

Fuzzy Concept Lattice Construction

A Basis for Building Fuzzy Ontologies

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Abstract— Fuzzy concept lattices are being used as the basis for creating fuzzy ontologies. Fuzzy formal contexts serve as the starting point for which a variety of proposed methods have been used to create fuzzy concept lattices from them. This paper reviews two of these methods: the one-sided threshold approach and the fuzzy closure operator approach and presents the first comparison between these two approaches. Some simple examples are used and then bioinformatics data, specifically several gene annotation data files. The results show that the fuzzy closure approach produces huge numbers of concepts as compared to the threshold approach, and the extents produced by the threshold approach are a subset of the extents produced by the fuzzy closure approach.

Keywords—Fuzzy concept lattices; threshold method; fuzzy closure method; fuzzy ontologies; fuzzy formal concept analysis

I. INTRODUCTION

Ontologies have become accepted as the main knowledge representation method for making domain knowledge available on the web. Ontologies typically are specified in a machine-readable format and represent domain knowledge by defining concepts, attributes of the concepts and relationships between the concepts with the taxonomic is-a relationship creating the ontology structure. More formal ontology languages such as OWL [1] also allow axioms to be specified in the definition of the ontology. Ontologies facilitate the use of web resources and knowledge sharing by web applications.

One of the bottlenecks in the use of ontologies, however, is the time and effort to create them. Much research has been undertaken to automate as much as possible the building of ontologies [2]. More recently formal concept analysis (FCA) is being used to aid in constructing ontologies [3][4]. FCA was developed in the early 1980's [5] and has grown into a widely used tool for knowledge discovery [6]. FCA is an unsupervised learning method to cluster data to create concepts where a formal concept has two components, the instance objects which are its extent, and the attributes describing the objects, which are its intent.

Since real-world knowledge can be vague or uncertain, researchers have focused on integrating fuzzy logic into the modeling of domain knowledge and the result is the definition of fuzzy ontologies [7]. A natural extension to the use of FCA for creating ontologies is the use of fuzzy formal concept

analysis (FFCA) to build fuzzy ontologies. In crisp FCA, the association of an attribute with an object is either complete or not at all. In FFCA an attribute may only be partially associated with an object, i.e., an attribute has a degree of association or membership when used to describe the object. The fuzzy formal context is typically specified by a matrix with a set of objects for its rows and a set of attributes or properties for its columns as shown in Fig. 1. It is basically a formal context where the relation R between the objects and the attributes is a fuzzy binary relation; that is, instead of a crisp 0 or 1 degree, the degree is taken from the interval $[0, 1]$ and indicates that the object x_i possesses the attribute y_j to the specified degree.

	y_1	y_2	y_3	y_4	y_5
x_1	1	0.5	0.5	1	1
x_2	1	1	1	1	0.5
x_3	0	0	0.5	0.5	1

Fig. 1. A fuzzy formal context [11].

Numerous researchers have developed approaches for creating fuzzy concept lattices that are then used to develop fuzzy ontologies [8] [9] [10]. These approaches use an α -cut on the fuzzy formal context to first convert to a crisp formal context and then use standard FCA to both create the formal concepts and the structure of the concept lattice. The fuzziness is introduced back into concept lattice using several fuzzy operators on the fuzzy formal context. This method is referred to as the one-sided α -cut thresholding approach to fuzzy concept lattice creation. Recently a new approach to creating a fuzzy concept lattice that works directly on the fuzzy formal context has been introduced [11]. It uses a fuzzy closure operator to calculate all fixpoints in a finite set over a finite chain of truth degrees. These fixpoints are the fuzzy formal concepts. The partial order on the set of all fixpoints is also determined to create the structure of the fuzzy concept lattice.

This paper extends the work in [12] and presents a more detailed comparison of these two approaches. Both algorithms have been implemented in Java and various test cases have been used to compare their results, first using a simple example and then using a gene annotation data set with three different protein families. Currently, these two different approaches have not been directly compared. For the most part, the fuzzy

closure approach has simply been used on experimentally generated fuzzy formal contexts with a focus on its practical performance [11]. The threshold approach has been experimented with to examine its usefulness for creating fuzzy concept lattices as a foundation for developing fuzzy ontologies [10]. The focus in this present paper is examining the differences in the interpretation of the two approaches and actually applying them to the same real world fuzzy formal context based on gene annotation data which plays an important role in bioinformatics research. Section 2 describes the one-sided α -cut threshold approach to fuzzy concept lattice creation. Section 3 describes the fuzzy closure approach. Both of these sections use the simple fuzzy formal context shown in Fig. 1 to illustrate the two different approaches. Section 4 outlines the experimental data used to compare the two approaches. The results of these experiments are analyzed in section 5. Section 6 summarizes the research and outlines plans to further experiment with these two approaches.

II. ONE-SIDED THRESHOLD APPROACH

A fuzzy formal context is basically a formal context where the relation between the objects and the attributes is a fuzzy binary relation R between a set of objects X and a set of attributes Y as illustrated in Fig. 1. For example, object x_1 possesses attribute y_3 to degree 0.5, i.e., $R(x_1, y_3) = 0.5$. The fuzzy formal context is viewed as a set of fuzzy sets where each fuzzy set describes either an object or an attribute, i.e., the side that is selected or given preference. If the object is given preference, then an object $x_i \in X$ in a fuzzy formal context K with attribute set Y is represented by a fuzzy set $\Phi(x_i)$ as $\{R(x_i, y_1)/y_1, R(x_i, y_2)/y_2, \dots, R(x_i, y_n)/y_n\}$ where $R(x_i, y_j)$ is the degree to which object x_i possesses attribute y_j . For example, the row for object x_3 specifies its fuzzy set description over the attributes, $\{0.5/y_3, 0.5/y_4, 1.0/y_5\}$. An attribute can also be represented as a fuzzy set over the objects, i.e., the column for attribute y_3 specifies its fuzzy set description over the objects, $\{0.5/x_1, 0.5/x_2, 1.0/x_3\}$. With this view, algorithms for creating the fuzzy formal concepts that include the extent and intent, the concept lattice structure, and the strength of the links from a parent concept to its children concepts have been developed.

In [10], the threshold approach is used to generate fuzzy ontologies from fuzzy formal context by taking α -cuts on the fuzzy formal context where the object is given preference. Then a fuzzy formal concept is defined by specifying an α value, also referred to as a confidence threshold. For $A \subseteq X$, then the crisp definition simply is modified to require $R(x, y) \geq \alpha$ instead of $R(x, y) = 1$ as given:

$$A' = \{y \in Y \mid \forall x \in A, R(x, y) \geq \alpha\} \quad (1)$$

and similarly, $B \subseteq Y$

$$B' = \{x \in X \mid \forall y \in B, R(x, y) \geq \alpha\} \quad (2)$$

The threshold α restricts the fuzzy concept lattice generation since it eliminates from the original fuzzy formal context all (x, y) pairs whose degree is less than the threshold and basically converts the fuzzy formal context into a crisp context for all those remaining pairs. This method then simply

uses standard FCA to create the concepts and the concept lattice structure, but no fuzziness exists in the extent or intent at that point. The crisp formal concept must be transformed into a fuzzy formal concept.

The fuzzy formal concept of a fuzzy formal context K with a confidence threshold α is a pair (A_f, B) , where $A' = B$ and $B' = A$ and each $x \in A$ has a membership degree defined as

$$A_f(x) = \min_{y \in B} R(x, y). \quad (3)$$

If B is the empty set, then $A_f(x) = 1$ for every x . A is the extent and B is the intent of the fuzzy formal concept. If attributes are given preference then the formula becomes

$$B_f(x) = \min_{x \in A} R(x, y). \quad (4)$$

The term fuzzy concept is used in the remainder of the paper as an abbreviation for fuzzy formal concept. Note that when the object is given preference, the extent of the fuzzy concept is a fuzzy set. The membership degree belonging to the object in the fuzzy extent is interpreted as the degree to which the object belongs to the fuzzy concept and specifies the minimum degree to which the object possesses all the attributes in the intent. The partial ordering of fuzzy concepts $\leq$ is defined such that $(A1_f, B1) \leq (A2_f, B2)$ if and only if $A1_f \subseteq A2_f$ and equivalently $B2 \subseteq B1$. This approach creates a set of fuzzy concepts and a fuzzy concept lattice based on the $\leq$ ordering.

Although the fuzzy concept lattice can be created from the fuzzy concepts and the ordering, the subsumption or ordering relation does not provide a degree of similarity or strength between the parent fuzzy concept and the child fuzzy concept. Thus a measure of fuzzy concept similarity between concepts $c1 = (A1, B1)$ and $c2 = (A2, B2)$ is defined using the Jaccard fuzzy set similarity measure between two fuzzy sets as

$$E(c1, c2) = \frac{|A1_f \cap A2_f|}{|A1_f \cup A2_f|}. \quad (5)$$

This approach has been used in recent papers focusing on fuzzy ontology generation [8]-[10]. If the attribute set is given preference, then the intents are fuzzy sets and the A sets would be replaced with corresponding B sets in (5).

For the fuzzy formal context in Fig. 1 and selecting the object side with a threshold of the minimum degree in the fuzzy formal context, i.e., 0.5, produces the following two concepts labeled 0 and 1 followed by the intent set and the extent set of each one:

$$\begin{aligned} 0: & [0.5/x_1, 0.5/x_2, 0.5/x_3]; [y_3, y_4, y_5] \\ 1: & [0.5/x_1, 0.5/x_2]; [y_1, y_2, y_3, y_4, y_5] \end{aligned}$$

Since the minimum membership degree in the fuzzy formal context is selected as the threshold, any nonzero membership degree indicates that the object possess the attribute. The top concept 0 shows all objects in X possess attributes y_3, y_4 and y_5 to some degree, but the minimum degree of possession over all attributes in the intent of concept 0 is 0.5 for each of the objects. The bottom concept 1 shows that all attributes are possessed by both objects x_1 and x_2 .

Again, the minimum degree of possession over all attributes in the intent of concept 1 is 0.5 for both x_1 and x_2 . Concept 0 has concept 1 as its child with a link strength of 0.667 between them.

Using a threshold of 1.0 greatly changes the results. Only pairs of objects and attributes which have a 1.0 membership degree in the fuzzy formal context are considered. The following concepts are produced:

- 0: $[1.0/x_1, 1.0/x_2, 1.0/x_3]; []$
- 1: $[1.0/x_1, 1.0/x_2]; [y_1, y_4]$
- 2: $[1.0/x_2]; [y_1, y_2, y_3, y_4]$
- 3: $[1.0/x_1, 1.0/x_3]; [y_5]$
- 4: $[1.0/x_1]; [y_1, y_4, y_5]$
- 5: $[]; [y_1, y_2, y_3, y_4, y_5]$

The top concept 0 in the lattice shows that not a single attribute is possessed by all objects at the 1.0 degree. Concept 5, the bottom concept of the lattice, shows that not a single object possesses all attributes to degree 1.0. Concept 1 which is a child of concept 0 with a link strength of 0.667 shows that object x_1 and x_2 possess both attributes y_1 and y_4 . Concept 3 which is also a child of concept 0 with a link strength of 0.667 shows that objects x_1 and x_3 possess attribute y_5 . Concept 2 which is a child of concept 1 with a link strength of 0.5 shows that object x_2 possesses the attributes y_1, y_2, y_3 , and y_4 . Finally concept 4 which is a child of concept 1 with a link strength of 0.5 and a child of concept 3 with a link strength of 0.5 shows object x_1 possesses attributes y_1, y_4 , and y_5 . Concept 5 as the bottom concept is a child of both concepts 2 and 4 but with a link strength of 0.0 since its extent is empty and the intersection of it with the extents of concepts 2 and 4 is empty.

III. FUZZY CLOSURE APPROACH

The threshold approach can give preference to either the objects or the attributes. In its fuzzy concepts, a fuzzy extent exists if the objects are given preference, or a fuzzy intent exists if the attributes are given preference. In the fuzzy closure approach, however, the fuzzy concepts contain both fuzzy extents and fuzzy intents since it looks at the complete fuzzy formal context without giving preference to either objects or attributes. It also differs from the threshold approach in that it does not convert the fuzzy formal context into a crisp formal context in order to create the initial formal concepts and concept lattice structure. It also does not propose a link strength between fuzzy concepts although the fuzzy similarity measure given in (5) could be used between the intent fuzzy sets, the extent fuzzy sets or both to determine a link strength between a parent and child concept. A detailed description of the fuzzy closure approach and its algorithm can be found in [11] and an explanation and insights into the workings of the algorithm are provided in [12]. Here a summary and discussion of the important points of this approach are presented.

The fuzzy closure approach requires that the $R(x,y)$ values in the fuzzy formal context be taken from a complete residuated lattice L in the real unit interval $[0,1]$, well-known in fuzzy logic [13]. L is assumed to be finite and linearly ordered. The lattice structure requires the two operations of

fuzzy conjunction $\otimes$ and a residuum $\rightarrow$, a special kind of fuzzy implication operator defined as

$$a \rightarrow b = \max\{c \text{ in } L \mid a \otimes c \leq b\}.$$

A residuum satisfies an adjointness condition: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$. With the operations $\otimes$ and $\rightarrow$, L forms a complete residuated lattice. In [11], the example uses the Lukasiewicz operators

$$\begin{aligned} a \otimes b &= \max(0, a + b - 1) \\ a \rightarrow b &= \min(1 - a + b, 1) \end{aligned} \quad (5)$$

The set of all fuzzy sets using L in a universe X is denoted by L^X where a fuzzy set A means a mapping $A: X \rightarrow L$ assigning to any $x \in X$ a truth degree $A(x) \in L$ to which x belongs to A . Given the fuzzy formal context $\langle X, Y, R \rangle$ where X is the set of objects, Y is the set of attributes, and R is the fuzzy binary relation, for fuzzy sets $A \in L^X$ and $B \in L^Y$, fuzzy sets $A \uparrow \in L^Y$ and $B \downarrow \in L^X$ are defined as

$$\begin{aligned} A \uparrow &= \min_{y \in Y} (A(x) \rightarrow R(x,y)) \\ B \downarrow &= \min_{x \in X} (B(y) \rightarrow R(x,y)) \end{aligned} \quad (6)$$

The interpretation of $A \uparrow(y)$ based on fuzzy logic is that the membership degree of y in $A \uparrow$ specifies the truth degree that the attribute y is possessed by all objects in the fuzzy set A . Similarly, the membership degree of x in $B \downarrow$ specifies the truth degree that the object x possesses all attributes in the fuzzy set B . $\langle A, B \rangle$ is a fuzzy formal concept in the fuzzy concept lattice if $A \uparrow = B$ and $B \downarrow = A$. Denote $\mathbf{B} \langle X, Y, R \rangle = \{ \langle A, B \rangle \mid A \uparrow = B, B \downarrow = A \}$ as the set of all fuzzy concepts for the fuzzy formal context $\langle X, Y, R \rangle$.

The conceptual hierarchy is modeled by the relation $\leq$ defined on $\mathbf{B} \langle X, Y, R \rangle$ by $\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle$ if and only if $A_1 \subseteq A_2$ (or, equivalently $B_1 \supseteq B_2$). The set $\mathbf{B} \langle X, Y, R \rangle$ under the relation $\leq$ is a fuzzy concept lattice. Since fuzzy concepts are simply the fixpoints of a particular fuzzy closure operator [14], computing all fuzzy concepts reduces to computing all fixpoints of a fuzzy closure operator.

A fuzzy closure operator C on a universe X is a mapping $C: L^X \rightarrow L^X$ such that the following three conditions are satisfied: $A \subseteq C(A)$ (extensivity), $A \subseteq B$ implies $C(A) \subseteq C(B)$ (monotony), and $C(A) = C(C(A))$ (idempotency). If $A = C(A)$, then A is called a fixpoint of C . The set of all fixed points of C is denoted by $\text{fix}(C)$, such that $\text{fix}(C) = \{A \in L^X \mid A = C(A)\}$. Given a formal context $\langle X, Y, I \rangle$, the compound mapping $\uparrow \downarrow: L^X \rightarrow L^X$ is a fuzzy closure operator in X and $\downarrow \uparrow: L^Y \rightarrow L^Y$ is a fuzzy closure operator in Y . The fixpoints of $\uparrow \downarrow$ and the fixpoints of $\downarrow \uparrow$ are just the extents and intents respectively of the fuzzy formal concepts of $\langle X, Y, I \rangle$.

A brute-force method to determine all the fixpoints would be to simply generate all $A \in L^X$ and then output those where $A = C(A)$. This method has exponential time complexity and, therefore, is of no use. The research in [11] presents an algorithm to compute a fuzzy concept lattice, i.e., all the fixpoints of a fuzzy closure operator as well as the structure of the fuzzy concept lattice. It is based on extending the NEXTNEIGHBOR algorithm [15] which computes crisp concept lattices.

This recursive algorithm begins by calculating the fuzzy closure $\downarrow\uparrow$ on the empty attribute set to produce the first intent B_0 of the fuzzy concept lattice. B_0 is the intent of the top concept $(B_0\downarrow, B_0)$ in the fuzzy concept lattice. The algorithm proceeds by generating from B_0 (GenerateFrom procedure) the intents of its set of upper neighbors (Neighbors procedure), i.e., the intents of the fuzzy concepts that are subconcepts of the fuzzy concept $(B_0\downarrow, B_0)$. A neighbor generated for fuzzy concept $(B_0\downarrow, B_0)$ has an intent that is a superset of B_0 and an extent that is a subset of the extent of $B_0\downarrow$, where the standard definition of inclusion on fuzzy sets is used, i.e., $F_1 \subseteq F_2$ if $\forall x \in X, \mu_{F_1}(x) \leq \mu_{F_2}(x)$. Each of these neighbors is then linked to $(B_0\downarrow, B_0)$. The generated intents of the neighbors not already in the set of intents for $\mathbf{B} \langle X, Y, R \rangle$ are then added to the set. Each intent added to the set of intents is then input to the GenerateFrom procedure to generate its neighbors. This process repeats recursively up to the intent consisting of Y , the complete attribute set, which is the greatest fixpoint of C .

The challenging part is finding the upper neighbor intents for a given input intent B . The *upper neighbor* $D \in \text{fix}(C)$ w.r.t C , written as $B \prec^C D$, is established if $B \in \text{fix}(C)$ such that i) $B \subset D$ ii) there is no $D' \in \text{fix}(C)$ such that $B \subset D' \subset D$. The Neighbors procedure relies on several lemmas and theorems proven in [11] that show each upper neighbor intent is in fact produced by some $y \in Y$. Two orderings are required by the Neighbors procedure: the ordered membership degrees in $L = \{a_1, a_2, \dots, a_k\}$ such that $0 = a_1 < a_2 < \dots < a_k = 1$ and a fixed ordering of attributes from $Y = \{y_1, y_2, \dots, y_n\}$ and $y_i < y_j$ if $i < j$. The Neighbors procedure systematically proceeds through this fixed ordering checking each y_i that does not already have a membership degree of 1.0 in B as a possible generator of an upper neighbor of B . It does this by first modifying B to produce B' . Assume $B(y_i) = a_j$, then B' becomes $B \cup \{a_{j+1}/y_i\}$, that is, the membership degree of y_i in B' becomes the next membership degree in the ordering that is greater than the membership of y_i in B . Then the potential upper neighbor D which must be a fixpoint of C is calculated as $D = C(B')$. The problem, however, is that there might exist $D' \in \text{fix}(C)$ such that $B \subset D' \subset D$. The algorithm uses the fixed ordering of the attributes Y to make sure that the calculated D is selected as an upper neighbor of B only if no other D' exists.

The fuzzy formal context in Fig. 1 with $L = \{0, 0.5, 1.0\}$ and the implication operator $\rightarrow$ given in (5) was used to verify our implementation of the fuzzy closure operator approach to creating fuzzy concept lattices. The results produced are identical to those provided in [11]

- 0: [1.0/x1, 1.0/x2, 1.0/x3]; [0.5/y3, 0.5/y4, 0.5/y5]
- 1: [1.0/x1, 1.0/x2, 0.5/x3]; [0.5/y1, 0.5/y2, 0.5/y3, 1.0/y4, 0.5/y5]
- 2: [1.0/x1, 0.5/x2, 1.0/x3]; [0.5/y3, 0.5/y4, 1.0/y5]
- 3: [1.0/x1, 1.0/x2]; [1.0/y1, 0.5/y2, 0.5/y3, 1.0/y4, 0.5/y5]
- 4: [0.5/x1, 1.0/x2, 0.5/x3]; [0.5/y1, 0.5/y2, 1.0/y3, 1.0/y4, 0.5/y5]
- 5: [1.0/x1, 0.5/x2, 0.5/x3]; [0.5/y1, 0.5/y2, 0.5/y3, 1.0/y4, 1.0/y5]
- 6: [0.5/x1, 1.0/x2]; [1.0/y1, 1.0/y2, 1.0/y3, 1.0/y4, 0.5/y5]

- 7: [1.0/x1, 0.5/x2]; [1.0/y1, 0.5/y2, 0.5/y3, 1.0/y4, 1.0/y5]
- 8: [0.5/x1, 0.5/x2]; [1.0/y1, 1.0/y2, 1.0/y3, 1.0/y4, 1.0/y5]
- 9: [0.5/x1, 0.5/x2, 0.5/x3]; [0.5/y1, 0.5/y2, 1.0/y3, 1.0/y4, 1.0/y5]

The concept lattice hierarchy produced by our implementation shows the first the concept number followed by concept numbers of its children and agrees with that given in [11]:

- 0: 1; 2;
- 1: 3; 4; 5;
- 2: 5;
- 3: 6; 7;
- 4: 6; 9;
- 5: 7; 9;
- 6: 8;
- 7: 8;
- 8:
- 9: 8;

The results from the threshold approach for 0.5 threshold with object preference produced two very simple fuzzy concepts; whereas the fuzzy closure approach produces a more complex concept lattice with 10 fuzzy concepts where both the extents and intents are fuzzy sets. Notice that (3) used to produce the fuzzy extent in the threshold approach is basically using $B\downarrow$ given in (6); however, since the threshold approach assumes a membership degree of 1 for all attributes in the intents, $B\downarrow$ in (6) simply becomes $(1 - 1 + R(x, y))$ or simply $R(x, y)$ with the $\rightarrow$ operation given in (5). Notice that the results of the threshold approach with a 0.5 value produce a top concept 0 that is similar to the top concept 0 of the fuzzy closure approach. Both have the same objects in the extent and the same attributes in the intent. A major difference is that in threshold approach with object preference, an object is described as a fuzzy set over attributes; therefore, an object's membership degree in the extent of the fuzzy concept indicates the minimum degree to which the object possesses all of the attributes in the intent. For the fuzzy closure approach the top concept 0 has an extent where each object fully belongs. Here performing $\uparrow$ on this extent produces the corresponding fuzzy intent. Each attribute's membership degree indicates the truth degree that the attribute is shared by all objects in the extent. This truth degree is affected by how strongly the objects themselves belong to the fuzzy extent.

Similarly, the bottom concept 1 for the threshold approach agrees with the bottom concept 8 for the fuzzy closure approach with respect to the objects in the extent and the attributes in the intent. However, in this case the membership degrees also match. As pointed out in the fuzzy closure algorithm the greatest fixpoint of C is the set Y which is the intent of the bottom concept. All membership degrees are 1.0 so that performing $B\downarrow$ on this intent of Y simply reduces to using (3) with the Lukasiewicz implication operator in (5).

Notice that when the threshold is 1.0, the results for the top and bottom concepts are quite different from that of the fuzzy closure approach. This difference is to be expected since the threshold approach ignores all entries in the fuzzy formal context that do not meet the threshold value while the fuzzy closure approach uses all the non-zero entries.

IV. EXPERIMENTAL TEST CASES

In [11], the experiments that were run using the fuzzy closure approach focused on its practical performance in terms of the algorithm's running time as influenced by characteristics of the formal context such as the size and distribution of the truth degrees. The formal contexts used in these performance experiments were not real world data but randomly generated. Another characteristic considered in the performance evaluation is the number of truth degrees in L . The performance of the fuzzy closure approach is compared to that of a fuzzy extension to the NEXTCLOSURE presented in [16]. The results of these experiments showed that the algorithm in [16] performs best when only the fuzzy concepts need to be computed. The fuzzy closure approach, however, performs considerably better when the structure of the fuzzy concept lattice is also needed.

In [10] the experiment using the threshold approach has as its objective to determine the usefulness of the fuzzy concept lattice as a foundation for developing a fuzzy ontology and how to represent the fuzzy ontology using OWL. There is no comparison of other algorithms for creating fuzzy concept lattices. The experiments described in this present paper differs from the previous ones done separately with these two approaches since the objective is to compare the resulting fuzzy concepts and fuzzy concept lattices produced by these two different approaches. The following discusses the various test cases used in the experiments.

A. Olympic Data

The first test case is the formal context developed using Olympic data for several athletes and is taken from [11]. It was selected since it could be used as another verification test case for the software implementation of the fuzzy closure approach algorithm and provides a smaller fuzzy formal context for comparing the two approaches. It also provides a more complex formal context than the x objects and y attributes formal context and a larger L set with $L = \{0.0, 0.25, 0.5, 0.75, 1.0\}$. Here the five objects represent athletes and the ten attributes are Olympic events. The fill ratio, the fraction of non-zero degrees in the fuzzy formal context, is 100%. The membership degree can be interpreted as an indication of how well the athlete performs in the event.

	100m	long Jump	shot Put	high Jump	400m	110m Hurdle	discus Throw	pole Vault	Javekin Throw	1500m
Sebrle	0.5	1	1	1	0.75	1	0.75	0.75	1	0.75
Clay	1	1	0.75	0.75	0.5	1	0.75	0.5	1	0.5
Karpov	1	1	0.75	0.75	1	1	1	0.25	0.25	0.75
Macey	0.5	0.5	0.75	1	0.75	0.5	0.75	0.25	0.5	1
Warners	0.75	0.75	0.5	0.5	0.75	1	0.25	0.5	0.25	0.75

Figure 2. Olympic data [11].

B. Gene Annotation Data

A gene can be annotated with concepts from the Gene Ontology (GO) [17]. An evidence code is provided with an annotation, for example, Traceable Author Statement (TAS). The evidence code is translated into a degree of evidence strength between a GO concept and a gene product. In the following, the evidence codes are translated into membership degrees as done in [18]. Traceable Author Statement (TAS) is

1.0. Inferred from Sequence Similarity (ISS) is 0.8. Inferred from Electronic Annotation (IEA) is 0.6. Non-traceable Author Statement (NAS) is 0.4. Not Documented (ND) and Not Recorded (NR) were modified from 0.1 to 0.2 since the fuzzy closure operator approach requires that the set of membership degrees in L must form a closed lattice, i.e., they must be closed on the implication operator so that $L = \{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$ for all the annotation data test cases.

A fuzzy formal context can be created from an annotation data file with the genes as the objects, the GO terms as the attributes. The numeric evidence codes are interpreted as the degree to which a gene is annotated by a GO concept. For the experiments, the gene annotation data file GPD 194 data set [18] is used. It contains 194 human gene products (proteins). Here genes are classified into three families: collagen, myotubularin, and receptor precursor.

For the collagen annotation data shown in Fig. 3, there are 13 genes annotated by 23 GO terms. The fill-ratio is 20%.

GO t e r m	5	5	8	1	5	5	7	8	7	5	7	8	5	5	3	5	8	9	7	5	0	5	5
gene	1A2	.6	.6	.2	1	.2																	
21A 1							.6	.6															
24A 1	.6	.6						.6															
27A 1	.6	.6																					
2A1	.6	.6	.2	1	.2				1	.6	.2												
3A1	.6	.6									.2	1	1										
4A1	.6												.2										
4A2	1												1	.4									
4A3	.6						.6		1			1	1	.6	1	1							
4A6	.4						.6						.4	.4	.6								
5A3	.6	.6					.6			1								1	1				
9A1	.4						.6			1										.2	.6	1	
9A2	.6			1																		.6	1

Figure 3. Fuzzy formal context for the Collagen Family.

Fig. 4 shows the fuzzy formal context for the myotubularin family that has 7 (genes) and 14 (GO terms). The fill ratio is 35%.

Go Term	4722	4725	16787	6470	8372	4727	7517	8151	8138	5737	16020	4721	8270	6118
gene	MTMR1	.8	.8	.6	.8	.2								
MTMR2	.4		.6	.4	.2	.6	1	1	.4					
MTMR3			.6	1		.6			1	1	1			
MTMR4				.6								.6	.6	
MTMR6	.4	.4	.6	.4	.2									
MTMR7		1	.6	1										.6
MTMR8			.6	.6								.6		

Figure 4. Fuzzy formal context for the Myotubularin Family.

For the receptor precursor annotation, there are 7 genes annotated by 35 GO terms with a fill-ratio of 33%. This data set is not shown due to space limitations. In the combined gene annotation data, all three families, collagen, myotubularin and receptor, are considered as a single test. It contains 27 (13+7+7) genes as objects and 63 GO terms

(23+14+35 – 9 common terms) as attributes with a fill ratio of 10%. Due to the large size of this combined fuzzy formal context, it is not provided here. The rows are simply the union of all three sets of genes in each of the families and its columns are simply the union of all GO terms in the above three previous tables.

V. ANALYSIS OF THE EXPERIMENTAL RESULTS

The fuzzy closure approach uses the complete fuzzy formal context; thus, the results for the threshold approach are only given using the minimum membership degree in each of the fuzzy formal contexts in section IV. Other threshold values instead of the minimum degree alters the fuzzy formal context from the one used by the fuzzy closure approach so that an unbiased comparison is not possible.

A. Olympic Data

The membership degrees in L can be interpreted as 0.0 does not perform in the event, 0.25 performs but not very well, 0.5 performs average in this event, 0.75 performs well in the event, and 1.0 performs exceptionally well in the event.

Threshold Approach: With $\alpha \geq 0.25$, only one concept which is both the top and bottom concept is produced :

0: [0.5/Sebrle, 0.5/Clay, 0.25/Karpov, 0.25/Macey, 0.25/Warners];
[100m, longJump, shotPut, highJump, 400m, 110mHurdle, discusThrow, poleVault, javelinThrow, 1500m]

The extent of concept 0 indicates that for this fuzzy formal context all of the objects possess all of the attributes to at least a degree value of 0.25. This result is expected since the fill ratio is 100%. The membership degrees in the extent, however, indicate for each object (athlete) the minimum degree of possession across all the attributes (events) in the intent. For example, the athlete Sebrle performs at least fair in all the events (his minimum degree in any event is 0.5). An examination of the formal context for the Olympic data confirms that indeed the 100m event has 0.5 degree, i.e., his weakest Olympic event.

Fuzzy Closure Approach: The Lukasiewicz implication operator produces 128 concepts. Fig. 5 provides a summary of this result since providing the complete listing of the concepts and the concept hierarchy would require too much space. The complete results on this data set were verified by requesting from the authors of [11] their fuzzy concept lattice results since these were not provided in their original paper. An exact match existed with their produced fuzzy concepts and fuzzy concept lattice hierarchy.

Notice that the results of the threshold approach with a 0.25 value produce a top concept 0 that is similar to the top concept 0 of the fuzzy closure approach as seen for the simple fuzzy formal context in Fig. 1. For the threshold approach, the membership degree of the object in the extent of the fuzzy concept indicates the minimum degree to which the athlete performs in all the events in the intent. For the fuzzy closure

Detail	Concept Id	Concepts
Top Concept	0	0: [1.0/Sebrle, 1.0/Clay, 1.0/Karpov, 1.0/Macey, 1.0/Warners]; [0.5/100m, 0.5/longJump, 0.5/shotPut, 0.5/highJump, 0.5/400m, 0.5/110mHurdle, 0.25/discusThrow, 0.25/poleVault, 0.25/javelinThrow, 0.5/1500m]
Children of top concept	1, 2, 3	1: [1.0/Sebrle, 1.0/Clay, 1.0/Karpov, 0.75/Macey, 1.0/Warners]; [0.5/100m, 0.75/longJump, 0.5/shotPut, 0.5/highJump, 0.5/400m, 0.75/110mHurdle, 0.25/discusThrow, 0.25/poleVault, 0.25/javelinThrow, 0.5/1500m] 2: [1.0/Sebrle, 1.0/Clay, 1.0/Karpov, 1.0/Macey, 0.75/Warners]; [0.5/100m, 0.5/longJump, 0.75/shotPut, 0.75/highJump, 0.5/400m, 0.5/110mHurdle, 0.5/discusThrow, 0.25/poleVault, 0.25/javelinThrow, 0.5/1500m] 3: [1.0/Sebrle, 0.75/Clay, 1.0/Karpov, 1.0/Macey, 1.0/Warners]; [0.5/100m, 0.5/longJump, 0.5/shotPut, 0.5/highJump, 0.75/400m, 0.5/110mHurdle, 0.25/discusThrow, 0.25/poleVault, 0.25/javelinThrow, 0.75/1500m]
Bottom concept	31	31: [0.5/Sebrle, 0.5/Clay, 0.25/Karpov, 0.25/Macey, 0.25/Warners]; [1.0/100m, 1.0/longJump, 1.0/shotPut, 1.0/highJump, 1.0/400m, 1.0/110mHurdle, 1.0/discusThrow, 1.0/poleVault, 1.0/javelinThrow, 1.0/1500m]
Parents of Bottom Concept	30, 35, 55	30: [0.5/Sebrle, 0.5/Clay, 0.25/Karpov, 0.5/Macey, 0.25/Warners]; [1.0/100m, 1.0/longJump, 1.0/shotPut, 1.0/highJump, 1.0/400m, 1.0/110mHurdle, 1.0/discusThrow, 0.75/poleVault, 1.0/javelinThrow, 1.0/1500m] 35: [0.5/Sebrle, 0.5/Clay, 0.25/Karpov, 0.25/Macey, 0.5/Warners]; [1.0/100m, 1.0/longJump, 1.0/shotPut, 1.0/highJump, 1.0/400m, 1.0/110mHurdle, 0.75/discusThrow, 1.0/poleVault, 0.75/javelinThrow, 1.0/1500m] 55: [0.75/Sebrle, 0.5/Clay, 0.25/Karpov, 0.25/Macey, 0.25/Warners]; [0.75/100m, 1.0/longJump, 1.0/shotPut, 1.0/highJump, 1.0/400m, 1.0/110mHurdle, 1.0/discusThrow, 1.0/poleVault, 1.0/javelinThrow, 1.0/1500m]

Figure 5. Olympic Data Summary for Fuzzy Closure Approach.

approach the top concept 0 has an extent where each athlete fully belongs to the extent. Here performing $\uparrow$ on this extent produces the corresponding fuzzy intent. Each event's membership degree in the intent indicates the truth degree that the event is shared by all athletes in the extent.

Similarly, the bottom concept 0 (also the same as the top) for the threshold approach agrees with the bottom concept 31 for the fuzzy closure approach with respect to the athletes in the extent and the events in the intent. However, in this case the membership degrees also match as seen for the simple fuzzy formal context given in Fig. 1. As pointed out in the fuzzy closure algorithm the greatest fixpoint of C is the set Y which is the intent of the bottom concept, i.e. all events. All membership degrees for the events are 1.0 so that performing $B\downarrow$ on this intent Y simply reduces to using (3) with the Lukasiewicz implication operator in (5).

Fig. 5 also shows the children fuzzy concepts of the top concept 0, that is fuzzy concepts 1, 2, and 3 and the parent fuzzy concepts of the bottom concept 31, that is, fuzzy concepts 30, 35, and 55. For the children of the top concept, notice that each child has one or more athletes in the extent whose memberships is decreased by 0.25 from the extent of

the top concept and one or more events in the intent that are increased by 0.25 from the intent of the top concept. For example, fuzzy concept 3 has Clay's membership decreased by 0.25 in the fuzzy extent and has the 400m and the 1500 memberships increased by 0.25 in the fuzzy intent. For the parents of the bottom concept, each parent has one or more athletes in the extent whose memberships is increased by 0.25 and one or more events in the intent that are decreased by 0.25. For example, fuzzy concept 30 has Macy's membership increased by 0.25 in the fuzzy extent and has the pole vault membership decreased by 0.25 in the fuzzy intent. From this example, one can see that the fuzzy closure operator approach generates a vast number of fuzzy concepts, many of which are very close or similar to already existing fuzzy concepts.

B. Gene Annotation Data

For all analysis, $\alpha \geq 0.2$ threshold is used. To illustrate and explain the similarities between the two approaches the collagen data set is selected. Fig. 6 shows the concepts produced for this data set. The concept lattice hierarchy is not shown due to space limitations but is described here. The top concept 0 has child concept 1 whose extent shows that almost all genes (one is missing) can be described by GO_0005201 with at least the minimum evidence code of 0.2. The extent of concept 7, the other child concept of the top concept 0, shows six genes can all be described by GO_0007155. The bottom concept 6 has several parent concepts 5, 10, 12, 16, 18, 22, 23, and 25, each having one gene described by a set of several GO terms. Concepts 10 and 12 have the largest intents with 12 GO terms; the single gene in their extents is described by the greatest number of GO terms at the minimum evidence code or better.

Since 1160 concepts are produced using the fuzzy closure approach, the results are not presented, but the similarities between the two approaches are discussed. For the collagen data set fuzzy closure top concept 0 has an empty intent and an extent with every gene having a 1.0 membership degree. Similarly, the bottom concept has an empty extent and an intent with every GO term having a 1.0 membership degree. No GO term is shared by all genes and no gene is shared by all GO terms. This result is similar to that of the threshold approach.

The two children of the top concept are concepts 1 and 2 and given below.

1: [1.0/COL1A2, 0.8/COL21A1, 1.0/COL24A1, 1.0/COL27A1, 1.0/COL2A1, 1.0/COL3A1, 1.0/COL4A1, 1.0/COL4A2, 1.0/COL4A3, 1.0/COL4A6, 1.0/COL5A3, 1.0/COL9A1, 1.0/COL9A2]; [0.2/GO_0005201]
2: [0.8/COL1A2, 1.0/COL21A1, 1.0/COL24A1, 0.8/COL27A1, 0.8/COL2A1, 0.8/COL3A1, 0.8/COL4A1, 0.8/COL4A2, 1.0/COL4A3, 1.0/COL4A6, 1.0/COL5A3, 1.0/COL9A1, 0.8/COL9A2]; [0.2/GO_0007155]

They have the same two GO terms found for the two children concepts of the top concept in the threshold approach, concepts 1 and 7; however, they disagree in the membership degrees with a 0.2 membership for both GO_0005201 and GO_0007155. This degree represents the truth degree that the specified GO term is possessed by all genes in its respective intent. The genes in the extents differ between the two

approaches. For the threshold concept 1, the missing gene is COL21A1. In Fig. 3, one can see that this gene is not annotated by this GO term GO_0005201. Looking at concept 1 in the fuzzy closure approach, one sees that COL21A has a 0.8 membership degree. Similarly, for threshold concept 7, there are 7 missing genes in the extent that can be found with a 0.8 membership degree in the extent of concept 2 for the fuzzy closure approach. A similar analysis occurs for all the other gene data sets when comparing the threshold approach to the fuzzy closure approach. The top and bottom concepts agree with respect to the genes in the extent and the GO terms in the intent. They may have different membership degrees.

Concepts:
0: [1.0/COL1A2, 1.0/COL21A1, 1.0/COL24A1, 1.0/COL27A1, 1.0/COL2A1, 1.0/COL3A1, 1.0/COL4A1, 1.0/COL4A2, 1.0/COL4A3, 1.0/COL4A6, 1.0/COL5A3, 1.0/COL9A1, 1.0/COL9A2]; []
1: [0.6/COL1A2, 0.6/COL24A1, 0.6/COL27A1, 0.6/COL2A1, 0.6/COL3A1, 0.6/COL4A1, 1.0/COL4A2, 0.6/COL4A3, 0.4/COL4A6, 0.6/COL5A3, 0.4/COL9A1, 0.6/COL9A2]; [GO_0005201]
2: [0.6/COL1A2, 0.6/COL24A1, 0.6/COL27A1, 0.6/COL2A1, 0.6/COL3A1, 0.6/COL5A3]; [GO_0005581, GO_0005201]
3: [0.2/COL1A2, 0.2/COL2A1]; [GO_0008147, GO_0005201, GO_0005581, GO_0001501, GO_0005584]
4: [0.6/COL1A2, 0.6/COL2A1, 0.6/COL9A2]; [GO_0001501, GO_0005201]
5: [0.6/COL21A1]; [GO_0005198, GO_0007155]
6: []; [GO_0005198, GO_0005201, GO_0005581, GO_0008147, GO_0001501, GO_0005584, GO_0007155, GO_0008544, GO_0007605, GO_0005585, GO_0007397, GO_0008015, GO_0005586, GO_0005587, GO_0030198, GO_0005194, GO_0008285, GO_0009405, GO_0007517, GO_0005588, GO_0000004, GO_0005578, GO_0005594]
7: [0.6/COL21A1, 0.6/COL24A1, 0.6/COL4A3, 0.6/COL4A6, 0.6/COL5A3, 0.6/COL9A1]; [GO_0007155]
8: [0.6/COL24A1, 0.6/COL4A3, 0.4/COL4A6, 0.6/COL5A3, 0.4/COL9A1]; [GO_0007155, GO_0005201]
9: [0.6/COL24A1, 0.6/COL5A3]; [GO_0007155, GO_0005201, GO_0005581]
10: [0.2/COL2A1]; [GO_0008544, GO_0005201, GO_0005581, GO_0008147, GO_0001501, GO_0005584, GO_0007605, GO_0005585]
11: [0.6/COL2A1, 0.6/COL4A3]; [GO_0007605, GO_0005201]
12: [0.6/COL4A3]; [GO_0007605, GO_0005201, GO_0007155, GO_0008015, GO_0005587, GO_0005194, GO_0008285, GO_0009405]
13: [0.2/COL3A1, 0.6/COL5A3, 0.4/COL9A1]; [GO_0007397, GO_0005201]
14: [0.2/COL3A1, 0.6/COL5A3]; [GO_0007397, GO_0005201, GO_0005581]
15: [0.6/COL5A3, 0.4/COL9A1]; [GO_0007397, GO_0005201, GO_0007155]
16: [0.6/COL5A3]; [GO_0007397, GO_0005201, GO_0005581, GO_0007155, GO_0007517, GO_0005588]
17: [0.6/COL3A1, 0.6/COL4A3]; [GO_0008015, GO_0005201]
18: [0.2/COL3A1]; [GO_0008015, GO_0005201, GO_0005581, GO_0007397, GO_0005586]
19: [0.2/COL4A1, 1.0/COL4A2, 0.6/COL4A3, 0.4/COL4A6]; [GO_0005587, GO_0005201]
20: [0.6/COL4A3, 0.4/COL4A6]; [GO_0005587, GO_0005201, GO_0007155, GO_0005194]
21: [0.4/COL4A2, 0.4/COL4A6]; [GO_0030198, GO_0005201, GO_0005587]
22: [0.4/COL4A6]; [GO_0030198, GO_0005201, GO_0007155, GO_0005587, GO_0005194]
23: [0.2/COL9A1]; [GO_0000004, GO_0005201, GO_0007155, GO_0007397, GO_0005578, GO_0005594]
24: [0.4/COL9A1, 0.6/COL9A2]; [GO_0005578, GO_0005201, GO_0005594]
25: [0.6/COL9A2]; [GO_0005578, GO_0005201, GO_0001501, GO_0005594]

Figure 6. Collagen data threshold approach with $\alpha \geq 0.2$.

Fig. 7 compares the approaches. Column 1 gives the test data set. Columns 2 and 3 gives the number of concepts produced by the respective approaches. Column 4 gives the number of α -cut concepts exactly matching a fuzzy closure concept. For this column, the extent of the α -cut concept must agree with that of the fuzzy closure concept both with respect to the set of objects and their membership degrees. The intent of the α -cut concept, however, only has to agree with the set of attributes in the intent of the fuzzy closure concept since the α -cut intents assume degree of 1.0 for each attribute. Column 5 gives the number of concepts from the fuzzy closure approach that match an α -cut concept ignoring the membership degrees. Column 6 gives the number of concepts in the threshold approach having exactly the same extent including matching degrees. Column 7 gives the number of extents from the fuzzy closure approach that match to an α -cut extent, ignoring the membership degrees.

Column 8 gives the number of intents from the fuzzy closure approach that match to an α -cut intent, ignoring the membership degrees.

Test Cases	# concepts		# concepts exact matches	# concepts matches	# extent exact matches	# extent matches	# intent matches
	Alpha-cut	Fuzzy Closure					
x-object, y-attribute	2(0.5)	10	1	6	2	10	10
Olympic	1(0.25)	128	1	128	1	128	128
Collagen	26(0.2)	1160	2	12	26	1160	1160
Myotubularin	13(0.2)	470	1	5	13	470	470
Receptor	17(0.2)	913	1	19	17	913	913
Combined gene	62(0.2)	3495	4	30	62	3495	3495

Figure 7. Summary comparing threshold to fuzzy closure approach.

Both column 3 and 7 have identical values. That is, each extent produced by threshold approach maps exactly to an extent in one of the concepts produced by the fuzzy closure approach. The conclusion is that extents produced by the threshold approach are a subset of the extents produced by the fuzzy closure approach. Secondly, the numbers in column 7 and column 8 are identical to that of column 3. Ignoring degrees, the extents produced by the closure approach result in the same set of extents produced by the threshold approach. A large number of extents produced by the closure approach are identical in the set of objects they contain. For example, for the 1160 extents of the collagen data, 26 sets of objects are unique and repeatedly used with varying membership degrees and similarly, for the intents of the collagen data. These test cases show that the fuzzy closure operator approach generates a vast number of fuzzy concepts, many of which are very close or similar to already existing fuzzy concepts within the lattice. Threshold approach using the smallest membership degree in the fuzzy formal context as the α -cut value is much simpler and produces all the extent sets and intent sets of the fuzzy closure approach ignoring membership degrees. The fuzzy closure approach provides additional information with respect to the intents membership degrees. Examining such large fuzzy concept lattices to uncover information hidden in the fuzzy formal context, would be very challenging.

VI. CONCLUSIONS AND FUTURE WORK

The threshold approach and the fuzzy closure approach to creating fuzzy concept lattices from a fuzzy formal context have been implemented and verified using examples provided in [11]. GO annotation data files have also been used to experiment on larger fuzzy formal contexts. The two approaches have been compared using these data sets. The fuzzy closure approach produces an explosion in the number of fuzzy concepts compared to the threshold approach. The extents produced by the threshold approach are a subset of those produced by the fuzzy closure approach. Currently, the recommended factor analysis method [11] is being

implemented to identify the significant fuzzy concepts from a fuzzy formal context. Different implication operators are currently being tested for the closure operator to compare the differences in the resulting lattices. Future plans are to apply this software to other real world application data to validate and enhance its usability.

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