \in L$ such that $A(m) = c$ for any $m \in M$. For simplicity, a constant fuzzy set is denoted by the corresponding element of L , e.g., a, b, c .² Let us denote $\text{Supp}(A) = \{m \mid m \in M \ \& \ A(m) > \perp\}$ and $\text{Core}(A) = \{m \mid m \in M \ \& \ A(m) = \top\}$ the *support* and *core* of a fuzzy set A , respectively. Obviously, $\text{Supp}(1_X) = \text{Core}(1_X) = X$ for any crisp fuzzy set. A fuzzy set A is called *normal*, if $\text{Core}(A) \neq \emptyset$.

Let $\{A_i \mid i \in I\}$ be a non-empty family of fuzzy sets on M . Then the *union* of A_i is defined by

$$\left(\bigcup_{i \in I} A_i \right) (m) = \bigvee_{i \in I} A_i(m) \quad (2)$$

for any $m \in M$ and the *intersection* of A_i is defined by

$$\left(\bigcap_{i \in I} A_i \right) (m) = \bigwedge_{i \in I} A_i(m) \quad (3)$$

for any $m \in M$. Let A, B be fuzzy sets on M . The *difference* of A and B is a fuzzy set $A \setminus B$ on M defined by

$$(A \setminus B)(m) = A(m) \otimes (B(m) \rightarrow \perp) \quad (4)$$

for any $m \in M$ and the *complement* of A is a fuzzy set $\bar{A} = 1_M \setminus A$. Finally, an extension of the operations $\otimes$ and $\rightarrow$ on L to the operations on $\mathcal{F}(M)$ is given by

$$(A \otimes B)(m) = A(m) \otimes B(m), \quad (5)$$

$$(A \rightarrow B)(m) = A(m) \rightarrow B(m) \quad (6)$$

for any $A, B \in \mathcal{F}(M)$ and $m \in M$, respectively.

Since we will deal with fuzzy sets over different universes, let us introduce the ordering relation between fuzzy sets on arbitrary universes. We say that a fuzzy set A on M is an *fuzzy subset* of a fuzzy set B on M' and denote by $A \subseteq B$, if $A(m) \leq B(m)$ for any $m \in \text{Supp}(A)$. The set of all fuzzy subsets of A on M is denoted by $\mathcal{F}(A)$. Thus

$$\mathcal{F}(A) = \{B \mid B \in \mathcal{F}(M) \text{ and } B \subseteq A\}. \quad (7)$$

Further, we say that a fuzzy set A on M is *equal* to a fuzzy set B on M' and denote by $A = B$, if $A \subseteq B$ and $B \subseteq A$. Obviously, if A and B are fuzzy sets on the same universe M , then the proposed relations coincide with the common definitions of the ordering and equality relations.

C. Fuzzy measures and integrals

In the following, we will consider algebras of fuzzy sets as a base for defining fuzzy measures of fuzzy sets.

DEFINITION 1 Let A be a non-empty fuzzy set on M . A subset $\mathcal{F}$ of $\mathcal{F}(A)$ is an *algebra of fuzzy sets on A* , if the following conditions are satisfied

²We suppose that the meaning of this symbol will be unmistakable from the context, that is, it should be clear when an element of L is considered and when a constant fuzzy set is assumed.

- (i) $1_\emptyset, A \in \mathcal{F}$,
- (ii) if $X \in \mathcal{F}$, then $A \setminus X \in \mathcal{F}$,
- (iii) if $X, Y \in \mathcal{F}$, then $X \cup Y \in \mathcal{F}$.

We denote $\mathbf{Alg}(A)$ the set of all algebras of fuzzy sets on A . A pair $(A, \mathcal{F})$ is called a *fuzzy measurable space* (on A), if $\mathcal{F}$ is an algebra of fuzzy sets on A . Let $(A, \mathcal{F})$ be a fuzzy measurable space and $X \in \mathcal{F}(A)$. We say that X is *$\mathcal{F}$ -measurable*, if $X \in \mathcal{F}$.

REMARK 2 In order to refer to the universe of discourse M of A , we will sometimes write $\text{Dom}(A)$ instead of M .

REMARK 3 For a non-empty set M , one checks easily that the proposed definition of algebra of fuzzy sets on 1_M coincides with the standard definition of algebra of fuzzy sets on M (cf. [30]).

EXAMPLE 4 $\{1_\emptyset, A\}$ and $\mathcal{F}(A)$ are trivial examples of algebras of fuzzy sets on A . The set of all crisp subsets of a fuzzy set A is not, in general, an algebra of fuzzy sets on A .

Let us introduce the concept of fuzzy measure as follows. The definition is a modification of the definition of a normed measure with respect to truth values (see e.g. [10]).

DEFINITION 2 Let $(A, \mathcal{F})$ be a fuzzy measurable space. A mapping $\mu : \mathcal{F} \rightarrow L$ is called a *fuzzy measure* on $(A, \mathcal{F})$, if

- (i) $\mu(1_\emptyset) = \perp$ and $\mu(A) = \top$,
- (ii) if $B, C \in \mathcal{F}$ such that $B \subseteq C$, then $\mu(B) \leq \mu(C)$.

A triplet $(A, \mathcal{F}, \mu)$ is called the *fuzzy measure space*, if $(A, \mathcal{F})$ is a fuzzy measurable space and μ is a fuzzy measure on $(A, \mathcal{F})$.

In the following text, a fuzzy measure space $(A, \mathcal{F}, \mu)$ may be shortly denoted by the same bold symbol $\mathbf{A}$, i.e. $\mathbf{A} = (A, \mathcal{F}, \mu)$. We denote $\mathbf{Fms}(M)$ the class of all fuzzy measure spaces defined on a non-empty universe M , i.e. a fuzzy measure space $(A, \mathcal{F}, \mu)$ belongs to $\mathbf{Fms}(M)$, if $A \in \mathcal{F}(M)$.

EXAMPLE 5 Let $\mathbf{L}_T$ be an algebra from Example 1, where T is a continuous t -norm. Let M be a finite non-empty set and $A \in \mathcal{F}(M)$, $A \neq 1_\emptyset$. Then we can define fuzzy measure space $\mathbf{A}^\sigma = (A, \mathcal{F}(A), \mu^\sigma)$, where

$$\mu^\sigma(X) = \frac{\sum_{m \in M} X(m)}{\sum_{m \in M} A(m)}$$

for all $X \in \mathcal{F}(A)$. For motivation and more general definition of this fuzzy measure, we refer to Example 3.5 of our paper [5].

Now, we can introduce a type of fuzzy integral that is definable on an arbitrary fuzzy measure space $(A, \mathcal{F}, \mu)$. The integrated functions are fuzzy sets on $\text{Dom}(A)$. For a convention with the classical measure theory, we will prefer, in this part, the denotation f, g for the integrated functions instead

of X, Y . This integral is defined over a general operation $\odot$ which substitutes one of the operations $\wedge$ and $\otimes$, i.e., $\odot \in \{\wedge, \otimes\}$. This integral will be used as a basis for the definition of fuzzy quantifiers of type $\langle 1, 1 \rangle$ determined by fuzzy measures.

DEFINITION 3 Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space with $M = \text{Dom}(A)$, $f : M \rightarrow L$ be a mapping and X be an $\mathcal{F}$ -measurable fuzzy set. Then $\odot$ -fuzzy integrals of f on X is given by

$$\int_X^\odot f \, d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \text{Supp}(Y)} (f(m) \odot \mu(Y)), \quad (8)$$

where $\mathcal{F}_X^- = \{Y \mid Y \in \mathcal{F} \text{ and } 1_\emptyset \neq Y \subseteq X\}$. If $X = A$, then we write $\int^\odot f \, d\mu$.

One may see that no measurability of the mapping f is supposed in the definition, in contrast to the standard definition of fuzzy integral (cf. [16], [15]).

III. OPERATIONS BETWEEN FUZZY SETS DETERMINED BY RL-OPERATIONS

In this section, we will define so-called residuated lattice (rl-)operations and operations between fuzzy sets determined by them, which will subsequently be used as a means for a combination of two arguments (the restriction and scope, see Introduction) of a type $\langle 1, 1 \rangle$ fuzzy quantifier. For example, the standard quantifier *some* is defined using the intersection operation, i.e.,

$$\text{some}(A, B) = \bigvee_{m \in M} (A \cap B)(m), \quad (9)$$

and *every* by the residuum operation $\rightarrow$, i.e.,

$$\text{every}(A, B) = \bigwedge_{m \in M} (A \rightarrow B)(m). \quad (10)$$

Hence, for the modeling of various natural language quantifiers we need different operations on fuzzy sets (cf. [11]).

DEFINITION 4 Let L be a set of constants. The set $\mathbf{P}(L)$ of *binary (residuated lattice) polynomials* with constants from L is the smallest set satisfying the following conditions

- (i) $c \in \mathbf{P}(L)$ for any $c \in L$,
- (ii) $x, y \in \mathbf{P}(L)$,
- (iii) if $\psi_1, \psi_2 \in \mathbf{P}(L)$, then $\psi_1 \square \psi_2 \in \mathbf{P}(L)$ for any $\square \in \{\wedge, \vee, \otimes, \rightarrow\}$.

REMARK 6 In the previous definition, we refer the polynomials to a set L of constants. In our work, the set L will be identified with a support of a residuated lattice to deal with its elements and, hence, to extend the scale of models of fuzzy quantifiers.

Polynomials are abstract expressions of functions in all residuated lattices in which the considered constants are interpretable. As we have mentioned above, we identify a set L of

constants with a support L of a residuated lattice. Thus, if we write $\mathbf{P}(L)$, then we understand that all values of the support L are constants used in the constructions of polynomials of $\mathbf{P}(L)$. For simplicity, the support of a residuated lattice will be always denoted by the italic letter corresponding to the bold denotation of the residuated lattice. For example, the support of $\mathbf{L}$ is denoted by L . Using polynomials we can define operations on residuated lattices as follows (cf. [11]).

DEFINITION 5 Let $\mathbf{L}$ be a residuated lattice (with the support L). A binary residuated lattice polynomial $\psi \in \mathbf{P}(L)$ defines a binary *residuated lattice operation* (shortly *rl-operation*) on L by the following rules ($a, b, c \in L$)

- (i) if $\psi = c$, then $\psi(a, b) = c$,
- (ii) if $\psi = x$ or $\psi = y$, then $\psi(a, b) = a$ or $\psi(a, b) = b$, respectively,
- (iii) if $\psi = \psi_1 \square \psi_2$, then $\psi(a, b) = \psi_1(a, b) \square \psi_2(a, b)$ for any operation $\square \in \{\wedge, \vee, \otimes, \rightarrow\}$.

The set of all rl-operations in a residuated lattice $\mathbf{L}$ will be denoted by $\mathbf{Ldo}(\mathbf{L})$ or shortly $\mathbf{Ldo}$, if the residuated lattice $\mathbf{L}$ is unmistakably determined by the context and thus no confusion can occur.

DEFINITION 6 Let $\psi_1, \psi_2 \in \mathbf{P}(L)$. We say that ψ_1 is *equal to* ψ_2 in $\mathbf{L}$ and denote it by $\psi_1 = \psi_2$, if $\psi_1(a, b) = \psi_2(a, b)$ holds for any $a, b \in L$.

EXAMPLE 7 The unary operation of the negation can be defined by the polynomial $\psi_1 = x \rightarrow \perp$ and thus $\neg a = \psi_1(a, b) = a \rightarrow \perp$ for any $a, b \in L$. The binary operation of the meet can be obtained from the polynomial $\psi_2 = x \wedge y$ and thus, $\psi_2(a, b) = a \wedge b$ for any $a, b \in L$. The difference between a and b can be defined using $\psi_3 = x \otimes (y \rightarrow \perp)$ and $a \setminus b = \psi_3(a, b) = a \otimes (b \rightarrow \perp)$.

DEFINITION 7 Let M be a non-empty universe and $\psi \in \mathbf{Ldo}(\mathbf{L})$. We say that an operation $\varphi_M : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow \mathcal{F}(M)$ is *determined by* ψ , if

$$\varphi_M(A, B)(m) = \psi(A(m), B(m)) \quad (11)$$

holds for any $A, B \in \mathcal{F}(M)$ and $m \in M$.

In this paper, we will suppose that all operations between fuzzy sets are determined by rl-operations.

IV. TYPE $\langle 1, 1 \rangle$ FUZZY QUANTIFIERS DETERMINED BY $\odot$ -FUZZY INTEGRALS

This section is devoted to fuzzy quantifiers that are defined using fuzzy measures and integrals introduced in [3] and summarized in Subsection II-C. Before commencing our introduction of fuzzy quantifiers, let us recall a common definition of limited and unlimited type $\langle 1, 1 \rangle$ fuzzy quantifiers presented in [8], [9], [13].

In general, a mapping $Q_M : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow L$ is called the *monadic fuzzy quantifier of type $\langle 1, 1 \rangle$ limited to*

M , where M is a (possibly empty³) universe. A value of a fuzzy quantifier on fuzzy sets $A, B \in \mathcal{F}(M)$ is written as $Q_M(A, B)$.⁴ A type $\langle 1, 1 \rangle$ *unlimited (finite, countable) monadic fuzzy quantifier* is a functional Q assigning to each (finite, countable) universe M a type $\langle 1, 1 \rangle$ monadic fuzzy quantifier Q_M limited to M .

In order to define fuzzy quantifiers determined by fuzzy measures in a reasonable generality, let us consider a functional $\mathcal{S}$ assigning to each non-empty universe M a mapping

$$\mathcal{S}(M) : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow \mathbf{Fms}(M). \quad (12)$$

For example, we can define $\mathcal{S}(M)(A, B) = (M, \mathcal{M}, \mu)$, where $(M, \mathcal{M}, \mu)$ is an arbitrary fuzzy measure space from $\mathbf{Fms}(M)$. One can see that $\mathcal{S}(M)$ is a constant mapping and the fuzzy sets A, B do not play any role in this definition. Further examples are $\mathcal{S}(M)(A, B) = (A, \mathcal{M}_A, \mu_A)$ or $\mathcal{S}(M)(A, B) = (A \cap B, \mathcal{M}_{A \cap B}, \mu_{A \cap B})$. Here, the fuzzy sets A, B have an active role in the definition of these fuzzy measure spaces.

To exclude a confusion, let us denote by

$$\int_{(A, \mathcal{F}, \mu)}^{\odot} f \, d\mu \quad \text{or simply} \quad \int_A^{\odot} f \, d\mu$$

the fuzzy integral from Subsection II-C defined on a fuzzy measure space $\mathbf{A} = (A, \mathcal{F}, \mu)$.

DEFINITION 8 Let M be a non-empty universe, $\mathcal{S}(M)$ be a mapping of the form (12) and φ_M be an operation on $\mathcal{F}(M)$ determined by an rl-operation ψ . A *fuzzy quantifier of type $\langle 1, 1 \rangle$ limited to M determined by $(\mathcal{S}(M), \varphi_M)$* is a mapping

$$Q_M : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow L$$

defined by

$$Q_M(A, B) = \int_{\mathcal{S}(M)(A, B)}^{\odot} \varphi_M(A, B) \, d\mu. \quad (13)$$

Since there is no fuzzy measure space on the empty (fuzzy) set, an extension of limited quantifiers to quantifiers over arbitrary universes cannot comprise a reasonably general stipulation how the quantifiers have to look for the empty universe. Therefore, we omit this situation in the following definition and introduce unlimited fuzzy quantifiers determined by fuzzy measures for the non-empty universes, whereas these fuzzy quantifiers can take arbitrary values for the empty universe.

³To define the behavior of generalized quantifiers for the empty universe is important in some situations. It happens, for example, when we study type $\langle 1, 1 \rangle$ quantifiers which are obtained from type $\langle 1 \rangle$ quantifiers by means of *relativization*. Then it is vital to have values of e.g. $\text{some}_{\emptyset}(1_{\emptyset})$ defined, see discussion in [26], p. 137.

⁴Note that, in the literature (see e.g. [13], [26]), we can also find the denotation $Q_M(A)(B)$ to emphasize what is the property modifying a common noun like “to be a young student” (A) and what is the considered predicate like “to have a good result” (B) in the sentence “Many young students have good results”. In this conception, a type $\langle 1, 1 \rangle$ fuzzy quantifier limited to M is a mapping $Q_M : \mathcal{F}(M) \rightarrow [\mathcal{F}(M) \rightarrow L]$, where $[\mathcal{F}(M) \rightarrow L]$ denotes the set of all mappings (type $\langle 1 \rangle$ fuzzy quantifiers) of $\mathcal{F}(M)$ to L characterizing the predicates like “to have a good result”.

DEFINITION 9 Let $\mathcal{S}$ and φ be functionals assigning to each non-empty universe M a mapping $\mathcal{S}(M)$ of the form (12) and an operation φ_M determined by an rl-operation ψ , respectively. An *unlimited fuzzy quantifier of type $\langle 1, 1 \rangle$ determined by $(\mathcal{S}, \varphi)$* is an unlimited fuzzy quantifier Q of type $\langle 1, 1 \rangle$ assigning a fuzzy quantifier Q_M limited to M determined by $(\mathcal{S}(M), \varphi_M)$ to each non-empty universe M .

Let us consider the sentence “Many young students have good results”,. Obviously, the fuzzy quantifier “many” expresses a quantity on the fuzzy set of “young students” and would be independent, in some sense, from the property “to have good results”. More precisely, if we replace the property “to have good results” by “to have bad results”, the interpretation of the fuzzy quantifier “many” would stay the same. This motivate us to establish the fuzzy quantifiers definable by fuzzy measures as follows.

DEFINITION 10 We say that an unlimited fuzzy quantifier Q is *definable by fuzzy measures*, if there exists a pair of functionals $(\mathcal{S}, \varphi)$ such that Q is determined by $(\mathcal{S}, \varphi)$ and $\mathcal{S}(M)(A, -)$ is a constant mapping for any non-empty universe M and $A \in \mathcal{F}(M)$.

REMARK 8 We can see that the class of fuzzy quantifiers definable by fuzzy measures is a subclass of the class of fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$. Hence, if a property is satisfied by fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$, then it is also satisfied by fuzzy quantifiers definable by fuzzy measures. In what follows, we will usually formulate statements for fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$.

Before we show several examples and start our study of monotonicity of fuzzy quantifiers definable by fuzzy measures (or determined by a pair of functionals $(\mathcal{S}, \varphi)$ in general), let us introduce two important notions saying when one fuzzy quantifier is less than or equal to the second one, and when two fuzzy quantifiers are equal.

DEFINITION 11 Let P, Q be two quantifiers. We say that Q is *less than or equal to P* and denote it by $Q \leq P$, if

$$Q_M(A, B) \leq P_M(A, B) \quad (14)$$

holds for arbitrary non-empty universe M and $A, B \in \mathcal{F}(M)$. Further, we say that Q is *equal to P* and denote it by $Q = P$, if $Q \leq P$ and $P \leq Q$.

A reason why the empty universe is omitted in the previous definition is that fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$ are not a priori defined for the empty universe. In the following part, for simplicity, we will write only “ Q is determined by $(\mathcal{S}, \varphi)$ ” instead of “ Q is a fuzzy quantifier of type $\langle 1, 1 \rangle$ determined by $(\mathcal{S}, \varphi)$ ”.

EXAMPLE 9 Let M be a non-empty universe and $\mathcal{S}_i(M)(A, B) = \{(M, \mathcal{F}(M), \mu_i)\}$, where, for $i = 1, 2$,

we have

$$\mu_1(X) = \begin{cases} \perp, & \text{if } X = 1_\emptyset, \\ \top, & \text{otherwise} \end{cases} \quad (15)$$

and

$$\mu_2(X) = \begin{cases} \top, & \text{if } X = M, \\ \perp, & \text{otherwise.} \end{cases} \quad (16)$$

If Q is determined by $(\mathcal{S}_1, \varphi_1)$, where $\varphi_{M,1}(A, B) = A \cap B$, then $Q = \text{some}$, i.e., it is the interpretation of the standard quantifier some A are B . If Q is determined by $(\mathcal{S}_2, \varphi_2)$ for all $M \neq \emptyset$, where $\varphi_{M,2}(A, B) = A \rightarrow B$, then it is equal to every.

EXAMPLE 10 Let φ be a functional assigning φ_M to each non-empty universe and $\mathcal{S}_i$, $i = 1, 2$, be functionals defined in the previous example. We may introduce “boundary” fuzzy quantifiers $\forall_\varphi$ and $\exists_\varphi$ as follows

$$\forall_{\varphi, M}(A, B) = \bigwedge_{m \in M} \varphi_M(A, B)(m), \quad (17)$$

$$\exists_{\varphi, M}(A, B) = \bigvee_{m \in M} \varphi_M(A, B)(m). \quad (18)$$

Obviously, $\text{every} = \forall_\varphi$ for $\varphi_M(A, B) = A \rightarrow B$ and $\text{some} = \exists_\varphi$ for $\varphi_M(A, B) = A \cap B$. One can see that $\forall_\varphi \leq \exists_\varphi$.

EXAMPLE 11 Let $\mathbf{L}_L$ be the Łukasiewicz algebra from Example 1. Let M be a finite non-empty set and $A \in \mathcal{F}(M)$, $A \neq 1_\emptyset$ and $\mathbf{A}^\sigma = (A, \mathcal{F}(A), \mu^\sigma)$ be a fuzzy measure space from Example 5. Let us define $\mathcal{S}(M)(A, B) = \mathbf{A}^\sigma$ for $A \neq 1_\emptyset$, $\mathcal{S}(M)(1_\emptyset, B) = (M, \mathcal{F}(M), \mu_2)$ and $\varphi_M(A, B) = A \rightarrow B$. Then, for any $A, B \in \mathcal{F}(M)$, we may define an interpretation of fuzzy quantifier many as

$$\begin{aligned} Q_M^{\text{many}}(A, B) &= \int_{\mathcal{S}(M)(A, B)}^{\otimes} \varphi_M(A, B) d\mu^\sigma = \\ &= \int_{(A, \mathcal{F}(A), \mu^\sigma)}^{\otimes} A \rightarrow B d\mu^\sigma = \\ &= \bigvee_{Y \in \mathcal{F}(A)^-} (\mu^\sigma(Y) \otimes \bigwedge_{m \in \text{Supp}(Y)} (A \rightarrow B)(m)), \end{aligned}$$

if $A \neq 1_\emptyset$, and $Q_M^{\text{many}}(1_\emptyset, B) = 1$.

EXAMPLE 12 We can construct important fuzzy quantifiers, e.g., *few*, *no*, *at most half*, etc., from already defined quantifiers using so-called external and inner negations (see [13], Definition 14). Let Q be a fuzzy quantifier of type $\langle 1, 1 \rangle$. The external or inner negation of Q is the fuzzy quantifier $\sim Q$ or $Q \sim$ defined as $(Q \sim)_M(A, B) = \neg Q_M(A, B)$ or $(\sim Q)_M(A, B) = Q_M(A, \bar{B})$, respectively. Hence, we can define fuzzy quantifiers, e.g., $\text{no} = \sim \text{some}$ and $\text{few} = \text{many} \sim$, where some and many are defined in Examples 9 and 11, respectively.

V. MONOTONICITY

The concept of monotonic quantifiers belongs among the basic properties which is observed for a large scale of natural language quantifiers as *all*, *some*, *few*, *many*, *nearly all*, *at least five*, *at most ten* etc. Of course, not all fuzzy quantifiers are monotonic. Typical examples are *about half*, *at least one and at most ten*. The monotonicity for the type $\langle 1, 1 \rangle$ fuzzy quantifiers is defined as follows.⁵

DEFINITION 12 Let Q be a fuzzy quantifier. We say that Q is monotonically non-decreasing in its first (second) component, if

$$Q_M(A, C) \leq Q_M(B, C) \quad (Q_M(C, A) \leq Q_M(C, B)) \quad (19)$$

holds for any universe M , $A, B, C \in \mathcal{F}(M)$, $A \subseteq B$, and Q is monotonically non-increasing in its first (second) component, if

$$Q_M(A, C) \geq Q_M(B, C) \quad (Q_M(C, A) \geq Q_M(C, B)) \quad (20)$$

holds for any universe M , $A, B, C \in \mathcal{F}(M)$, $A \subseteq B$.

REMARK 13 Notice that the mentioned inequalities are trivially satisfied for the empty universe. Hence, the empty universe can be excluded from our investigation of the monotonicity of fuzzy quantifiers.

One can see that *all*, *some*, *many*, *at least k* are monotonically non-decreasing and *not all*, *no*, *few*, *at most k* are monotonically non-increasing in the second component. Analogously to [26], we will use, for simplicity, the denotation $\uparrow\text{Mon}$, $\downarrow\text{Mon}$ and $\text{Mon}\uparrow$, $\text{Mon}\downarrow$ to denote that a fuzzy quantifier is monotonically non-decreasing, non-increasing in the first component and monotonically non-decreasing, non-increasing in the second component, respectively. Sometimes it is useful a combination of the previous denotations as $\uparrow\text{Mon}\downarrow$ saying that a quantifier is non-decreasing in the first component and non-increasing in the second one. In general, the monotonicity of fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$ is closely related to the monotonicity of φ . Let us denote the monotonicity of the functional φ analogously by $\text{Mon}\uparrow$, $\text{Mon}\downarrow$ and $\uparrow\text{Mon}$, $\downarrow\text{Mon}$ and similarly for their combinations, e.g. $\uparrow\text{Mon}\downarrow$.

THEOREM 1 A fuzzy quantifier definable by fuzzy measures determined by $(\mathcal{S}, \varphi)$ is $\text{Mon}\uparrow$ ($\text{Mon}\downarrow$) if and only if φ is $\text{Mon}\uparrow$ ($\text{Mon}\downarrow$).

Proof: Let Q be a fuzzy quantifier definable by fuzzy measures determined by $(\mathcal{S}, \varphi)$ which is $\text{Mon}\uparrow$. Let M be a non-empty universe and suppose that φ_M is determined by an rl-operation ψ . From the definition of φ , one can derive that the monotonicity of ψ implies the same form of the monotonicity of φ . Moreover, if a and b denote the constant fuzzy sets in $\mathcal{F}(M)$ with values a and b , respectively, then we may write $\varphi(a, b)(m) = \psi(a, b)$ for any $m \in M$. Recall that $\mathcal{S}(A, -)$

is a constant mapping from $\mathcal{F}(M)$ to $\mathbf{Fms}(M)$. Let a, b, c be constant fuzzy sets such that $b \leq c$. Since $\int_{\mathcal{S}(M)(a, b)}^\odot c \, d\mu = c$, we obtain

$$\psi(a, b) = \varphi_M(a, b)(m) = \int_{\mathcal{S}(M)(a, b)}^\odot \varphi_M(a, b) \, d\mu =$$

$$Q_M(a, b) \leq Q_M(a, c) =$$

$$\int_{\mathcal{S}(M)(a, c)}^\odot \varphi_M(a, c) \, d\mu = \varphi_M(a, c)(m) = \psi(a, c),$$

where m is an arbitrary element of M . Hence, ψ is a monotonically non-decreasing mapping and, from the consideration above, we obtain that φ is $\text{Mon}\uparrow$. Analogously, we can prove $\text{Mon}\downarrow$ of φ under the presumption that Q is $\text{Mon}\downarrow$.

The opposite implication is a trivial consequence of the fact that the fuzzy integral is a non-decreasing mapping. ■

In the previous theorem, we cannot suppose the cases of $\uparrow\text{Mon}$ and $\downarrow\text{Mon}$, since $\mathcal{S}(M)(b, -)$ and $\mathcal{S}(M)(c, -)$ are different fuzzy measure spaces and thus the presented inequality cannot be guaranteed here. Nevertheless, supposing that $\mathcal{S}(M)(-, -)$ is a constant mapping, we may state the analogous result also for $\uparrow\text{Mon}$ and $\downarrow\text{Mon}$.

THEOREM 2 A fuzzy quantifier determined by $(\mathcal{S}, \varphi)$ such that $\mathcal{S}(-, -)$ is a constant mapping for any non-empty universe M is $\uparrow\text{Mon}$ ($\downarrow\text{Mon}$) if and only if φ is $\uparrow\text{Mon}$ ($\downarrow\text{Mon}$).

Proof: This is analogous to the proof of the cases $\text{Mon}\uparrow$ and $\text{Mon}\downarrow$. ■

REMARK 14 It is easy to see that the operations $\wedge$, $\vee$, $\otimes$ are $\uparrow\text{Mon}\uparrow$ and $\rightarrow$ is $\downarrow\text{Mon}\uparrow$. One can simply verify that, for example, $\varphi_1 \square \varphi_2$ is $\text{Mon}\uparrow$, whenever φ_1 and φ_2 are $\text{Mon}\uparrow$ and $\square$ is $\uparrow\text{Mon}\uparrow$.

REMARK 15 Obviously, the external negation and the inner negation of fuzzy quantifiers mentioned in the first part of our contribution (see also [8], [9], [13], [14], [26]) change the polarity of monotonicity in both components and in the second component, respectively. For example, let us consider every and some defined in Example 3.2. of the first part. Then every is $\downarrow\text{Mon}\uparrow$ and some is $\uparrow\text{Mon}\uparrow$, but $\sim\text{every} = \text{some}\sim = \text{not every}$ is $\uparrow\text{Mon}\downarrow$ and $\text{every}\sim = \sim\text{some} = \text{no}$ is $\downarrow\text{Mon}\downarrow$.

In [5], we showed that each type $\langle 1 \rangle$ fuzzy quantifier determined by a fuzzy measure lays among the fuzzy quantifiers all and some. The question is, which boundaries hold for fuzzy quantifiers determined by $(\mathcal{S}, \varphi)$, especially, for fuzzy quantifiers definable by fuzzy measures.

THEOREM 3 For each fuzzy quantifier Q determined by $(\mathcal{S}, \varphi)$, we have $\forall_\varphi \leq Q \leq \exists_\varphi$.

Proof: This is a straightforward consequence of the definition of fuzzy measures on which the fuzzy quantifiers $\forall_\varphi$ and $\exists_\varphi$ are defined. ■

⁵A more general definition for monadic fuzzy quantifiers of type $\langle 1^n, 1 \rangle$ can be found in [13].

Naturally, we can find out fuzzy quantifiers (definable by fuzzy measures) which are not monotonic. For example, let us consider the Łukasiewicz algebra and operations between fuzzy sets determined by the following rl-operations

$$\begin{aligned}\psi_1(a, b) &= a \otimes (a \rightarrow (b \vee \neg b)), \\ \psi_2(a, b) &= a \leftrightarrow b.\end{aligned}$$

Then fuzzy quantifiers which use the operations ψ_1 and ψ_2 are not monotonic. In fact, it is sufficient to consider the constant fuzzy sets $0 \subset 0.5 \subset 1$ defined on a non-empty universe. Since $\int^\odot c \, d\mu = c$, we may write, for ψ_1 ,

$$Q_M(1, 0) = Q_M(1, 1) = 1 > 0.5 = Q_M(1, 0.5)$$

and, for ψ_2 ,

$$Q_M(0.5, 0) = Q_M(0.5, 1) = 0.5 < 1 = Q_M(0.5, 0.5).$$

The proposed operations could be used in the evaluation of the quantified sentences:

“Every students is or is not prepared on the exam.”
“Only blondes are pretty.”

Obviously, the first statement is trivially fulfilled in the classical logic, but practically there are many students who know something but we cannot say that they are perfectly prepared on the exam. Hence, this statement has a sense in fuzzy logic.

Although, we are able to find some “reasonable” models of non-monotonic linguistic quantifiers definable by fuzzy measures, much more natural it is to define non-monotonic fuzzy quantifiers as a conjunction of two fuzzy quantifiers (definable by fuzzy measures), where one is $\text{Mon}\uparrow$ ($\uparrow\text{Mon}$) and the second one is $\text{Mon}\downarrow$ ($\downarrow\text{Mon}$). For example, a fuzzy quantifier *roughly between eight and ten* is a conjunction of *about eight or more than eight* ($\text{Mon}\uparrow$) and *about ten or less than ten* ($\text{Mon}\downarrow$).

An interesting question is, which non-monotonic fuzzy quantifiers can be described by the conjunction of two fuzzy quantifiers (not necessary definable by fuzzy measures) having the opposite polarity, i.e., one is $\text{Mon}\uparrow$ and the second one $\text{Mon}\downarrow$. In the generalized quantifier theory, such quantifiers are called *continuous*.⁶ A generalization of the continuity for fuzzy quantifiers is a concavity defined as follows.

DEFINITION 13 We say that a fuzzy quantifier Q is concave in the first or the second component, if

$$Q_M(A', B) \wedge Q_M(A'', B) \leq Q_M(A, B) \quad (21)$$

or

$$(Q_M(B, A') \wedge Q_M(B, A'')) \leq Q_M(B, A) \quad (22)$$

⁶Note that a quantifier Q of type $\langle 1 \rangle$ is continuous, if the following holds: if $Q_M(A) = Q_M(C) = \top$ for a non-empty universe M and $A, C \in \mathcal{F}(M)$ such that $A \subseteq C$, then $Q_M(B) = \top$ for any $B \in \mathcal{F}(M)$ with $A \subseteq B \subseteq C$ (cf. [26]). For other types, one can analogously define the continuity of quantifiers in the given component as in the definition for the type $\langle 1 \rangle$.

hold for any non-empty universe M and fuzzy sets $A, A', A'', B \in \mathcal{F}(M)$ such that $A' \subseteq A \subseteq A''$, respectively.

Let Q be a fuzzy quantifier and denote $Q_{M,a} : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow \{\perp, \top\}$ a fuzzy quantifier defined by $Q_{M,a}(A, B) = \top$ if and only if $Q_M(A, B) \geq a$. A fuzzy quantifier Q_a is the a -cut of the fuzzy quantifier Q .⁷ The following lemma shows a relation between the concavity of fuzzy quantifiers and the continuity of a -cut fuzzy quantifiers.

LEMMA 4 A fuzzy quantifier Q is concave in the first (second) component if and only if each a -cut fuzzy quantifier Q_a is continuous in the first (second) component.

Proof: Here, we will prove only the concavity in the first component. The proof for the second component is analogous. Let Q be a concave in the first component fuzzy quantifier, $a \in L$, M be a non-empty universe and B be arbitrary fuzzy set of $\mathcal{F}(M)$. If $Q_{M,a}(A', B) = Q_{M,a}(A'', B) = \top$ and $A' \subseteq A \subseteq A''$, then $Q_M(A', B) \wedge Q_M(A'', B) \geq a$ and using the concavity of Q we obtain $Q_M(A, B) \geq a$ which implies $Q_{M,a}(A, B) = \top$. Hence, Q is continuous in the first component for any $a \in L$.

On the contrary, let Q_a be continuous fuzzy quantifier for any $a \in L$, $A' \subseteq A \subseteq A''$ and B be arbitrary fuzzy sets of $\mathcal{F}(M)$. Denote $a = Q_M(A', B) \wedge Q_M(A'', B)$. From the definition of Q_a , we obtain $Q_{M,a}(A', B) = Q_{M,a}(A'', B) = \top$. Since Q_a is continuous in the first component for any $a \in L$, then also $Q_{M,a}(A, B) = \top$. Hence, $Q_M(A, B) \geq a = Q_M(A', B) \wedge Q_M(A'', B)$ which means that Q is concave. ■

THEOREM 5 Let Q and Q' be $\uparrow\text{Mon}$ and $\downarrow\text{Mon}$ ($\text{Mon}\uparrow$ and $\text{Mon}\downarrow$). Then $Q \wedge Q'$ is concave in the first (second) component.

Proof: We restrict ourselves to the proof of the case of the first component. For any non-empty universe M and $A, A', A'', B \in \mathcal{F}(M)$ such that $A' \subseteq A \subseteq A''$, we have $Q_M(A', B) \leq Q_M(A, B) \leq Q_M(A'', B)$ and $Q'_M(A'', B) \leq Q'_M(A, B) \leq Q'_M(A', B)$. Hence, we obtain

$$\begin{aligned}(Q \wedge Q')_M(A', B) \wedge (Q \wedge Q')_M(A'', B) &= \\ Q_M(A', B) \wedge Q'_M(A', B) \wedge Q_M(A'', B) \wedge Q'_M(A'', B) &\leq \\ Q_M(A', B) \wedge Q'_M(A'', B) \leq Q_M(A, B) \wedge Q'_M(A, B) &= \\ (Q \wedge Q')_M(A, B).\end{aligned}$$

and thus $Q \wedge Q'$ is concave fuzzy quantifier. ■

THEOREM 6 Let L be a distributive residuated lattice. Each concave in the first (second) component fuzzy quantifier is expressible as the conjunction of fuzzy quantifiers which are $\uparrow\text{Mon}$ and $\downarrow\text{Mon}$ ($\text{Mon}\uparrow$ and $\text{Mon}\downarrow$).

Proof: Again, we restrict ourselves to the proof of the case of the first component. Let Q be concave in the first

⁷Obviously, each fuzzy quantifier can be simply derived from its a -cuts by $Q_M(A, B) = \bigvee \{a \mid a \in L \text{ and } Q_{M,a}(A, B) = \top\}$.

component. Put

$$P_M(A, B) = \bigvee_{A' \in \mathcal{F}(M), A' \subseteq A} Q_M(A', B).$$

$$P'_M(A, B) = \bigvee_{A'' \in \mathcal{F}(M), A \subseteq A''} Q_M(A'', B).$$

One can verify easily that P and P' are $\uparrow\text{Mon}$ and $\downarrow\text{Mon}$, respectively. Moreover, $P \wedge P'$ is a concave fuzzy quantifier in the first component according to the previous theorem. Let M be an arbitrary non-empty universe and $A' \subseteq A \subseteq A''$ be arbitrary fuzzy sets from $\mathcal{F}(M)$. Obviously, we have

$$(P \wedge P')_M(A, B) = P_M(A, B) \wedge P'_M(A, B) =$$

$$\bigvee_{\substack{A' \in \mathcal{F}(M) \\ A' \subseteq A}} Q_M(A', B) \wedge \bigvee_{\substack{A'' \in \mathcal{F}(M) \\ A \subseteq A''}} Q_M(A'', B) \geq Q_M(A, B).$$

Since $\mathbf{L}$ is a distributive residuated lattice and $Q_M(A, B) \geq Q_M(A', B) \wedge Q_M(A'', B)$ for any $A' \subseteq A \subseteq A''$, then we may write

$$\begin{aligned} Q_M(A, B) &\geq \bigvee_{\substack{A' \in \mathcal{F}(M) \\ A' \subseteq A}} \bigvee_{\substack{A'' \in \mathcal{F}(M) \\ A \subseteq A''}} (Q_M(A', B) \wedge Q_M(A'', B)) \\ &= \bigvee_{\substack{A' \in \mathcal{F}(M) \\ A' \subseteq A}} Q_M(A', B) \wedge \bigvee_{\substack{A'' \in \mathcal{F}(M) \\ A \subseteq A''}} Q_M(A'', B) = \\ &P_M(A, B) \wedge P'_M(A, B) = (P \wedge P')_M(A, B). \end{aligned}$$

Hence, Q is expressible as the conjunction $P \wedge P'$, where P is $\text{Mon}\uparrow$ and P' is $\text{Mon}\downarrow$. ■

VI. CONCLUSION

In this contribution, we defined fuzzy quantifiers of type $\langle 1, 1 \rangle$ using fuzzy integrals and studied their monotonicity. Aside from the theoretical importance of generalized fuzzy quantifiers, we are also motivated by the applications of fuzzy logic e.g. in time series analysis, decision making and other fields where it can be advantageous to be able to use other than classical quantifiers “for all” and “exists”. For example, in time series analysis and modeling, we can use quantifiers like “many” for filtering off of outliers, i.e. values that are numerically distant from (the course of) the data.

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