

Dualities and Isomorphisms between Indistinguishabilities and Related Concepts

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Abstract—Given a T -indistinguishability operator, the corresponding set H_E of extensional sets and the operators ϕ_E and ψ_E that provide upper and lower approximations of a fuzzy subset by extensional ones can be defined. Reciprocally, given any of these, the initial indistinguishability operator can be retrieved. In order to understand in depth how this relation works, it will be shown that there is an isomorphism between the corresponding lattices and how this isomorphism acts over the operations and orderings of the lattices will be studied.

Keywords: *indistinguishability operator; extensional set; upper approximation, lower approximation; transitive closure; extensional closure*

I. INTRODUCTION

The set H_E of extensional fuzzy subsets of an indistinguishability operator E has been widely studied in the literature. In [1] it has been studied from an algebraic point of view and in [2] its topological behaviour has been analyzed. It has also been characterized as the set of generators of E [3], the set of logical states and the set of fixed points of two operators known as ϕ_E and ψ_E [2]. These two operators give upper and lower approximations respectively of a fuzzy subset by extensional ones with respect to E . Although the results obtained in relation to H_E , ϕ_E and ψ_E show a very similar family of properties these concepts fulfill separately, up to this point a real study of the depth of this relation has not been done yet.

In [4] the operators that can be upper and lower approximations of an indistinguishability operators have been characterized and [1] did the same for a set of fuzzy subsets of a universe to be the extensional set of an indistinguishability operator providing as a consequence bijections between the set $\mathcal{E}$ of indistinguishability operators on a given universe X , the set $\mathcal{H}$ of sets of extensional sets with respect to indistinguishability operators on X and the sets $\mathcal{U}$ and $\mathcal{L}$ of upper and lower approximations of fuzzy subsets by extensional ones with respect to an indistinguishability operator on X . Taking this into account and also the fact that these concepts verify a very similar collection of properties, it seems natural to ask if

the connection between these sets is more than a simple bijection or if some kind of structure is kept.

This work has focused on this point and the results obtained have been that these bijections are actually isomorphisms, as the ordering and the structure given by the operations is kept. Furthermore, these sets can be regarded as lattices and, therefore, what we obtain is that all these lattices are isomorphic.

This fact has both a theoretical and practical relevance. It is theoretically interesting because this work provides some order and relation between the studies on indistinguishability operators, extensional sets, and upper and lower approximations that have been done up to this point. Furthermore, as all these concepts are isomorphic, the result shows that the only difference between them is the way in which are approached and the notation; this fact allowing us to translate all the results obtained on one lattice to the others through the isomorphisms presented on this paper.

From a practical point of view these isomorphisms are interesting too because several times multiple attributes are considered in a given problem and some kind of aggregation has to be done between them. This problem has been studied in [8] restricted to the aggregation of different indistinguishability operators. What if some attributes are more easily modelled by an indistinguishability operator, as they come from matching objects with a prototype, for instance, while other attributes are better seen as extensional sets because they come from direct observation? The isomorphisms we present in this paper allow us to codify different attributes into the concept that shows less difficulty to do so, and translate them into the same concept through these isomorphisms.

This work is structured in the following way:

In the next section of preliminaries the concepts related with the work are defined and the main properties and results related to them are given.

In Section III the main results of this work are proved. It is separated into three subsections that will show separately the isomorphisms between the sets $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$.

The isomorphisms will be explicitly given and how they work on the operations will be showed.

Finally, the main results of this work are summarized at the Conclusion.

II. PRELIMINARIES

In this section the basic definitions and results that underly the present work will be provided. Firstly the concepts related to indistinguishability operators will be given: the definition of an indistinguishability operator and the transitive closure of a reflexive and symmetric fuzzy relation. Furthermore, the definition and basic properties of the set H_E of extensional fuzzy subsets sets of an indistinguishability operator E and the upper and lower approximations (ϕ_E and ψ_E respectively) related to E will be given.

Definition II.1. Let T be a t -norm.

- The residuation $\overrightarrow{T}$ of T is defined for all $x, y \in [0, 1]$ by

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0, 1] | T(\alpha, x) \leq y\}.$$

- The birresiduation $\overleftarrow{T}$ of T is defined for all $x, y \in [0, 1]$ by

$$\overleftarrow{T}(x, y) = \min\{\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)\} = T(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

Definition II.2. Let T be a t -norm. A fuzzy relation E on a set X is a T -indistinguishability operator if and only if for all $x, y, z \in X$

- $E(x, x) = 1$ (Reflexivity)
- $E(x, y) = E(y, x)$ (Symmetry)
- $T(E(x, y), E(y, z)) \leq E(x, z)$ (T -transitivity).

It is well known that given two indistinguishability operators E, F , the intersection $E \cap F$ is also an indistinguishability operator. On the contrary, the union $E \cup F$ is not generally an indistinguishability operator. The concept of transitive closure will prove to be useful further to be able to use the union operator as well.

Definition II.3. Let T be a t -norm and R a fuzzy proximity (a reflexive and symmetric fuzzy relation) on a set X . The transitive closure $\overline{R}$ of R is the T -indistinguishability operator defined by

$$\overline{R} = \bigcap_{E \in A} E$$

where A is the set of T -indistinguishability operators on X greater than or equal to R .

Now we can give a lattice structure to the set $\mathcal{E}$ of T -indistinguishability operators on a set X , with the natural ordering $\leq$, and two operations: the intersection $E \cap F$ and the closed union $\overline{E \cup F}$.

Indistinguishability operators can be generated in several ways. The simplest way is by obtaining the indistinguishability related to a fuzzy subset.

Definition II.4. Let X be a set, T a t -norm and μ a fuzzy subset of X . The fuzzy relation E on X defined by

$$E_\mu(x, y) = \overleftarrow{T}(\mu(x), \mu(y))$$

for all $x, y \in X$ is a T -indistinguishability operator.

One of the main results of the theory of indistinguishability operators is the Representation Theorem. This theorem proves that any family of fuzzy subsets on a set X generates a T -indistinguishability operator E on X and, reciprocally, that every T -indistinguishability operator on X can be generated by a family of fuzzy subsets.

Theorem II.5. [12] *Representation Theorem.* Let R be a fuzzy relation on a set X and T a continuous t -norm. Then R is an T -indistinguishability operator if and only if there exists a family $(\mu_i)_{i \in I}$ of fuzzy subsets of X such that for all $x, y \in X$

$$R(x, y) = \inf_{i \in I} E_{\mu_i}(x, y).$$

An indistinguishability operator E determines a set of observable sets or granules. These are given by the generators of E in the sense of the Representation Theorem and it is well known that they coincide with the set of extensional sets we define now.

Definition II.6. Let X be a set and E a T -indistinguishability operator on X . A fuzzy subset μ of X is called extensional if and only if:

$$\forall x, y \in X \quad T(E(x, y), \mu(y)) \leq \mu(x).$$

We will denote H_E the set of all extensional fuzzy subsets of X with respect to E .

Proposition II.7. Let X be a set, E a T -indistinguishability operator on X and μ a fuzzy subset of X . Then:

$$\mu \in H_E \Leftrightarrow E_\mu \geq E$$

As it has been proved in [1], extensional fuzzy subsets verify the properties of the next proposition. Furthermore, these properties completely determine the set of extensional fuzzy subsets.

Proposition II.8. Let E be a T -indistinguishability operator and H_E its set of extensional fuzzy sets. Then, $\forall \mu \in H_E, (\mu_i)_{i \in I}$ a family of extensional sets and $\forall \alpha \in [0, 1]$ the following properties hold:

- 1) $\bigvee_{i \in I} \mu_i \in H_E$
- 2) $\bigwedge_{i \in I} \mu_i \in H_E$
- 3) $T(\alpha, \mu) \in H_E$
- 4) $\overrightarrow{T}(\mu|\alpha) \in H_E$
- 5) $\overleftarrow{T}(\alpha|\mu) \in H_E$

Given a fuzzy subset $\mu \in [0, 1]^X$, it is an interesting topic to find the closest extensional set that contains μ . This is done using the operator ϕ_E , defined below. This operator can be also seen as a topological closure operator on the set $[0, 1]^X$ [2].

Definition II.9. Let X be a set and E a T -indistinguishability operator on X . The map $\phi_E: [0, 1]^X \rightarrow [0, 1]^X$ is defined $\forall x \in X$ by:

$$\phi_E(\mu)(x) = \sup_{y \in X} T(E(x, y), \mu(y)).$$

The operator ϕ_E verifies the following properties.

Proposition II.10. For all $\mu, \mu' \in [0, 1]^X$ the following properties are satisfied:

- 1) $\mu \leq \mu' \Rightarrow \phi_E(\mu) \leq \phi_E(\mu')$
- 2) $\mu \leq \phi_E(\mu)$
- 3) $\phi_E(\mu \vee \mu') = \phi_E(\mu) \vee \phi_E(\mu')$
- 4) $\phi_E(\phi_E(\mu)) = \phi_E(\mu)$
- 5) $\phi_E(\{x\})(y) = \phi_E(\{y\})(x)$
- 6) $\phi_E(T(\alpha, \mu)) = T(\alpha, \phi_E(\mu))$

Analogously, given a T -indistinguishability operator E , the operator ψ_E can be defined that provides extensional lower approximations of fuzzy subsets of X with respect to E . The operator ψ_E is dual to ϕ_E and can be seen topologically as an opening or interior operator.

Definition II.11. Let E be a T -indistinguishability operator on a set X . The map $\psi_E: [0, 1]^X \rightarrow [0, 1]^X$ is defined $\forall x \in X$ by:

$$\psi_E(\mu)(x) = \inf_{y \in X} \overrightarrow{T}(E(x, y) | \mu(y)).$$

ψ_E satisfies the properties below (dual to Properties II.10) and is determined by them.

Proposition II.12. For all $\mu, \mu' \in [0, 1]^X$ the following properties are satisfied:

- 1) $\mu \leq \mu' \Rightarrow \psi_E(\mu) \leq \psi_E(\mu')$
- 2) $\psi_E(\mu) \leq \mu$
- 3) $\psi_E(\mu \wedge \mu') = \psi_E(\mu) \wedge \psi_E(\mu')$
- 4) $\psi_E(\psi_E(\mu)) = \psi_E(\mu)$
- 5) $\psi_E(\overrightarrow{T}(x | \alpha)(y)) = \psi_E(\overrightarrow{T}(y | \alpha)(x))$
- 6) $\psi_E(\overrightarrow{T}(\alpha | \mu)) = \overrightarrow{T}(\alpha, \psi_E(\mu))$

An interesting issue concerning the operators ϕ_E and ψ_E is that $Im(\phi_E) = Im(\psi_E) = H_E$, and hence the set of extensional fuzzy subsets H_E can also be characterized as the set of fixed points of ϕ_E and ψ_E . This fact is actually a direct consequence of the definitions and is very natural when ϕ_E and ψ_E is regarded from the topological approach.

Finally, we remind the concept of Galois connection between two ordered sets, as it will be used further to understand the relations between the concepts introduced previously:

Definition II.13. Let $(A, \leq), (B, \leq)$ two partially ordered sets. A Galois connection between these sets is a pair (F, G) of functions $F: A \rightarrow B$ and $G: B \rightarrow A$ satisfying:

$$\begin{aligned} F(a) \leq b &\Leftrightarrow a \leq G(b) \text{ (monotone Galois connection)} \\ b \leq F(a) &\Leftrightarrow a \leq G(b) \text{ (antitone Galois connection)} \end{aligned}$$

III. ISOMORPHISMS AND DUALITIES BETWEEN INDISTINGUISHABILITY OPERATORS, EXTENSIONAL SETS, UPPER AND LOWER APPROXIMATIONS

It is noticeable that the three concepts introduced (extensional sets, upper approximations and lower approximations) related to an indistinguishability operator E satisfy very similar families of properties. It is natural to ask then what is exactly the relation between these concepts.

In this section it will be shown that there actually exist bijections between the four stated concepts $(\mathcal{E}, \mathcal{H}, \mathcal{U}, \mathcal{L})$ and furthermore the structure of these is preserved through these bijections. This will lead us to the final fact that there are isomorphisms of lattices between the stated concepts, and these isomorphisms will be explicitly given.

This section is divided into three subsections. The first one will study the isomorphism between indistinguishability operators and extensional sets. The second one will face indistinguishability operators and upper approximations and the third one will deal with lower approximations.

A. Indistinguishability Operators and Extensional Sets

This section will focus on studying the isomorphism between the lattice $\mathcal{E}$ of indistinguishability operators on a set X and the lattice $\mathcal{H}$ of sets of extensional fuzzy subsets.

Analogously as in the case of indistinguishability operators, given H_E and H_F two collections of extensional sets, the intersection of them $H_E \cap H_F$ is also the set of extensional sets of a certain indistinguishability operator, but the union of them $H_E \cup H_F$ is not. In order to close the set of extensional sets under the union operator, and therefore give to it the structure of a lattice, we have to introduce, as done with indistinguishability operators, the concept of extensional closure of a subset of $[0, 1]^X$.

Definition III.1. Let $J \subseteq [0, 1]^X$. Its extensional closure $\overline{J}$ is defined by:

$$\overline{J} := \bigcap_{H \in [0, 1]^X, J \subseteq H} H$$

where H is a set of fuzzy subsets of X satisfying the properties of Proposition II.8:

- 1) $\bigvee_{i \in I} \mu_i \in H$
- 2) $\bigwedge_{i \in I} \mu_i \in H$
- 3) $\overrightarrow{T}(\alpha, \mu) \in H$
- 4) $\overrightarrow{T}(\alpha | \mu) \in H$
- 5) $\overrightarrow{T}(\mu | \alpha) \in H$

The lattice $\mathcal{E}$ of indistinguishability operators on a set X has a natural ordering given by $\leq$, while the lattice $\mathcal{H}$ of sets extensional fuzzy subsets is ordered by the inclusion $\subseteq$. The following proposition states how these orders are inverted when we pass from one lattice to the other one.

Proposition III.2. Let E, F be two indistinguishability operators on X with extensional sets H_E and H_F respectively. Then:

$$E \leq F \Leftrightarrow H_F \subseteq H_E$$

Proof: $E \leq F \Leftrightarrow \forall x, y \in X \ E(x, y) \leq F(x, y)$, which is equivalent to $\forall x, y \ \forall \mu \in H_F \ T(E(x, y), \mu(y)) \leq T(F(x, y), \mu(y)) \leq \mu(x) \Leftrightarrow H_F \subseteq H_E$ ■

The following theorem proves that there is a bijection between both lattices. This theorem was proved in [1].

Theorem III.3. *Let H be a subset of $[0, 1]^X$ satisfying the properties of Proposition II.8. Then there exists a unique T -indistinguishability operator E such that $H = H_E$.*

Proof: Let E be the indistinguishability operator generated by H using the Representation Theorem (Theorem II.5).

Let us proof first that $H = H_E$.

Since H is a generating family, its fuzzy subsets are extensional with respect to E and $H \subseteq H_E$.

Let us see the other inequality.

Given $\mu \in H_E$ and $y \in X$, we define the fuzzy subset $\mu_y(x) = T(E(x, y), \mu(y))$. We will show that $\mu = \sup_{y \in X} \mu_y$.

$$\begin{aligned} \mu_y(x) &= T(\inf_{\nu \in H} E_\nu(x, y), \mu(y)) \\ &= \inf_{\nu \in H} T(\min\{\vec{T}(\nu(x)|\nu(y)), \vec{T}(\nu(y)|\nu(x))\}, \mu(y)). \end{aligned}$$

Since $\mu(y)$ and $\nu(y)$ are constants, applying properties 2, 3, 4 and 5 of Proposition II.8 we have $\mu_y \in H$.

Since μ is extensional, $\mu_y(x) = T(E(x, y), \mu(y)) \leq \mu(x)$. As $\mu_y(y) = \mu(y)$ we obtain $\mu = \sup_{y \in X} \mu_y$ which belongs to H due to property 1 of Proposition II.8. So, $H_E \subseteq H$.

The uniqueness is a direct consequence of Proposition III.2. ■

Corollary III.4. $(\Gamma : E \rightarrow H_E, \Sigma : H_E \rightarrow E)$ define an antitone Galois connection between $\mathcal{E}$ and $\mathcal{H}$.

We have shown that there is a bijection between $\mathcal{E}$ and $\mathcal{H}$ and how the orderings of the lattices are affected by it. Let us finally show how the structure is preserved (inverting the order) and therefore we will have proved that the bijection is an isomorphism of lattices. In fact we will have proved that given an indistinguishability operator E on X , H_E is actually its dual space.

Theorem III.5. *Let E, F be two indistinguishability operators on a set X with sets of extensional fuzzy subsets H_E and H_F respectively. Then:*

- 1) $H_{\overline{E \cup F}} = H_E \cap H_F$
- 2) $H_{E \cap F} = \overline{H_E \cup H_F}$

Proof:

- 1) $\mu \in H_E \cap H_F \Leftrightarrow \mu \in H_E$ and $\mu \in H_F$. This is equivalent to $E_\mu \geq E$ and $E_\mu \geq F$ or $E_\mu \geq E \cup F$ and by the definition of the extensional closure, this is equivalent to $E_\mu \geq \overline{E \cup F} \Leftrightarrow \mu \in H_{\overline{E \cup F}}$
- 2) $\supseteq$
 $E \cap F \leq E \Rightarrow H_E \subseteq H_{E \cap F}$.
 $E \cap F \leq F \Rightarrow H_F \subseteq H_{E \cap F}$.
Then $H_E \cup H_F \subseteq H_{E \cap F} \Rightarrow \overline{H_E \cup H_F} \subseteq H_{E \cap F}$.

$\subseteq$

Let $\mu \in H_{E \cap F}$. Let us suppose $\mu \notin \overline{H_E \cup H_F}$. Then there exists an extensional set H_G with indistinguishability operator G such that $H_G \supseteq H_E \cup H_F$ and $\mu \notin H_G \Rightarrow \exists(x, y)$ such that $E_\mu(x, y) < G(x, y)$. Without loss of generality we can assume $E(x, y) \leq F(x, y)$. As $H_G \supseteq H_E \cup H_F$, $H_E \subseteq H_G$ and $H_F \subseteq H_G \Rightarrow G \leq E$ and $G \leq F$. So $E_\mu(x, y) < G(x, y) \leq E(x, y) \leq F(x, y)$. Hence $E_\mu(x, y) < (E \cap F)(x, y)$. Therefore, $\mu \notin H_{E \cap F}$ which contradicts our hypothesis!

So $\mu \in \overline{H_E \cup H_F}$. ■

B. Indistinguishability Operators and Upper Approximations

This subsection will be faced on proving that the lattice $\mathcal{U}$ is isomorphic to the lattice $\mathcal{E}$ of indistinguishability operators on a set X . The path that will take us to this result will be analogous to the one followed in the previous subsection.

As we did with $\mathcal{E}$ and $\mathcal{H}$, to give the set $\mathcal{U}$ of upper approximations ϕ_E ($E \in \mathcal{E}$) a lattice structure, we have to define a ϕ -closure.

Definition III.6. *Let f be a map defined on $[0, 1]^X \rightarrow [0, 1]^X$. The ϕ -closure $\bar{f}$ of f is defined as:*

$$\bar{f} = \bigwedge_{f \leq \phi, \phi \in U} \phi$$

where U is the set of maps $\phi : [0, 1]^X \rightarrow [0, 1]^X$ satisfying the properties of Proposition II.10:

- 1) $\mu \leq \mu' \Rightarrow \phi(\mu) \leq \phi(\mu')$
- 2) $\mu \leq \phi(\mu)$
- 3) $\phi(\mu \vee \mu') = \phi(\mu) \vee \phi(\mu')$
- 4) $\phi(\phi(\mu)) = \phi(\mu)$
- 5) $\phi(\{x\})(y) = \phi(\{y\})(x)$
- 6) $\phi(T(\alpha, \mu)) = T(\alpha, \phi(\mu))$

On the contrary with respect to what happened with extensional sets, in the case of upper approximations the ordering relation is directly preserved, instead of dually.

Proposition III.7. *Let E, F be two T -indistinguishability operators on X with upper approximations ϕ_E and ϕ_F respectively. Then:*

$$E \leq F \Leftrightarrow \phi_E \leq \phi_F$$

Given a ϕ operator, one can define an indistinguishability operator in the following way.

Proposition III.8. *Let $\phi : [0, 1]^X \rightarrow [0, 1]^X$ be a map satisfying the properties of Proposition II.10. Then the fuzzy relation E_ϕ defined by*

$$E_\phi(x, y) = \phi(\{x\})(y)$$

is a T -indistinguishability operator on X .

Proof: Straightforward ■

The following theorem proves that there is a bijection between indistinguishability operators and upper approximations.

Theorem III.9. *There is a bijection between the set $\mathcal{E}$ of T -indistinguishability operators on X and the maps $\phi : [0, 1]^X \rightarrow [0, 1]^X$ satisfying the properties of Proposition II.10.*

Proof: We have to prove that $\phi_{E_\phi} = \phi$ and $E_{\phi_E} = E$

- Let ϕ be a map satisfying the properties of Proposition II.10. For all $x, y \in X$ and for any fuzzy subset $\mu \in [0, 1]^X$ we have:

$$\begin{aligned}\phi_{E_\phi}(\mu)(x) &= \sup_{y \in X} T(E_\phi(x, y), \mu(y)) \\ &= \sup_{y \in X} T(\phi(\{y\})(x), \mu(y)) \\ &= \phi(\mu)(x).\end{aligned}$$

- Let E be a T -indistinguishability operator. Then for all $x, y \in X$ we have:

$$\begin{aligned}E_{\phi_E}(x, y) &= \phi(\{x\})(y) \\ &= \sup_{z \in X} T(E(y, z), \{x\}(z)) \\ &= E(x, y).\end{aligned}$$

Corollary III.10. $(\Gamma : E \rightarrow \phi_E, \Sigma : \phi_E \rightarrow E)$ define a monotone Galois connection between $\mathcal{E}$ and $\mathcal{U}$.

Proof: Trivial due to the monotonicity of a t-norm ■

We have construct a bijection between $\mathcal{E}$ and $\mathcal{U}$ and analyzed how the order was affected by it. Again, let us finish by studying how the structure of the operations is directly preserved and therefore by proving that the bijection is an isomorphism.

Theorem III.11. *Let E, F be two indistinguishability operators with upper approximations ϕ_E and ϕ_F respectively. Then:*

- 1) $\phi_{E \cup F} = \overline{\phi_E \vee \phi_F}$
- 2) $\phi_{E \cap F} = \phi_E \wedge \phi_F$

Proof:

- 1) $\geq$

Let $\mu \in [0, 1]^X$.

$$\phi_{E \cup F}(\mu)(x) = \sup_{y \in X} T(\overline{E \cup F}(x, y), \mu(y)).$$

As $\overline{E \cup F} \geq E, F \Rightarrow \phi_{E \cup F}(\mu)(x) \geq \phi_E(\mu)(x)$ and $\phi_{E \cup F}(\mu)(x) \geq \phi_F(\mu)(x)$. Hence:

$$\phi_{E \cup F}(\mu)(x) \geq (\phi_E \vee \phi_F)(\mu(x)) \Rightarrow \phi_{E \cup F} \geq \overline{\phi_E \vee \phi_F}.$$

$\leq$

By Definition III.6 and Proposition II.10, there exists a T -indistinguishability operator G on X such that $\overline{\phi_E \vee \phi_F} = \phi_G$. We have to prove that $\overline{E \cup F} \leq G$. Indeed, $\phi_E \leq \overline{\phi_E \vee \phi_F} = \phi_G$ and by Lemma III.7 $E \leq G$. Analogously we get $F \leq G$, and then $E \cup F \leq G \Rightarrow \overline{E \cup F} \leq G$, which proves the inequality.

- 2) $\phi_{E \cap F}(\mu)(x) = \sup_{y \in X} T(\min\{E(x, y), F(x, y)\}, \mu(y)) = \sup_{y \in X} \min\{T(E(x, y), \mu(y)), T(F(x, y), \mu(y))\}$. And, as the min and the sup operators are interchangeable in the previous expression, this is equal to $(\phi_E \wedge \phi_F)(\mu)(x)$ ■

C. Indistinguishability Operators and Lower Approximations

This subsection will deal with indistinguishability operators and lower approximations. The final result of the subsection will be, analogously as the previous subsections, the construction of the isomorphism between both lattices.

Firstly, to give the set $\mathcal{L}$ of lower approximations a lattice structure we have to introduce the ψ -closure of a map.

Definition III.12. *Let f be a map defined on $[0, 1]^X \rightarrow [0, 1]^X$. The ψ -closure $\tilde{f}$ of f is defined as:*

$$\tilde{f} = \bigwedge_{\psi \in L, f \leq \psi} \psi$$

where L is the set of maps $\psi : [0, 1]^X \rightarrow [0, 1]^X$ satisfying the properties of Proposition II.12:

- 1) $\mu \leq \mu' \Rightarrow \psi_E(\mu) \leq \psi_E(\mu')$
- 2) $\psi_E(\mu) \leq \mu$
- 3) $\psi_E(\mu \wedge \mu') = \psi_E(\mu) \wedge \psi_E(\mu')$
- 4) $\psi_E(\psi_E(\mu)) = \psi_E(\mu)$
- 5) $\psi_E(\overrightarrow{T}(x|\alpha)(y)) = \psi_E(\overrightarrow{T}(y|\alpha)(x))$
- 6) $\psi_E(\overrightarrow{T}(\alpha|\mu)) = \overrightarrow{T}(\alpha, \psi_E(\mu))$

This way, the lattice structure is obtained with the ordering $\leq$, and the operations $\wedge$ and $\tilde{\vee}$.

Let us prove now that the ordering of the lattices is directly preserved when we pass from one to the other.

Proposition III.13. *Let E, F be two indistinguishability operators on X with upper approximations ψ_E and ψ_F respectively. Then:*

$$E \leq F \Leftrightarrow \psi_E \leq \psi_F.$$

Analogously to the case of upper approximations, an indistinguishability operator can be defined given a lower approximation ψ .

Proposition III.14. *Let $\psi : [0, 1]^X \rightarrow [0, 1]^X$ be a map satisfying the properties of Proposition II.12. Then the fuzzy relation E_ψ defined by*

$$E_\psi(x, y) = \inf_{\alpha \in [0, 1]} \overrightarrow{T}(\psi(\overrightarrow{T}(\{x\}|\alpha)(y)|\alpha)).$$

is a T -indistinguishability operator on X .

From this proposition, it can be proved that there is a bijection between the lattices $\mathcal{E}$ and $\mathcal{L}$.

Theorem III.15. [4] *There is a bijection between the set of T -indistinguishability operators and the maps ψ satisfying the properties of Proposition II.12.*

Corollary III.16. $(\Gamma : E \rightarrow \psi_E, \Sigma : \psi_E \rightarrow E)$ define a monotone Galois connection between $\mathcal{E}$ and $\mathcal{L}$.

Proof: Trivial by applying the properties of the residuation. ■

We finally have to see that the stated bijection between the lattice of indistinguishability operators on X and the lattice of lower approximations on X preserves the structure given by $\wedge$ and ∇ . This is proved in the following Theorem. With this result, we have the stated isomorphism between both lattices.

Theorem III.17. *Let E, F be two T -indistinguishability operators on X with lower approximations ψ_E and ψ_F respectively. Then:*

- 1) $\psi_{\overline{E \cup F}} = \psi_E \vee \psi_F$
- 2) $\psi_{E \cap F} = \phi_E \wedge \phi_F$

Proof:

- 1) $\geq$

$$E \leq \overline{E \cup F} \Rightarrow \psi_E \leq \psi_{\overline{E \cup F}}$$

$$F \leq \overline{E \cup F} \Rightarrow \psi_F \leq \psi_{\overline{E \cup F}}$$

$$\text{Therefore, } \psi_E \vee \psi_F \leq \psi_{\overline{E \cup F}} \Rightarrow \widetilde{\psi_E \vee \psi_F} \leq \psi_{\overline{E \cup F}} \leq \psi_{\overline{E \cup F}}$$

Let us assume that there exist $x \in X$ and a fuzzy subset μ of X such that:

$\psi_{\overline{E \cup F}}(\mu)(x) > \psi_E \vee \psi_F(\mu)(x)$. Then there exists a T -indistinguishability operator G on X such that $\psi_E \vee \psi_F \leq \psi_G$ and $\psi_G(\mu)(x) < \psi_{\overline{E \cup F}}(\mu)(x)$.

But $\psi_E \leq \psi_E \vee \psi_F \leq \psi_G$. By Lemma III.13 we have $E \leq G$. Analogously $F \leq G$. So $E \cup F \leq G \Rightarrow \overline{E \cup F} \leq G$, which contradicts $\psi_G(\mu)(x) < \psi_{\overline{E \cup F}}(\mu)(x)$. So we have $\psi_{\overline{E \cup F}} \leq \psi_E \vee \psi_F$.

- 2) $(\psi_E \wedge \psi_F)(\mu)(x) =$
 $\min\{\inf_{y \in X} \overrightarrow{T}((E(x, y)|\mu(x)),$
 $\inf_{y \in X} \overrightarrow{T}((F(x, y)|\mu(x)))\}$
 $= \inf_{y \in X} \overrightarrow{T}(\min\{E(x, y), F(x, y)\}|\mu(x)) =$
 $\psi_{E \cap F}(\mu)(x).$

and the isomorphisms have been given explicitly. These results are summarized below.

$$\begin{array}{ccccccc} \mathcal{L} & \approx & \mathcal{U} & \approx & \mathcal{E} & \approx & \mathcal{H} \\ \psi_E \wedge \psi_F & \leftrightarrow & \phi_E \wedge \phi_F & \leftrightarrow & E \cap F & \leftrightarrow & \overline{H_E \cup H_F} \\ \widetilde{\psi_E \vee \psi_F} & \leftrightarrow & \overline{\phi_E \vee \phi_F} & \leftrightarrow & \overline{E \cup F} & \leftrightarrow & H_E \cap H_F \end{array}$$

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REFERENCES

- [1] Castro, J.L., Klawonn, F.: "Similarity in Fuzzy Reasoning," *Mathware & Soft Computing*, vol. 2, 1996, pp. 197-228.
- [2] Jacas, J., Recasens, J.: "Fuzzy T-transitive relations: eigenvectors and generators," *Fuzzy Sets and Systems*, vol. 72, 1995, pp. 147-154.
- [3] Jacas, J., Recasens, J.: "Fixed points and generators of fuzzy relations," *J. Math. Anal. Appl.*, vol. 186, 1994, pp. 21-29.
- [4] Morsi, N.N., Yakout, M.M.: "Axiomatics for fuzzy rough sets," *Fuzzy Sets and Systems*, vol. 100, 1998, pp. 327-342.
- [5] Boixader, D., Jacas, J., Recasens, J.: "Upper and lower approximation of fuzzy sets," *Int. J. of General Systems*, vol. 29, 2000, pp. 555-568.
- [6] Bělohlávek, R.: *Fuzzy Relational Systems: Foundations and Principles*. Kluwer Academic Publishers, New York, 2002.
- [7] Boixader, D., Jacas, J., Recasens, J.: Fuzzy equivalence relations: advanced material. In: Dubois, D., Prade, H. (eds.) *Fundamentals of Fuzzy Sets*, pp. 261-290. Kluwer Academic Publishers, New York, 2000.
- [8] Jacas, J., Recasens, J.: "Aggregation of T-Transitive Relations," *Int J. of Intelligent Systems*, vol. 18, 2003, pp. 1193-1214.
- [9] Klement, E.P., Mesiar, R., Pap E.: *Triangular norms*. Kluwer Academic Publishers, Dordrecht, 2000.
- [10] Recasens, J. *Indistinguishability Operators. Modelling Fuzzy Equalities and Fuzzy Equivalence Relations Series: Studies in Fuzziness and Soft Computing*, vol. 260., 2011.
- [11] Trillas, E.: "Assaig sobre les relacions d'indistingibilitat," *Proc. Primer Congrés Català de Lògica Matemàtica*, Barcelona, 1982, pp. 51-59. (In Catalan).
- [12] Valverde, L.: "On the Structure of F-indistinguishability Operators," *Fuzzy Sets and Systems*, vol. 17, 1985, pp. 313-328.
- [13] Zadeh, L.A. "Similarity relations and fuzzy orderings," *Inform. Sci.*, vol. 3, 1971, pp. 177-200.
- [14] Zadeh, L.A.: Fuzzy sets and information granularity. In: Gupta, M.M., Ragade, R.K., Yager, R.R. (eds.) *Advances in Fuzzy Set Theory and Applications*, pp. 3-18. Amsterdam, North-Holland, 1979.
- [15] Zadeh, L.A.: "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic," *Fuzzy Sets and Systems*, vol. 90, 1997, pp. 111-127.

IV. CONCLUSION

In this work, the relation between the set $\mathcal{E}$ of T -indistinguishability operators on a set X , the set $\mathcal{H}$ of sets of extensional fuzzy subsets with respect to a T -indistinguishability operator E on X and the sets $\mathcal{U}$ and $\mathcal{L}$ of upper and lower approximations of fuzzy subsets by extensional ones with respect to a T -indistinguishability operator respectively has been studied and stated. It has been proved that these sets are in fact isomorphic lattices