

Approximate Confidence Interval for Generalized Taguchi Process Capability Index

Abbas Parchami and Mashaallah Mashinchi

Department of Statistics, Faculty of Mathematics and
Computer Sciences, Shahid Bahonar University of Kerman,
Kerman, Iran
parchami@mail.uk.ac.ir, mashinchi@mail.uk.ac.ir

M. Hadi Mashinchi

Department of Computing, Macquarie
University, NSW 2112, Australia
hadi.mashinchi@mq.edu.au

Abstract— The impreciseness happens in the quality control same as other statistical problems. In quality control where specification limits are better expressed by fuzzy sets, the generalized process capability indices $C_{\tilde{p}}$, $C_{\tilde{pk}}$ and $C_{\tilde{pm}}$ can be helpful and necessary for measuring the capability. We propose a generalized form of Taguchi index $C_{\tilde{pm}}$ to assess the ability of the fuzzy process to be clustered around the target value. An approximate $100(1-\gamma)\%$ confidence interval for the generalized process capability index $C_{\tilde{pm}}$ by using fuzzy specification limits is also presented.

Keywords– Fuzzy quality; Taguchi process capability index; Confidence interval component

I. BACKGROUND

Traditional process capability indices (PCIs) can be classified into two major kinds. The first one, such as index C_p , deals solely with the process variation in relation to a specified tolerance region. The second one, such as indices C_{pk} and C_{pm} , considers not only the process variation relative to the tolerance region but also the deviation of process mean from target value. In [3] it is pointed out that the exact sampling properties of the estimates of C_{pk} were more complex than those for C_{pm} . The process capability index C_{pm} , named Taguchi capability index, is defined as

$$C_{pm} = \frac{USL - LSL}{6\sqrt{E(X - T)^2}} = \frac{USL - LSL}{6\sigma_T} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} \quad [1,$$

3, 20]. Here, T , LSL and USL are the target value, lower and upper specification limits (SLs), respectively.

In quality control, we may confront imprecise concepts. Ordinary capability indices cannot measure the capability if for example upper and lower SLs are imprecise. The impreciseness exists in SLs can be express by fuzzy terms. Recently, several PCIs developed deal with impreciseness [9, 19, 24, 25, 26, 27, 28, 32, 34, 38, 42] and they are applied in real cases [12, 13, 14, 15, 21]. We further investigate the point

and interval estimators of the generalized capability index C_{pm} which has been introduced in [16, 17, 18].

In this paper, the problems of point and interval estimations of the generalized process capability index $C_{\tilde{pm}}$ are considered which are based on the observed random sample for an in-control fuzzy process. Statistical inference based on fuzzy arithmetic is used to introduce a $100(1-\gamma)\%$ confidence interval for generalized capability index $C_{\tilde{pm}}$, where we have two membership functions for SLs.

The organization of this paper is as follows. In Section II, some preliminaries on fuzzy arithmetic are presented and then generalized PCIs are introduced. Tow point estimators and an interval estimator is introduced in Section III for $C_{\tilde{pm}}$. The final section is the conclusions part.

II. ARITHMETIC OPERATIONS ON FUZZY SPECIFICATION LIMITS

Let R be the set of all real numbers and $F(R) = \{\tilde{A} \mid \tilde{A} : R \rightarrow [0,1], \tilde{A} \text{ is a continuous function}\}$ be the set of all fuzzy sets on R . The α -cut of $\tilde{A} \in F(R)$ is the crisp set defined by $\tilde{A}_\alpha = \{x \mid \tilde{A}(x) \geq \alpha\}$, for any $\alpha \in [0,1]$. Although fuzzy SLs are introduced in [24], but we introduce new flexible formulas for the fuzzy SLs in Definition 2.1, inspired LR fuzzy numbers [7, 23]. The motivation of introducing fuzzy SLs in fuzzy processes is already introduced in [24].

Definition 2.1 Let $\widetilde{USL}$ and $\widetilde{LSL}$ be two fuzzy sets with membership functions

$$\widetilde{USL}(x) = \begin{cases} 1 & \text{if } x < q \\ U(\frac{x-q}{b}) & \text{if } x \geq q \end{cases}$$

and

$$\widetilde{LSL}(x) = \begin{cases} L(\frac{p-x}{a}) & \text{if } x \leq p \\ 1 & \text{if } x > p \end{cases}$$

in which reference functions $L, U : [0, +\infty] \rightarrow [0, 1]$ are non-increasing functions, $U(0) = L(0) = 1$, and the upper and lower spreads $a, b > 0$. Then, $\widetilde{USL}$ and $\widetilde{LSL}$ are called upper fuzzy specification limit and lower fuzzy specification limit, and symbolically are denoted by $(q, b)_U$ and $(p, a)_L$, respectively. Also, $F_U(R)$ and $F_L(R)$ denote two sets of all upper and lower fuzzy SLs, respectively.

In Definition 2.1, the reference function L (or similarly U) can be described by any of the following functions:

$$(1) L(x) = 1 \text{ for } x \in [0, 1] \text{ and } 0 \text{ outside,}$$

$$(2) L(x) = \max\{0, 1 - |x|^r\},$$

$$(3) L(x) = e^{-|x|^r},$$

$$(4) L(x) = \frac{1}{1 + |x|^r},$$

$$(5) L(x) = \frac{1}{1 + r|x|},$$

in which $r > 0$.

Remark 2.1 The fuzziness increases in $(p, a)_L$ and $(q, b)_U$, with the increase in the spreads a and b .

Definition 2.2 A process with fuzzy SLs $(q, b)_U$ and $(p, a)_L$, which we call a fuzzy process for short, is one which satisfies the normal distribution condition, that is $X_1, \dots, X_n$ are independent, identically distributed random variables with $N(\mu, \sigma^2)$.

Definition 2.3 Let $\widetilde{USL} \in F_U(R)$ and $\widetilde{LSL} \in F_L(R)$ are the upper and the lower fuzzy SLs, such that $p \leq q$. Then

$$\widetilde{USL} \oplus \widetilde{LSL} = \int_0^1 g(\alpha)(u_\alpha + l_\alpha) d\alpha$$

and

$$\widetilde{USL} \ominus \widetilde{LSL} = \int_0^1 g(\alpha)(u_\alpha - l_\alpha) d\alpha,$$

are called addition and subtraction of $\widetilde{USL}$ and $\widetilde{LSL}$, respectively, in which

$$\widetilde{USL}_\alpha = (-\infty, u_\alpha], \quad \widetilde{LSL}_\alpha = [l_\alpha, +\infty), \quad (1)$$

u_0, l_0 are finite real numbers and g is a real valued non-decreasing function on $[0, 1]$ with $g(0) = 0$ and $\int_0^1 g(\alpha) d\alpha = 1$

(for instance $g(\alpha) = (m+1)\alpha^m$ where $m = 1, 2, 3, \dots$).

Note that in Definition 2.3, $u_\alpha + l_\alpha$ is the addition and $u_\alpha - l_\alpha$ is the subtraction of the left and the right end points

of the α -cut of $\widetilde{USL}$ and $\widetilde{LSL}$. The value of $g(\alpha)$ can be understood as the weight of $u_\alpha \pm l_\alpha$, and the non-decreasing property of g means that the higher the membership of α -cut, the more important it is in determining the addition and subtraction of $\widetilde{USL}$ and $\widetilde{LSL}$. These defined operations synthetically reflect the information on every membership degree. The application of the arithmetic operations given in Definition 2.3 on fuzzy SLs allows different α -cuts to play different roles.

For observing some properties of the arithmetic operations on fuzzy SLs and numerical examples on fuzzy SLs, see [24]. Now, by extending the addition and subtract operations on fuzzy SLs, one can use $\ominus$ and $\oplus$ operations to generalize the classical PCIs, when the SLs are fuzzy rather than crisp [24].

Definition 2.4 Let in a fuzzy process $\widetilde{USL} \in F_U(R)$ and $\widetilde{LSL} \in F_L(R)$ be the engineering upper and lower fuzzy SLs, respectively, where $p \leq q$. Then the extended process capability indices are defined as follows:

$$C_{\tilde{p}} = \frac{\widetilde{USL} \ominus \widetilde{LSL}}{6\sigma}, \quad (2)$$

$$C_{\tilde{pk}} = \frac{\widetilde{USL} \ominus \widetilde{LSL} - 2 \left| \mu - \frac{\widetilde{USL} \oplus \widetilde{LSL}}{2} \right|}{6\sigma} \quad (3)$$

and

$$C_{\tilde{pm}} = \frac{\widetilde{USL} \ominus \widetilde{LSL}}{6\sqrt{E(X-T)^2}} = \frac{\widetilde{USL} \ominus \widetilde{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}}, \quad (4)$$

where T is the target value. Note that $C_{\tilde{p}}$ is useful when

$$\mu = m \text{ in which } m = \frac{\widetilde{USL} \oplus \widetilde{LSL}}{2}.$$

In the next section, we will discuss on the point and interval estimations of $C_{\tilde{pm}}$.

III. ESTIMATION OF THE GENERALIZED TAGUCHI INDEX

A. Point estimation of $C_{\tilde{pm}}$

The rationale behind the point estimation is quite simple. When sampling is from a population described by a Probability Density Function (PDF) or Probabiliiy Mass Function (PMF) f_θ , knowledge of unknown parameter θ yields knowledge of the entire population. Hence it is natural to seek a good estimator of the unknown point θ , that is a good estimator as a function of sample. In this subsection, we

are going to obtain two point estimators for unknown parameter C_{pm} .

The sample mean of random sample $X_1, \dots, X_n$ from a process is $\bar{X} = \sum_{i=1}^n X_i / n$ and the sample variances are

$$S_{n-1}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \quad \text{and} \quad S_n^2 = \frac{n-1}{n} S_{n-1}^2. \quad \text{The}$$

estimators for $\sigma_T^2 = E(X - T)^2$ are

$$S_{T(n-1)}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - T)^2 \quad \text{and} \quad S_{T(n)}^2 = \frac{n-1}{n} S_{T(n-1)}^2.$$

Substituting the estimators of σ_T^2 in the formula of the Taguchi capability index will provides two point estimators for C_{pm} index. Therefore, the original estimate of C_{pm} is

defined as $\hat{C}_{pm(n-1)} = \frac{USL - LSL}{6S_{T(n-1)}}$ in [3]. Another estimator is

defined in [1] as

$$\hat{C}_{pm(n)} = \frac{USL - LSL}{6S_{T(n)}} = \sqrt{\frac{n}{n-1}} \hat{C}_{pm(n-1)}. \quad (5)$$

Considering the above discussion for the point estimations of C_{pm} and using Definition 2.4, one can easily define two

fuzzy estimators for C_{pm} . The below estimators (6) and (7)

are computed by substituting two estimators $S_{T(n-1)}^2$ and $S_{T(n)}^2$ of σ_T^2 in the formula of C_{pm} .

$$\hat{C}_{\widetilde{pm}(n-1)} = \frac{\widetilde{USL} \ominus \widetilde{LSL}}{6S_{T(n-1)}}, \quad (6)$$

$$\hat{C}_{\widetilde{pm}(n)} = \frac{\widetilde{USL} \ominus \widetilde{LSL}}{6S_{T(n)}} = \sqrt{\frac{n}{n-1}} \hat{C}_{\widetilde{pm}(n-1)}. \quad (7)$$

Like to traditional PCIs, for each of these generalized indices, a large value of estimate implies a better distribution of the quality characteristic. In sequel subsection, we only discuss the inference of the generalized index $C_{\widetilde{pm}(n)}$, since the case of $C_{\widetilde{pm}(n-1)}$ can be treated similarly.

B. Confidence interval for $C_{\widetilde{pm}}$

In this subsection, we obtain a confidence interval for the unknown parameter C_{pm} , based the sampling distributions of the corresponding estimators.

Substituting $S_{T(n)}^2$ and $S_{T(n-1)}^2$ in C_{pm} will provide two

fuzzy estimates for C_{pm} , which we denote them by $\hat{C}_{\widetilde{pm}(n)}$

and $\hat{C}_{\widetilde{pm}(n-1)}$, respectively. Like other statistics $\hat{C}_{\widetilde{pm}(n)}$ is also

subject to sampling variation and it is critical to compute a confidence interval. The confidence level provides a range which includes the true C_{pm} with high probability. The

confidence interval for C_{pm} is based on the distribution of

$S_{T(n)}^2$, where $S_{T(n)}^2$ is a uniformly minimum variance

unbiased estimator of $E(X - T)^2$ [35], and $\hat{C}_{\widetilde{pm}(n)}$ is defined

as a simple function of $S_{T(n)}^2$ in (7). Hence, to obtain any

100(1- γ)% confidence interval for C_{pm} index, we need a

statistical distribution of $S_{T(n)}^2$. This statistical distribution is presented in Chapter 29 of [10].

The approximate confidence interval for C_{pm} is given in Theorem 3.1.

Theorem 3.1 If $X_1, \dots, X_n$ is a random sample set from normal distribution with mean μ and variance σ^2 , then the approximate equal-tailed 100(1- γ)% confidence interval for C_{pm} is

$$\left[\hat{C}_{\widetilde{pm}(n)} \sqrt{\chi_{\frac{\alpha}{2}, \hat{\nu}_n}^2 / \hat{\nu}_n}, \hat{C}_{\widetilde{pm}(n)} \sqrt{\chi_{1-\frac{\alpha}{2}, \hat{\nu}_n}^2 / \hat{\nu}_n} \right],$$

where $\hat{\nu}_n = n(1 + \hat{\delta}_n^2)^2 / (1 + 2\hat{\delta}_n^2)$, $\hat{\delta}_n = (\bar{x} - T) / s_n$ and $\chi_{\alpha, \hat{\nu}_n}^2$ is the 100 α percentile of a central chi-square variable with $\hat{\nu}_n$ degree of freedom.

IV. CONCLUSIONS

This paper further improves the fuzzy process where it is more appropriate to define SLs by two fuzzy sets. In this paper, we presented two point estimators for C_{pm} . But as they are not exactly equal to the true value of parameter, so 100(1- γ)% confidence interval for the generalized process capability index C_{pm} is proposed and an approximate 100(1- γ)% confidence interval for C_{pm} is presented. For future research one can consider the following topics: (1) Generalizing the point and interval estimations of other precise process capability indices, (2) testing hypotheses on the generalized process capability indices, (3) validating of the generalized process capability indices for real-world industrial applications.

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