

Discretization of Fuzzy Transitive Relations.

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Abstract—Fuzzy transitivity is a key property for many fuzzy relational structures, such as Fuzzy Preorders and Equivalences. Theoretical models for practical problems which are based on fuzzy relations make use of continuous scales, mostly the unit interval $[0,1]$. Practical implementation of these models though, involves their discretization into finite scales, which generally results in some loss of transitivity. In this paper we study if there are any transitivity preserving discretization strategies. Also, we evaluate the loss of transitivity in some commonly used discretization approaches

Keywords - *fuzzy relation; fuzzy equivalence; T-transitivity; indistinguishability operator*

I. INTRODUCTION

Fuzzy relational structures play an important role in fuzzy modelling. Fuzzy transitivity is probably the most important property among those involved in the definition of the more complex relational structures, such as *Fuzzy Preorders* and *Fuzzy Equivalence Relations* [1,2].

Generally speaking, theoretical models of continuous problems make use of continuous scales. In the case of fuzzy modelling, the paradigm of such scales is the real unit interval $[0,1]$. For computing purposes though, discretization has to be inevitably introduced at some stage, resulting in a distortion of the original modelling structure.

The discrete, finitely-valued operations on the unit interval, such as t-norms, t-conorms and OWA operators, have been widely studied by G.Mayor et al. See [3] for an excellent survey. On the other hand, the topic of fuzzy relational structures based on finitely-valued operations has been developed by J. Recasens et al. [4]. A comprehensive and readable introduction can be found in [2].

The present paper is about the fuzzy relations which are obtained from continuously valued fuzzy relations through a process of discretization. Those relations are not necessarily *finitely valued* preorders or equivalences in the sense of [3,4] owing to two reasons: the involved continuous t-norm is not required to become a finitely valued one, and fuzzy transitivity is not always preserved through discretization.

Let us recall some basic definitions the present work relies upon.

Definition 1.1: An operation T on the unit interval $[0,1]$ is called a *t-norm* if it is associative, commutative and it satisfies the boundary conditions $T(x,0) = T(0,x) = 0$ and $T(x,1) = T(1,x) = x$ for all x in $[0,1]$.

Definition 1.2: A fuzzy relation R on a universe X , i.e. a mapping $R : X \times X \rightarrow [0,1]$, is:

Reflexive, if $R(x,x) = 1$ for every x in X .

Symmetric, if $R(x,y) = R(y,x)$ for every x and y in X .

Transitive with respect to a given t-norm T , or *T-transitive*, if $T(R(x,y), R(y,z)) \leq R(x,z)$ for every x, y and z in X .

Definition 1.3: A *Fuzzy Proximity* is a reflexive and symmetrical fuzzy relation.

Definition 1.4: A *Fuzzy Preorder* is a reflexive and *T-transitive* fuzzy relation.

Definition 1.5: A *Fuzzy Equivalence Relation*, or *Indistinguishability Operator* is a reflexive, symmetrical and *T-transitive* fuzzy relation.

There are other names in use such as Zadeh's original *Similarity Relations* [5] or *T-Indistinguishability Operators* [6].

The concept of t-norm has become a standard in fuzzy logic. The reader is referred to [7] for an excellent state of the art on the subject. Throughout this paper, we will assume t-norms to be continuous, which in practice means only $T=\text{MIN}$ (the *minimum* t-norm), $T=\text{LUK}$ (the *Lukasiewicz* t-norm) and $T=\text{PROD}$ (the *standard product* t-norm) are considered. Any other continuous t-norm is either isomorphic to one of these, or is isomorphic to an ordinal sum of many of these.

Fuzzy transitivity introduced via a t-norm, or *T-transitivity* for short, is closely related to the *triangle inequality* for distances, $d(x,z) \leq d(x,y) + d(y,z)$, where the standard sum of real numbers play a similar role to that of the t-norm T . The standard order is though reversed, so that equivalence degrees near 1 correspond to distances near 0. Thus, points linked by high degrees of equivalence become close points. Consequently, the results we present in this paper also apply to the discretization of metric spaces.

Let us state precisely what is to be understood by discretization in the context of this paper.

Definition 1.6: A discrete scale $L_N = \{a_N \dots a_0\}$, a_i in $[0,1]$

, is a finite ordered subset such that:

1. $a_0 = 1$ and $a_N \geq 0$
2. $a_j < a_k$ if $j > k$.

Definition 1.7: A *discretization function* is a mapping $\Delta : [0,1] \rightarrow L_N$ such that:

1. $\Delta(a) \leq \Delta(b)$ if $a \leq b$, for all a, b in $[0,1]$.
2. $\Delta(\Delta(a)) = \Delta(a)$ for all a in $[0,1]$.
3. $\Delta(1) = 1$

A discretization of a fuzzy relation E is simply the fuzzy relation $\Delta(E)$, that is, the image of E through a discretization function.

Note that no specific assumptions are made on E , and consequently no specific requirements can be stated regarding t-norms, not being any involved at this stage.

Having fuzzy transitivity in mind makes it very tempting to tie the previous definitions to a t-norm by asking, for example, that $T(a_i, a_j) \in L_N$ for all a_i, a_j in L_N (definition 1.6), or that for $T(\Delta(a), \Delta(b)) \in L_N$ for all a and b in $[0,1]$ (definition 1.7). Although for practical reasons this becomes very often the case, such an assumption would confine our research to the topics of finitely valued t-norms and finitely valued fuzzy preorders and equivalences, which is too restrictive a setting. Our only concern here has to do with assessing the regularity of the distorted relation $\Delta(E)$ with respect to its continuously valued original model E and therefore, the originally involved continuous t-norm.

This section ends with a simple yet fundamental example showing that T -transitivity is not necessarily preserved through discretization.

Example 1.8 Let X be the unit interval $[0,1]$, $T = T_{LUK}$ the Lukasiewicz t-norm, i.e. $T_{LUK}(a, b) = \max\{a + b - 1, 0\}$, the discrete scale $L_1 = \{a_0, a_1\} = \{0, 1\}$ and the discretization function $\Delta : [0,1] \rightarrow L_1$ defined by $\Delta(a) = 1$ if $a \geq 0.5$ and $\Delta(a) = 0$ otherwise, where a, b are in $[0,1]$.

The fuzzy relation considered will be $E(x, y) = 1 - |x - y|$ for all x, y in $X = [0,1]$. E is called the *residuated bimplication*, or *natural equivalence relation*, associated with the t-norm T_{LUK} so it is a fundamental example [2].

Consider any three real numbers such that $0 \leq x_1 \leq x_2 \leq x_3 \leq 1$, $x_2 - x_1 \leq 0.5$ and $x_3 - x_2 \leq 0.5$, but $x_3 - x_1 > 0.5$. Then $\Delta(E(x_1, x_2)) = 1$ and $\Delta(E(x_2, x_3)) = 1$, but $\Delta(E(x_1, x_3)) = 0$, so that

$$1 = T(\Delta(E(x_1, x_2)), \Delta(E(x_2, x_3))) > \Delta(E(x_1, x_3)) = 0$$

and thus the breach of transitivity. ■

II. PROXIMITY RELATIONS AND MIN-TRANSITIVE RELATIONS

Let us start with two kinds of fuzzy relations which behave in a way of their own with respect to discretization: proximity relations (definition 1.3) where transitivity is not involved, and those which are transitive with respect to the minimum t-norm, with Zadeh's *Similarity Relations* standing as their most prominent example [5].

Proposition 2.1 Let E be a proximity relation on a universe X . Then $\Delta(E)$ is also a proximity relation.

The result is an obvious one. More interesting is the case of *MIN*-transitive relations.

Proposition 2.2 Let E be a *MIN*-transitive relation on a set X . Then $\Delta(E)$ is also *MIN*-transitive.

Proof. We need to prove *MIN*-transitivity for $\Delta(E)$, that is, $\text{MIN}(\Delta(E(x, y)), \Delta(E(y, z))) \leq \Delta(E(x, z))$

Since $\text{MIN}(E(x, y), E(y, z)) \leq E(x, z)$ we apply 1.7.1 to obtain $\Delta(\text{MIN}(E(x, y), E(y, z))) \leq \Delta(E(x, z))$

Also, since $[0,1]$ is a totally ordered set $\text{MIN}(E(x, y), E(y, z))$ equals either $E(x, y)$ or $E(y, z)$, or both. Without loss of generality we may assume it is $\text{MIN}(E(x, y), E(y, z)) = E(x, y)$. Then $E(x, y) \leq E(y, z)$, $\Delta(E(x, y)) \leq \Delta(E(y, z))$ (1.7.1), and

$$\Delta(\text{MIN}(E(x, y), E(y, z))) = \text{MIN}(\Delta(E(x, y)), \Delta(E(y, z))).$$

As a consequence, $\text{MIN}(\Delta(E(x, y)), \Delta(E(y, z))) =$

$$\Delta(\text{MIN}(E(x, y), E(y, z))) \leq \Delta(E(x, z)) \quad \blacksquare$$

Next is a corollary of propositions 2.1 and 2.2.

Corollary 2.3 Let E be a similarity relation on a set X . Then $\Delta(E)$ is also a similarity relation.

Corollary 2.4 Let E be a *MIN*-transitive relation on a set X . Then $\Delta(E)$ is T -transitive for any continuous t-norm T .

Corollary 2.4 follows from the fact that $\text{MIN} \geq T$ for any t-norm T (a well known fact. See, for example, [3]).

III. THE CASE OF TRANSITIVITY WITH RESPECT TO ARCHIMEDEAN T-NORMS

Continuous Archimedean t-norms are either strict or non-strict. Strict Archimedean t-norms are isomorphic to the product t-norm, i.e. $T_{PROD}(a, b) = a \cdot b$ for all a, b in $[0,1]$, while the non-strict ones are isomorphic to the Lukasiewicz t-norm (introduced above). Archimedean t-norms can be obtained from decreasing mappings $f : [0,1] \rightarrow [0, f(0)] \subseteq [0, +\infty]$ called additive generators by $T(a, b) = f^{[-1]}(f(a) + f(b))$. The symbol $f^{[-1]}$ stands for the pseudo-inverse of f , that is, the mapping $f^{[-1]} : [-\infty, +\infty] \rightarrow [0,1]$ defined by $f^{[-1]}(a) = f^{-1}(a)$

if $a \in [0, f(0)]$, $f^{[-1]}(a) = 1$ if $a < 0$ and $f^{[-1]}(a) = 0$ if $a \geq f(0)$. Strict Archimedean t-norms are those with $f(0) = \infty$, in which case $f^{[-1]} = f^{-1}$ and they are isomorphic to the sum of positive real numbers, while the non-strict ones are those with $f(0) < \infty$ and their behaviour is that of a bounded sum of positive real numbers.

The following theorem characterizes those discretization functions that preserve T -transitivity, provided that T is Archimedean.

Theorem 3.1 Let T be an Archimedean t-norm, $L_N = \{a_N \dots a_0\}$ a discrete scale and Δ a discretization function. Then, $\Delta(E)$ is T -transitive for any T -transitive fuzzy relation E on some universe X if, and only if, $T(\Delta(a), \Delta(b)) \leq \Delta(T(a, b))$ for all a, b in $[0, 1]$.

Proof. $[\Rightarrow]$ Given a and b in $[0, 1]$ let E be the T -transitive relation defined on $X=[0, 1]$ by $E(x, y) = f^{[-1]}(f(x) - f(y))$ where f is an additive generator of T . E is T -transitive because $T(E(x, y), E(y, z)) =$

$$\begin{aligned} & f^{[-1]}(f(f^{[-1]}(f(x) - f(y))) + f(f^{[-1]}(f(y) - f(z)))) \leq \\ & f^{[-1]}(f(x) - f(y) + f(y) - f(z)) = \\ & f^{[-1]}(f(x) - f(z)) = E(x, z) \end{aligned}$$

and, under the hypothesis, so is $\Delta(E)$. We may then consider $x = 1$, $y = a$, $z = T(a, b)$ so that $E(x, y) = a$, $E(y, z) \geq b$ and $E(x, z) = T(a, b)$. Since $\Delta(E)$ is T -transitive we have that

$$\begin{aligned} T(\Delta(a), \Delta(b)) & \leq T(\Delta(E(x, y)), \Delta(E(y, z))) \leq \\ \Delta(E(x, z)) & = \Delta(T(a, b)) \end{aligned}$$

$[\Leftarrow]$ Let us suppose now that $T(\Delta(a), \Delta(b)) \leq \Delta(T(a, b))$ for all a, b in $[0, 1]$. If E is a T -transitive fuzzy relation on a set X , then $T(E(x, y), E(y, z)) \leq E(x, z)$ for all x, y, z in X . By taking $a = E(x, y)$ and $b = E(y, z)$ we obtain:

$$\begin{aligned} T(\Delta(E(x, y)), \Delta(E(y, z))) & = T(\Delta(a), \Delta(b)) \leq \\ \Delta(T(a, b)) & = \Delta(T(E(x, y), E(y, z))) \leq \Delta(E(x, z)) \end{aligned}$$

thus proving the T -transitivity of $\Delta(E)$ ■

An important consequence of theorem 3.1 is that arbitrarily chosen discretization functions and scales do not necessarily preserve transitivity with respect to Archimedean t-norms. We will focus on *homogeneous* (or *equidistant*) finite scales, where lower truncation is (almost) the only discretisation strategy that may be applied to the unit interval as a whole.

Definition 3.2 Let T be an Archimedean t-norm, and $L_N = \{a_N \dots a_0\}$, a_i in $[0, 1]$, a finite scale such that $a_N > 0$ if T is strict. In this situation, L_N is homogeneous or equidistant with respect to the t-norm T if

$f^{[-1]}(f(a_i) - f(a_{i+1})) = f^{[-1]}(f(a_{i-1}) - f(a_i))$, $i = 1 \dots N-1$ for some additive generator f of T .

NOTES: 1. Either all the additive generators of T satisfy the condition or all of them fail to satisfy it (see [3]).

2. $a_N > 0$ in the strict Archimedean case because $f(0) = \infty$, which makes equidistance impossible.

Example 3.3 Consider $L_N = \{a_N \dots a_0\}$ with $a_i = \frac{N-i}{N}$,

$i = 0 \dots N$. L_N is a homogeneous finite scale with respect to the Lukasiewicz t-norm T_{LUK} but it is not homogeneous with respect to the product t-norm T_{PROD} .

This is a consequence of the different behaviour of the two t-norms' additive generators with respect to L_N . Taking $f(a) = f_{LUK}(a) = 1 - a$ and $g(a) = f_{PROD}(a) = -\ln(a)$ as generators, while $g^{[-1]}(g(a_{i+1}) - g(a_i)) = \frac{N-i}{N-i+1}$ depends on i , $f^{[-1]}(f(a_{i+1}) - f(a_i)) = \frac{N-1}{N}$ remains constant. ■

Example 3.4 Consider $L_N = \{a_N \dots a_0\}$ with $a_i = p^i$ for a fixed p , $0 < p < 1$, $i = 0 \dots N$. L_N is a homogeneous finite scale with respect to the product t-norm T_{PROD} but it is not homogeneous with respect to the Lukasiewicz t-norm T_{LUK} .

In the present case $g^{[-1]}(g(a_{i+1}) - g(a_i)) = p$ does not depend on i , while $f^{[-1]}(f(a_{i+1}) - f(a_i)) = 1 - |p^{i+1} - p^i|$ obviously does. Also, note that $a_N > 0$ no matter how big the chosen N for the particular scale, which makes the cardinality of the scale as big as required. ■

Within any fixed interval $[a_{i+1}, a_i]$ there are only three possible discretization functions: lower truncation, upper truncation and rounding.

Upper truncation is defined by $\Delta_U(a) = \bar{a} = a_i$ if $a_{i+1} < a \leq a_i$. Lower truncation by $\Delta_L(a) = \underline{a} = a_{i+1}$ if $a_{i+1} \leq a < a_i$. In order to precisely define rounding, an additional b_{i+1} between a_i and a_{i+1} is to be considered. Then $\Delta_R(a) = \hat{a} = a_{i+1}$ if $a_{i+1} \leq a < b_i$, or $\Delta_R(a) = \hat{a} = a_i$ if $b_i \leq a \leq a_i$. The fact that no more discretization functions are possible inside $[a_{i+1}, a_i]$ is a simple consequence of 1.7.1 and 1.7.2.

When it is the whole unit interval that is taken into account, things become a bit more complicated. Roughly speaking, only lower truncation keeps T -transitivity in a general sense, although some especial situations may occur when slightly different discretization functions may perform as well.

Lemma 3.5 Let T be an Archimedean t-norm, and $L_N = \{a_N \dots a_0\} \subseteq [0, 1]$ a homogeneous finite scale. If $i + j \leq N$ then $T(a_i, a_j) = a_{i+j}$

Proposition 3.6 (strict case) Let $L_N = \{a_N \dots a_0\}$ be a homogeneous finite scale w.r.t. a strict Archimedean t-norm T . Then the discretization function

$$\Delta(a) = \begin{cases} \bar{a} & \text{if } 0 \leq a < a_N \\ \underline{a} & \text{if } a_N \leq a \leq a_0 \end{cases}$$

preserves T -transitivity.

Proof: For any a, b in $[0,1]$ we need to prove that $T(\Delta(a), \Delta(b)) \leq \Delta(T(a, b))$. There are two different situations, depending on the value of $T(a, b)$.

Case 1. If $T(a, b) \leq a_N$ then $\Delta(T(a, b)) = \overline{T(a, b)} = a_N$. Now, if $a \leq a_N$ or $b \leq a_N$ then $\Delta(a) = a_N$ or $\Delta(b) = a_N$ so that $T(\Delta(a), \Delta(b)) \leq a_N$ and therefore $T(\Delta(a), \Delta(b)) \leq \Delta(T(a, b))$. On the other hand, if $a > a_N$ and $b > a_N$ then $\Delta(a) = \underline{a} \leq a$ and $\Delta(b) = \underline{b} \leq b$ so that $T(\Delta(a), \Delta(b)) \leq T(a, b) \leq a_N$ and we also conclude $T(\Delta(a), \Delta(b)) \leq \Delta(T(a, b))$.

Case 2. If $T(a, b) > a_N$ then let i and j be such that $a_{i+1} \leq a < a_i$ and $a_{j+1} \leq b < a_j$. Then, by Lemma 3.5 we have $a_{i+j+2} = T(a_{i+1}, a_{j+1}) \leq T(a, b) < T(a_i, a_j) = a_{i+j}$ so that $\Delta(T(a, b)) \geq \underline{T(a, b)} \geq a_{i+j+2}$. On the other hand, since $\Delta(a) = \underline{a} = a_{i+1}$ and $\Delta(b) = \underline{b} = a_{j+1}$ then $T(\Delta(a), \Delta(b)) = T(a_{i+1}, a_{j+1}) = a_{i+j+2}$ follows (Lemma 3.5), and again $T(\Delta(a), \Delta(b)) \leq \Delta(T(a, b))$. ■

Proposition 3.7 (non-strict case) Let $L_N = \{a_N \dots a_0\}$, a_i in $[0,1]$, be a homogeneous finite scale w.r.t. a non-strict Archimedean t-norm T . Then the discretization function $\Delta(a) = \underline{a}$ preserves T -transitivity.

Proof: Similar to proposition 3.6, the only difference being $a_N = 0$ which hardly matters. ■

Now we will show that both upper truncation and rounding fail to preserve T -transitivity at some point.

Lemma 3.8 Let $L_N = \{a_N \dots a_0\}$ be a finite scale and Δ a discretization function such that $\Delta(a) = a_0 = 1$ for some $a_1 < a < a_0$. Then Δ does not preserve T -transitivity with respect to any Archimedean t-norm.

Proof: Consider $x_1 = a < x_2 < \dots < x_{k-1} \leq a_1 < x_k$ where $x_2 = T(a, a) = a^2$, ..., $x_k = T(a, a^{k-1}) = a^k$. Such a chain exists for T is Archimedean. Then $\Delta(E(x_1, x_k)) \leq a_1$ while

$T(\dots(T(\Delta(E(x_1, x_2)), \Delta(E(x_2, x_3)))) \dots) = T(\dots(T(1, 1)) \dots) = 1$ which is against the T -transitivity of $\Delta(E)$. ■

As a consequence the next two propositions naturally follow.

Proposition 3.9 Upper truncation discretization does not preserve T -transitivity with respect to any Archimedean t-norm.

Proposition 3.10 Rounding discretization does not preserve T -transitivity with respect to any Archimedean t-norm.

Propositions 3.9 and 3.10 exclude upper truncation and rounding as global discretization strategies when the whole unit interval is considered. However, they say nothing of their suitability to handle T -transitivity when mixed with lower truncation at specific, on-purpose chosen intervals $[a_i, a_{i+1}]$. A such ‘custom-made’ discretization is shown in the next example.

Example 3.11 Let L_4 be the finite scale $\{a_4 = 0, a_3 = 0.25, a_2 = 0.5, a_1 = 0.75, a_0 = 1\}$ which is homogeneous with respect to T_{LUK} , the Lukasiewicz t-norm.

The discretization function $\Delta(a) = \begin{cases} \underline{a} & \text{if } a_4 \leq a < a_2 \\ \bar{a} & \text{if } a_2 \leq a \leq a_0 \end{cases}$ preserves T_{LUK} -transitivity.

IV. TRANSITIVITY LOSS IN UPPER TRUNCATION AND ROUNDING

Fuzzy transitivity is not a crisp notion [8,9,10]. Even when a fuzzy relation E fails to be strictly T -transitive with respect to a given t-norm T , some degree of transitivity can still be assigned to it depending on how strongly the relation fails to comply with that property, or how far it falls from any truly T -transitive relation of reference.

Since we know that both upper truncation and rounding fail to preserve T -transitivity for Archimedean t-norms, it makes sense to ask ourselves about the extent of the transitivity loss through these two discretization functions.

Transitivity with respect to a t-norm can be turned into a gradual concept in mostly two different ways. The first way assigns a transitivity degree to any fuzzy relation on a triplet by triplet basis:

Definition 4.1: A fuzzy relation R on a universe X is a -Transitive with respect to a given t-norm T , or a - T -transitive, if $T(a, T(R(x, y), R(y, z))) \leq R(x, z)$ for a given a in $[0,1]$, and for every x, y and z in X .

The second way takes an overall criteria in order to assign a degree of transitivity, which is called the *strong T-transitivity* degree. Strong T-transitivity is based on the closeness between the actual fuzzy relation E and its transitive closure $\bar{E}$ [11]. The transitive closure $\bar{E}$ of an arbitrary fuzzy relation E is the lowest T -transitive relation which is above E with respect to the pointwise order ($E \leq_x \bar{E}$). The closer E is to $\bar{E}$, the higher its degree of transitivity. Closeness in that context is measured through the *natural indistinguishability operator*, defined as follows.

Definition 4.2: Let T be an Archimedean t-norm, and f one of its additive generators. The fuzzy relation

$E_T(x, y) = f^{[-1]}(|f(x) - f(y)|)$ for all x and y in $[0, 1]$ is called the natural indistinguishability associated with T .

NOTE: E_T does not depend on the particular f chosen.

Definition 4.3: The natural indistinguishability between two fuzzy relations E and F defined on a set X is $\mathcal{E}_T(R, S) = \inf_{x, y \in X} E_T(R(x, y), S(x, y))$

Definition 4.4: A fuzzy relation E on a universe X is *strong a -Transitive* with respect to a given t-norm T , or *a - T -strong transitive*, if $\mathcal{E}_T(E, \bar{E}) \geq a$ for a given a in $[0, 1]$.

The general idea is that both upper truncation and rounding behave quite similarly, showing a loss of transitivity degree (in the sense of definition 4.1) which is in accordance with the coarseness of the finite scale in use.

Proposition 4.5: Let E be a fuzzy T -transitive relation, f an additive generator of the t-norm T , and $L_N = \{a_N \dots a_0\}$ a homogeneous finite scale of $[0, 1]$. Then:

4.5.1 The fuzzy relation $\Delta_U(E)$ (upper truncation of E) is δ - T -transitive with $\delta \geq f^{[-1]}(f(a_i) - f(a_{i+1}))$

4.5.2 The fuzzy relation $\Delta_R(E)$ (rounding) is δ - T -transitive with $\delta \geq f^{[-1]}(f(a_i) - f(a_{i+1}))$

NOTE: The expression $\delta \geq f^{[-1]}(f(a_i) - f(a_{i+1}))$ includes the case when $\delta = 1$, i.e. E is fully T -transitive. This situation occurs, for example, when the fuzzy relation E is *MIN* transitive (Corollary 2.4).

Proof: 4.5.1 For a given tern x, y and z in X let i and j be such that $a_{i+1} < E(x, y) \leq a_i$ and $a_{j+1} < E(y, z) \leq a_j$. Then $\Delta_U(E(x, y)) = a_i$, $\Delta_U(E(y, z)) = a_j$ and

$$T(\Delta_U(E(y, z)), \Delta_U(E(x, y))) = a_{i+j}$$

On the other hand $a_{i+j+2} < T(E(x, y), E(y, z)) \leq a_{i+j}$. Since $T(E(x, y), E(y, z)) \leq E(x, z)$ (E is T -transitive), the inequality $a_{i+j+2} < E(x, z)$ also holds, and thus $a_{i+j+1} \leq \Delta_U(E(x, z))$.

Finally, $T(T(\Delta_U(E(y, z)), \Delta_U(E(x, y))), \delta_0) =$

$$T(a_{i+j}, \delta_0) = a_{i+j+1} \leq \Delta_U(E(x, z))$$

where $\delta_0 = f^{[-1]}(f(a_i) - f(a_{i+1}))$.

The proof of 4.5.2 involves a tedious case by case study of four different possible situations. We will skip that, as the ideas involved in there have already been shown in 4.5.1 ■

Proposition 4.5 provides an easy criterion to decide whether a finitely valued fuzzy relation can be obtained from a continuously valued T -transitive one. Finitely valued relations with a breach of transitivity heavier than the step of the mesh $\delta = f^{[-1]}(f(a_i) - f(a_{i+1}))$ are automatically discarded. The

question of whether this criterion is actually a characterization remains open, though.

However, when the strong transitivity degree is considered (definition 4.2), both upper truncation and rounding may end up in heavy loss of transitivity, even if precise, very fine scales are used. The problem is illustrated by the following example.

Example 4.6 (Upper truncation) Let us consider $X = [0, 1]$, $T = T_{LUK}$ the Lukasiewicz t-norm, E the fuzzy relation on X defined by $E(x, y) = 1 - |x - y|$, $L_N = \{a_N \dots a_0\}$ with

$$a_i = \frac{N-i}{N} \text{ and upper rounding, i.e. } \Delta(a) = \Delta_U(a) = a_i \text{ for all } a$$

in $(a_{i+1}, a_i]$, as the chosen discretization function. Then the transitive closure of $\Delta(E)$ is the relation $F = \overline{\Delta(E)} = 1$, and the strong transitivity degree is then 0.

Proof. The strong transitivity degree is (definition 4.2)

$$\mathcal{E}_T(\Delta(E), F) = \inf_{x, y \in X} E_T(\Delta(E(x, y)), F(x, y)) = \inf_{x, y \in X} E_T(\Delta(E(x, y)), 1) = 0$$

which accounts for the last part of the example.

To prove that $F = 1$ is the transitive closure of $\Delta(E)$, notice that for any pair x, y such that $E(x, y) > a_{N-1} = \frac{N-1}{N}$ its discretized value is $\Delta(E(x, y)) = a_N = 1$ and therefore $F(x, y) = 1$.

Now, if x and y are such that $E(x, y) \leq a_{N-1} = \frac{N-1}{N}$ with $x < y$, we may consider

$$x_0 = x, x_1 = x + \varepsilon, x_2 = x + 2\varepsilon \dots x_M = x + M\varepsilon$$

such that $E(x_i, x_{i+1}) = 1 - |x_i - x_{i+1}| = 1 - \varepsilon$, where

$$\frac{1}{N} > \varepsilon > 0 \text{ with } y < x_M = x + M\varepsilon. \text{ Since } F \text{ is } T\text{-transitive,}$$

$$F(x, y) \geq T(\dots T(T(F(x, x_1), F(x_1, x_2))) \dots F(x_{M-1}, y)) =$$

$$T(\dots T(1, 1) \dots 1) = 1 \text{ and thus } F(x, y) = 1 \text{ for all } x, y \text{ in } X.$$

■

The previous example can be easily adapted to the case of rounding by choosing an ε which is small enough in relation to the finite scale L_N .

More generally, the situation illustrated by the previous example occurs in all those situations where the elements of X may be linked to each other by chains of closely enough laid elements.

Theorem 4.7 Let T be an Archimedean t-norm, E a T -transitive relation on a set X and $L_N = \{a_N \dots a_0\} \subseteq [0, 1]$ an homogeneous finite scale. If for all x and y in X there is a chain $x = x_0, x_1, x_2 \dots x_M = y$ with $E(x_i, x_{i+1}) > \delta$ for all i , then $\overline{\Delta(E)} = 1$ ■

In that case, the strong transitivity degree may fall to 0.

V. CONCLUSIONS

Fuzzy transitive relations with respect to a continuous t-norm are a prominent class of fuzzy relations. They include fuzzy preorders and fuzzy equivalence relations, which are widely used in a variety of soft-computing areas ranging from approximate reasoning to clustering or pattern recognition.

From any practical purpose, discretization of the continuously valued models remains crucial. However, discretized T -transitivity fuzzy relations when T is an Archimedean t-norm do not always behave as expected. Only lower truncation discretization succeeds at preserving T -transitivity.

At the other end, MIN -transitivity is always preserved regardless the discretization strategy adopted, a fact that boosts the theoretical soundness of Similarity Relations, a class of fuzzy relations which is the choice of reference in many applied domains.

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