

Optimal Necessary Conditions for General SISO Mamdani Fuzzy Systems as Function Approximators within a Given Accuracy

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Abstract — In this paper, necessary conditions are investigated for a single input/single output (SISO) Mamdani fuzzy systems as function approximators of continuous functions within a given accuracy. Since general SISO Mamdani fuzzy systems are monotonic on subintervals, the optimal configuration of fuzzy systems is that the number of division points is at least the times of its monotonicity changes. Thus with the extreme of the desired continuous function, necessary conditions are obtained through generating intervals that contain division points and pruning redundant intervals. Furthermore, a dynamically constructive method is proposed to show the conditions are optimal. It has been shown that existing results concerning necessary conditions are only special cases of our results. Finally, simulation examples are given to illustrate the conclusions, the strength of the fuzzy systems as function approximators are analyzed.

Keywords— fuzzy systems; necessary conditions; fuzzy rules; approximation accuracy

I. INTRODUCTION

In the past decades, fuzzy systems were widely used and extensively studied for a variety of applications, especially in control applications because of their excellent performance in representing nonlinear functions in an intuitive and informative manner. In many applications of fuzzy theory, fuzzy systems involved can be viewed as rule-based function approximators. Many scholars have proved that fuzzy systems are universal approximators in different aspects [1-5]. To our knowledge, the smaller the approximation accuracy, the more fuzzy rules are required in order to approximate a certain continuous function, especially in high dimensions. Large numbers of rules render difficulty in designing fuzzy systems, such as high storage consumption, increased computational complexity, etc.

One always wants to construct a fuzzy system using less fuzzy rules to approximate a continuous function within a given accuracy. Therefore, many efforts have been made to reduce fuzzy rules. Some methods of these include orthogonal transformation, singular value decomposition[6-8] and heuristic methods such as genetic algorithms[9-10], clustering methods [11,12], and tree search[13], etc. Sufficient conditions for fuzzy systems have also been defined and solved by many scholars in term of the function approximation, which are mainly based on the uniform approximate polynomial [14-16] and the derivative

of the object function[17-18, 23]. However, from a practical viewpoint, necessary conditions for fuzzy systems are more important since they place restrictions on selection components for fuzzy systems. Despite its importance, only a few papers [19-22] have dealt with this problem. Generally, the number of the fuzzy rules needed increases with the number of extreme of the desired function. However, the existing conclusions are only available for the fuzzy system with perfect approximation to a continuous function. In most cases of fuzzy system applications, the core part is to design a fuzzy system to approximate a prospective continuous function within certain accuracy. As a result, necessary conditions for fuzzy systems as function approximators within a given accuracy deserve deep study.

In this paper, we will establish new necessary conditions for general SISO Mamdani fuzzy systems as function approximator within a prescribed accuracy. By generating the intervals that contain division points and pruning redundant intervals, a novel constructive algorithm is proposed to obtain the necessary conditions. Furthermore, it has been shown that the conditions are optimal and the existing results given in [19] are only a special case of ours. Finally, illustrative examples are used to demonstrate the strength and limitation of the general SISO Mamdani fuzzy systems.

II. PROBLEM FORMULATION

A. Configuration of general SISO Mamdani fuzzy systems

The interval $[a, b]$, on which the object function $f(x)$ is defined, is divided into N subintervals (S_i is endpoint of subintervals):

$$a = S_0 < S_1 < S_2 < \dots < S_N = b$$

Then there are $N + 1$ normal, consistent and continuous input fuzzy sets defined on $[a, b]$ to fuzzify the input variable (Fig. 1). Each input fuzzy set denoted by A_i ($0 \leq i \leq N$) has a membership function $\mu_i(x)$ defined as follows:

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respectively. By derivative's property, there must exist division points on $[S_j, S_{j+1}]$, which contradicted with Lemma 1. The proof is completed.

Lemma 2 has established some necessary conditions for Mamdani fuzzy systems as function approximators within a given approximation accuracy. In order to construct the necessary conditions in an intuitive manner, the construction algorithm is presented as follows, which consists of generating the intervals that contain division points and pruning redundant intervals.

Construction algorithm:

Step 1: Calculate the object function's extreme set ξ and create an empty set P .

Step 2: Traverse the extreme set ξ ,

WHILE there exist $\xi_p, \xi_e, \xi_q \in \xi$ with $\xi_p < \xi_e < \xi_q$ such that

$$\begin{aligned} f(\xi_p) - f(\xi_e) > 2\varepsilon, f(\xi_e) - f(\xi_q) < -2\varepsilon \\ \text{or } f(\xi_p) - f(\xi_e) < -2\varepsilon, f(\xi_e) - f(\xi_q) > 2\varepsilon \end{aligned}$$

DO $P = P \cup \{[\xi_p, \xi_q]\}$.

Step 3: $Q = P$,

WHILE there exists $C_1, C_2 \in Q$, such that $C_1 \subset C_2$.

DO remove C_2 from Q .

Step 4: The algorithm ends and the characteristic set Q is got.

The characteristic set Q determined by construction algorithm above has the following properties:

- (i) Given the extreme set ξ , Q is unique.
- (ii) There does not exist $C_j \in Q$ such that $P_i \subset C_j$ for arbitrary $P_i \in P$.
- (iii) There also does not exist $C_1, C_2 \in Q$ such that $C_1 \subset C_2$.

To analyze the characteristics of the elements in set Q , the following definitions are given.

Definition 1: For an interval $C_i = [\xi_j, \xi_k]$, let $C_i^{\min} = \xi_j$, $C_i^{\max} = \xi_k$.

Definition 2 (interval correlation): For arbitrary $C_1, C_2 \in Q$, C_1 and C_2 are said to be correlative if there exist extremes ξ_1 and ξ_2 with $\xi_1 < \xi_2$ such that $\xi_1, \xi_2 \in C_1$, $\xi_1, \xi_2 \in C_2$,

Definition 3 (interval consistency): For arbitrary $C_1, C_2 \in Q$, C_1 and C_2 are said to be consistent if there exist extremes $\xi_1 \in C_1$ and $\xi_2 \in C_2$ such that

$$f(C_1^{\min}) - f(\xi_1) > 2\varepsilon, f(C_1^{\max}) - f(\xi_1) > 2\varepsilon$$

$$\begin{aligned} f(C_2^{\min}) - f(\xi_2) > 2\varepsilon, f(C_2^{\max}) - f(\xi_2) > 2\varepsilon \\ \text{or } f(\xi_1) - f(C_1^{\min}) > 2\varepsilon, f(\xi_1) - f(C_1^{\max}) > 2\varepsilon \\ f(\xi_2) - f(C_2^{\min}) > 2\varepsilon, f(\xi_2) - f(C_2^{\max}) > 2\varepsilon; \end{aligned}$$

or else inconsistent.

Definition 4: For arbitrary $C_1, C_2 \in Q$, it is said $C_1 < C_2$ if $C_1^{\min} < C_2^{\min}$.

According to the construction algorithm, the following four corollaries could be obtained easily.

Corollary 1: For arbitrary $C_1, C_2 \in Q$, C_1 and C_2 could only be consistent or inconsistent exclusively.

Corollary 2: $C_1^{\min} \neq C_2^{\min}$, $C_1^{\max} \neq C_2^{\max}$ for arbitrary $C_1, C_2 \in Q$.

Corollary 3: For arbitrary $C_1, C_2 \in Q$, $C_1^{\max} < C_2^{\max}$ if $C_1^{\min} < C_2^{\min}$.

Corollary 4: For the elements $C_1, C_2, \dots, C_n$ in Q , there exists a new sorting $C_{i_1}, C_{i_2}, \dots, C_{i_n}$, such that $C_{i_1} < C_{i_2} < \dots < C_{i_n}$

$$C_{i_1} < C_{i_2} < \dots < C_{i_n} \quad (12)$$

In the following, relationship among elements in Q would be analyzed using reduction to absurdity.

Lemma 3: For arbitrary $C_1, C_2 \in Q$ with $C_1 < C_2$, C_1 and C_2 are not correlative if C_1 and C_2 are consistent.

Proof: Suppose C_1 and C_2 are correlative, we have $C_1^{\min} < C_2^{\min} < C_1^{\max} < C_2^{\max}$ by Corollary 3. Without losing generality, there exist extremes $\xi_1 \in C_1$, $\xi_2 \in C_2$ such that

$$\begin{aligned} f(C_1^{\min}) - f(\xi_1) > 2\varepsilon, f(C_1^{\max}) - f(\xi_1) > 2\varepsilon; \\ f(C_2^{\min}) - f(\xi_2) > 2\varepsilon, f(C_2^{\max}) - f(\xi_2) > 2\varepsilon \end{aligned} \quad (13)$$

Let $\xi_e = \arg_{\xi_i} \min\{f(\xi_1), f(\xi_2)\}$. If $\xi_e \in [C_1^{\min}, C_2^{\min}]$, we have

$$f(C_1^{\min}) - f(\xi_e) > 2\varepsilon, f(C_2^{\min}) - f(\xi_e) > 2\varepsilon.$$

Thus $[C_1^{\min}, C_2^{\min}] \in P$ is obtained. This contradicted with $[C_1^{\min}, C_2^{\min}] \subset C_1$ and $C_1 \in Q$ based on property (ii). Assuming $\xi_e \in [C_2^{\min}, C_1^{\max}]$ or $\xi_e \in [C_1^{\max}, C_2^{\max}]$ yields a similar contradiction. Therefore C_1 and C_2 are not correlative.

Lemma 4: For arbitrary $C_1 < C_2 \in Q$, if C_1 and C_2 are correlative,

- (1) C_1 and C_2 are inconsistent.
- (2) Extremes ξ_1 and ξ_2 defined in Definition 3 satisfy $C_1^{\min} < \xi_1 \leq C_2^{\min} < C_1^{\max} \leq \xi_2 < C_2^{\max}$.
- (3) There exist at least two division points in $[C_1^{\min}, C_2^{\max}]$ to achieve the approximation accuracy requirement in Eq. (4).

Proof: Conclusion (1) is obvious by Lemma 3. We now use a contradiction argument to show the other conclusions are also satisfied. Since C_1 and C_2 are correlative and $C_1 < C_2$, we can easily get

$$C_1^{\min} < C_2^{\min} < C_1^{\max} < C_2^{\max} \quad (14)$$

according to Corollary 3, Without losing generality, there is extreme $\xi_1 \in C_1$ such that

$$f(C_1^{\min}) - f(\xi_1) > 2\varepsilon, f(C_1^{\max}) - f(\xi_1) > 2\varepsilon \quad (15)$$

Because C_1 and C_2 are inconsistent, there exists $\xi_2 \in C_2$ such that

$$f(\xi_2) - f(C_2^{\min}) > 2\varepsilon, f(\xi_2) - f(C_2^{\max}) > 2\varepsilon \quad (16)$$

Assuming $C_2^{\min} < \xi_1 < C_1^{\max}$ and $\xi_1 < \xi_2$, we have

$$\begin{aligned} \max\{f(C_1^{\max}), f(\xi_2)\} - f(\xi_1) &> 2\varepsilon, \\ \max\{f(C_1^{\min}), f(\xi_2)\} - f(C_2^{\max}) &> 2\varepsilon \end{aligned} \quad (17)$$

Thus $[\xi_1, C_2^{\min}] \in P$ is obtained. This contradicted with $[\xi_1, C_2^{\min}] \subset C_2$ and $C_2 \in Q$ based on property (ii). Assuming $C_2^{\min} < \xi_2 < C_1^{\max}$ yields a similar contradiction. As a result, the conclusion (2) is satisfied.

Subsequently, let us analyze the conclusion (3). Without losing generality, there exist extremes $\xi_1 \in C_1$, $\xi_2 \in C_2$ such that

$$\begin{aligned} f(C_1^{\min}) - f(\xi_1) &> 2\varepsilon, f(C_1^{\max}) - f(\xi_1) > 2\varepsilon \\ f(\xi_2) - f(C_2^{\min}) &> 2\varepsilon, f(\xi_2) - f(C_2^{\max}) > 2\varepsilon \end{aligned} \quad (18)$$

By conclusion (2), we have

$$\begin{aligned} f(C_1^{\min}) - \min\{f(\xi_1), f(C_2^{\min})\} &> 2\varepsilon, \\ \min\{f(\xi_1), f(C_2^{\min})\} - \max\{f(C_1^{\max}), f(\xi_2)\} &< -2\varepsilon, \\ \max\{f(C_1^{\max}), f(\xi_2)\} - f(C_2^{\max}) &> 2\varepsilon. \end{aligned}$$

As a result, there exists at least two division points in $[C_1^{\min}, C_2^{\max}]$ according to Lemma 2.

From Lemma 4, we can conclude that two correlative intervals in Q could not have a common division point. Furthermore, the following Lemma which will help founding the relationship between the number of the elements in Q and the number of the division points for fuzzy systems could be obtained.

Lemma 5: Each element in Q determined by the construction algorithm has division point of its own if the fuzzy system $F(x)$ defined in (3) can approximate $f(x)$ within a given approximation accuracy ε .

Proof: For arbitrary $C_i \in Q$, if C_i is not correlative with the other element in Q , there exists division point in C_i according to Lemma 2. If C_i is correlative with $C_k \in Q$, C_i and C_k have their own division points by Lemma 4. Hence each element in Q determined by the proposed construction algorithm has division point of its own.

Suppose that the objective function $f(x)$ is defined on the interval $[a, b]$ in the framework of SISO Mamdani fuzzy system. If there are n elements in Q which is determined by the construction algorithm with respect to the function $f(x)$ above, n division points exist in the interval $[a, b]$ based on Lemma 5.

Furthermore, the interval is divided into at least $n+1$ subintervals, and at least $n+2$ fuzzy rules are needed to approximate the object function within a given approximation accuracy ε . Thus the necessary conditions for general SISO Mamdani fuzzy systems could be established intuitively.

Theorem 1: (Necessary conditions for fuzzy systems) Suppose there are n elements in Q determined by the construction algorithm. In order to approximate the object function within a given approximation accuracy, the input interval of the fuzzy system should be divided into at least $n+1$ subintervals; at least $n+2$ fuzzy rules are needed.

We will prove that the conditions established in Theorem 1 are also optimal.

Lemma 6: If $f(x_1) - f(x_2) \geq -2\varepsilon$ or $f(x_1) - f(x_2) \leq 2\varepsilon$ for arbitrary $x_1, x_2 \in [S_j, S_{j+1}]$ with $x_1 < x_2$, there exists a fuzzy system $F(x)$ in Eq.(4) which can approximate the object function $f(x)$ within a given approximation accuracy ε on subinterval $[S_j, S_{j+1}]$.

Proof: A constructive method will be used here to prove this lemma.

(a) If $f(x_1) - f(x_2) \geq -2\varepsilon$ for arbitrary $x_1, x_2 \in [S_j, S_{j+1}]$ with $x_1 < x_2$, the output $F(x)$ of a fuzzy system that can approximate $f(x)$ within a given accuracy ε on $[S_j, S_{j+1}]$ can be constructed as Fig. 2. And the output $F(x)$ of a fuzzy system can be constructed in details as follows.

Step 1 Calculate the extremes $\xi_0, \xi_1, \dots, \xi_{n+1}$ of the object function $f(x)$ on $[S_j, S_{j+1}]$ with $\xi_0 = S_j, \xi_{n+1} = S_{j+1}$.

Let $i = 0$, $e = \xi_0$.

Step 2 WHILE $i < n$ DO

$$F(x) = \min(f(x) + \varepsilon, f(e) + \varepsilon), x \in [\xi_i, \xi_{i+1}].$$

If $f(e) - f(\xi_{i+1}) \geq 0$

$$e = \xi_{i+1}.$$

$$i = i + 1 .$$

Step 3 If $f(\xi_n) - f(\xi_{n+1}) \geq 0$, on $[\xi_n, \xi_{n+1}]$

$$F(x) = f(x) - \varepsilon + \frac{\xi_{n+1} - x}{\xi_{n+1} - \xi_n} (f(e) - f(\xi_n) + 2\varepsilon) ,$$

else

$$F(x) = f(e) + \varepsilon + \frac{x - \xi_n}{\xi_{n+1} - \xi_n} (f(\xi_{n+1}) - f(e) - 2\varepsilon) .$$

(b) If $f(x_1) - f(x_2) \leq 2\varepsilon$ for arbitrary $x_1, x_2 \in [S_j, S_{j+1}]$ with $x_1 < x_2$. The corresponding fuzzy system output $F(x)$ on $[S_j, S_{j+1}]$ can be constructed in a similar way:

Step 1. Calculate the extremes $\xi_0, \xi_1, \dots, \xi_{n+1}$ of the object function $f(x)$ on $[S_j, S_{j+1}]$ with $\xi_0 = S_j, \xi_{n+1} = S_{j+1}$.

Let $i = 0, e = \xi_0$.

Step 2. WHILE $i < n$ DO

$$F(x) = \max(f(x) - \varepsilon, f(e) - \varepsilon) \text{ on } x \in [\xi_i, \xi_{i+1}] .$$

If $f(e) - f(\xi_{i+1}) \leq 0$

$$e = \xi_{i+1} .$$

$$i = i + 1 .$$

Step 3. If $f(\xi_n) - f(\xi_{n+1}) \leq 0$, on $[\xi_n, \xi_{n+1}]$,

$$F(x) = f(x) + \varepsilon + \frac{\xi_{n+1} - x}{\xi_{n+1} - \xi_n} (f(e) - f(\xi_n) - 2\varepsilon) .$$

else

$$F(x) = f(e) - \varepsilon + \frac{x - \xi_n}{\xi_{n+1} - \xi_n} (f(\xi_{n+1}) - f(e) + 2\varepsilon) .$$

Furthermore, the input/output fuzzy sets can be constructed as: $\mu_1(x) = (\sup F - F(x)) / (\sup F - \inf F)$, $\mu_2(x) = 1 - \mu_1(x)$, $y_1 = F(S_j)$ and $y_2 = F(S_{j+1})$. The proof is completed.

Remark 1: The constructive methods in Lemma 6 just show that there exists fuzzy system that can approximate the object function within a given approximation accuracy in Eq. (4). Since the output constructed above in the first case is $f(S_j) - \varepsilon$ at $x = S_j$ and decreases monotonically to $f(S_{j+1}) + \varepsilon$ at $x = S_{j+1}$, while the second output constructed above is $f(S_j) + \varepsilon$ at $x = S_j$ and increases monotonically to $f(S_{j+1}) - \varepsilon$ at $x = S_{j+1}$, this two cases can be used alternately on subintervals to form a continuous output on the whole domain.

Lemma 7: Arbitrary two adjacent $C_{i_j}, C_{i_{j+1}} \in Q$ ($1 \leq j < n$) in Corollary 4 are inconsistent.

Proof: If C_{i_j} and $C_{i_{j+1}}$ are correlative, C_{i_j} and $C_{i_{j+1}}$ are inconsistent according to Lemma 5. Otherwise, we have

$$C_{i_j}^{\max} \leq C_{i_{j+1}}^{\max} \quad (19)$$

If C_{i_j} and $C_{i_{j+1}}$ are consistent. Without losing generality, assume there exist extremes $\xi_1 \in C_{i_j}$, $\xi_2 \in C_{i_{j+1}}$ such that

$$\begin{aligned} f(C_{i_j}^{\max}) - f(\xi_1) &> 2\varepsilon, f(C_{i_j}^{\min}) - f(\xi_1) > 2\varepsilon \\ f(C_{i_{j+1}}^{\max}) - f(\xi_2) &> 2\varepsilon, f(C_{i_{j+1}}^{\min}) - f(\xi_2) > 2\varepsilon \end{aligned} \quad (20)$$

With (19) and (20),

$$\begin{aligned} \max\{f(C_{i_j}^{\max}), f(C_{i_{j+1}}^{\min})\} - f(\xi_1) &> 2\varepsilon, \\ \max\{f(C_{i_j}^{\max}), f(C_{i_{j+1}}^{\min})\} - f(\xi_2) &> 2\varepsilon. \end{aligned}$$

Thus $[\xi_1, \xi_2] \in P$ is obtained. It means there exists $C_s \in Q$ and $C_s \subseteq [\xi_1, \xi_2]$ such that $C_{i_j} < C_s < C_{i_{j+1}}$. It contradicted with Corollary 4. Therefore C_{i_j} and $C_{i_{j+1}}$ are inconsistent.

With Lemma 6 and Lemma 7, the optimal necessary condition for SISO Mamdani fuzzy systems is established as follows.

Theorem 2 (Optimal necessary conditions for fuzzy systems): To approximate a continuous function within a given accuracy, a general SISO Mamdani fuzzy system established on the number n of the characteristic set Q is optimal.

Proof: Let $C_j \in Q$ ($j = 1, \dots, n$), where Q is determined by the construction algorithm with respect to the function $f(x)$ defined on $[a, b]$. n division points exist in the interval $[a, b]$ by Lemma 5. Without losing generality, we select all the division points $S_j \in C_j$ ($j = 1, \dots, n$) at extremes of the function $f(x)$. And S_j satisfies

$$\begin{aligned} f(C_j^{\min}) - f(S_j) &> 2\varepsilon, f(C_j^{\max}) - f(S_j) > 2\varepsilon, \\ \text{or } f(S_j) - f(C_j^{\min}) &> 2\varepsilon, f(S_j) - f(C_j^{\max}) > 2\varepsilon. \end{aligned}$$

Suppose that the approximation accuracy ε can't be achieved on subinterval $[S_j, S_{j+1}]$. With Lemma 6, there exist extremes of $f(x)$ ξ_1, ξ_2, ξ_3 on $[S_j, S_{j+1}]$ with $S_j \leq \xi_1 < \xi_2 < \xi_3 \leq S_{j+1}$ such that

$$\begin{aligned} f(\xi_1) - f(\xi_2) &> 2\varepsilon, f(\xi_2) - f(\xi_3) < -2\varepsilon \\ \text{or } f(\xi_1) - f(\xi_2) &< -2\varepsilon, f(\xi_2) - f(\xi_3) > 2\varepsilon \end{aligned} \quad (21)$$

Thus $[\xi_1, \xi_3] \in P$ is obtained. It means there exists $C_i \in Q$, $C_i \subseteq [\xi_1, \xi_3]$ such that $C_j < C_i < C_{j+1}$. This contradicted with Corollary 4. Therefore the fuzzy system can achieve approximation accuracy on each subinterval. In other words, the inequality $f(x_1) - f(x_2) \geq -2\varepsilon$ or $f(x_1) - f(x_2) \leq 2\varepsilon$ holds for arbitrary $x_1, x_2 \in [S_j, S_{j+1}]$ with $x_1 < x_2$.

According to Lemma 7, if there exist $x_1, x_2 \in [S_j, S_{j+1}]$ with $x_1 < x_2$ such that $f(x_1) - f(x_2) > 2\varepsilon$, there must exist $x_3, x_4 \in [S_j, S_{j+1}]$ with $x_3 < x_4$ such that $f(x_3) - f(x_4) < -2\varepsilon$.

Hence, for arbitrary two adjacent subintervals $[S_{j-1}, S_j]$ and $[S_j, S_{j+1}]$, if $f(x_1) - f(x_2) \geq -2\varepsilon$ holds for arbitrary $x_1, x_2 \in [S_j, S_{j+1}]$ with $x_1 < x_2$; $f(x_3) - f(x_4) \leq 2\varepsilon$ holds for arbitrary $x_3, x_4 \in [S_j, S_{j+1}]$ with $x_3 < x_4$. A continuous output for fuzzy system can be constructed using the constructive methods in Lemma 6 alternately. It means that there exists a fuzzy system which can achieve the given approximation accuracy in Eq. (4). Therefore the necessary conditions are optimal under the same condition.

Remark 2: If n division points in Theorem 1 are selected appropriately, none other division points are necessary for the fuzzy systems to achieve the given approximation accuracy in Eq. (4).

Remark 3: In Theorems 1 and 2, the following assumptions are assumed to be known for the necessary conditions of the SISO Mamdani fuzzy system on a given approximation accuracy ε :

- (a) Approximation accuracy $\varepsilon \geq 0$;
- (b) The values of $f(x) \in C[a, b]$ at extremes $\xi_i : v_i = f(\xi_i)$, $1 \leq i \leq m$;
- (c) The values of $f(x)$ at endpoints: $v_0 = f(a), v_{m+1} = f(b)$.

These prerequisite conditions (a)-(c) will be the minimum information required to describe a well-behaved continuous function so that the fuzzy systems performs more efficiently than other classical functional approximators. And these conditions can be gained easily in practical process.

B. Comparision study on necessary conditions

Ying [19, 20] pointed out that, a fuzzy system as approximator, the number of division points is at least the number of extremes of the continuous function for the necessary condition. However, these results are not available for the fuzzy system as universal approximator within a given accuracy. In most cases, the necessary conditions for the fuzzy system as universal approximator within a given accuracy are more practical and useful. Here we will show that Ying's conclusion is only a special case of our results given above.

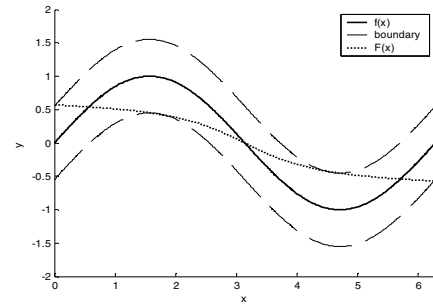


Figure 2. The fuzzy system using only two rules approximates $f(x)$.

Consider the function $f(x) = \sin x$, $x \in [0, 2\pi]$ in Fig. 3 with error bound 0.55 (dashed lines). A fuzzy system $F(x)$ with 2 fuzzy rules could be constructed (the dotted line) to achieve the given approximation accuracy by Lemma 7. However, according to Ying's conclusion, the fuzzy system requires at least 4 rules to achieve the given approximation accuracy. Consequently, necessary conditions proposed by Ying hold no longer when the approximation accuracy is prescribed.

In the construction algorithm, when the given approximation accuracy $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} f(\xi_{i-1}) - f(\xi_i) &> 2\varepsilon, f(\xi_i) - f(\xi_{i+1}) < -2\varepsilon \\ \text{or } f(\xi_{i-1}) - f(\xi_i) &< -2\varepsilon, f(\xi_i) - f(\xi_{i+1}) > 2\varepsilon \end{aligned} \quad (23)$$

for arbitrary $\xi_{i-1}, \xi_i, \xi_{i+1} \in \xi$, $i = 1, 2, \dots, m$, and $Q = \{[\xi_0, \xi_2], [\xi_1, \xi_3], \dots, [\xi_{m-1}, \xi_{m+1}]\}$.

According to Theorem 1, the input interval $[a, b]$ must be divided into at least $m+1$ subintervals, and the number of the fuzzy rules is $m+2$. This is just the necessary conditions for fuzzy systems as universal approximates proposed in [19]. As a result, the approach in [19] is a special case of our necessary conditions when ε approaches zero.

IV. ILLUSTRATIVE EXAMPLES

In this section, the developed approaches above will be illustrated by examples.

Example: Design a Mamdani fuzzy system to uniformly approximate the continuous function shown in Fig. 4. We can find easily that the extreme set of $f(x)$ is $\{0, 4, 6, 8, 12, 12.5, 13, 14.5, 16, 20, 21, 22, 26\}$ whose values are $\{1, -1, -0.4, -1, 1, 0.9, 1, 0.7, 1, -1, -0.8, -1, 1\}$.

The necessary conditions for a fuzzy system on various kinds of approximation accuracy are given as follows:

- (1) $0 \leq \varepsilon < 0.05$

Using the construction algorithm, the characteristic set Q is

$$Q = \{[0, 6], [4, 8], [6, 12], [8, 12.5], [12, 13], [12.5, 14.5], [13, 16], [14.5, 20], [16, 21], [20, 22], [21, 26]\}.$$

By using the necessary conditions for fuzzy systems in Theorem 1, 13 fuzzy rules are required to achieve approximation accuracy.

(2) $0.05 \leq \varepsilon < 0.1$

Using the construction algorithm, the characteristic set Q is

$$Q = \{[0, 6], [4, 8], [6, 12], [8, 14.5], [13, 16], [14.5, 20], [16, 21], [20, 22], [21, 26]\}.$$

We can conclude that 11 fuzzy rules can be used to achieve the given approximation accuracy by Theorem 1. Take $\varepsilon = 0.08$ as an example, the upper and lower boundaries of $f(x)$ are obtained by moving it vertically by ε upwards and downwards (the dashed lines in Fig. 5). A fuzzy system output whose monotonicity changes 9 times, is constructed in the tunnel encircled by the boundaries to approximate $f(x)$ (dotted line). Therefore 11 fuzzy rules are necessary for achieving the given accuracy.

(3) $0.1 \leq \varepsilon < 0.15$

Using the construction algorithm, the characteristic set Q is

$$Q = \{[0, 6], [4, 8], [6, 12], [8, 14.5], [13, 16], [14.5, 20], [16, 26]\}$$

And 9 fuzzy rules are required to achieve the given approximation accuracy by Theorem 1.

(4) $0.15 \leq \varepsilon < 0.2$

The construction algorithm is used to determine the characteristic set as

$$Q = \{[0, 6], [4, 8], [6, 12], [8, 20], [16, 26]\}.$$

And 7 fuzzy rules are required to achieve the given approximation accuracy by Theorem 1. Take $\varepsilon = 0.18$ as an example, the upper and lower bounds of $f(x)$ are obtained by moving it vertically by ε upwards and downwards (the dashed lines in Fig. 6). A fuzzy system output whose monotonicity changes 5 times, is constructed in the tunnel encircled by the boundaries to approximate $f(x)$ (dotted line). Therefore 7 fuzzy rules are necessary for achieving the given accuracy.

(5) $0.2 \leq \varepsilon < 0.5$

Using the construction algorithm, the characteristic set Q is

$$Q = \{[0, 12], [8, 20], [16, 26]\}.$$

5 fuzzy rules are required to achieve the given approximation accuracy by Theorem 1.

(6) $\varepsilon \geq 0.5$

The construction algorithm is used to determine the characteristic set Q as $Q = \Phi$. 2 fuzzy rules are required to achieve approximation accuracy by Theorem 1.

The strength as well as limitation of the general SISO Mamdani fuzzy system as function approximator is showed in the example. It is observed that the minimal number of fuzzy rules M increases with the number Ω of the desired continuous function's extremes whose swings are larger than the approximation accuracy. That is to say, on one hand, a small number of fuzzy rules suffice to achieve the approximation if Ω is a small number, even the desired function has a complicated formulation. On the other hand, large numbers of fuzzy rules are required to approximate a simple function with large Ω . It means that fuzzy systems are not ideal function approximators for applications involving highly oscillatory continuous functions with large amplitudes. And the necessary conditions are local constant with accuracy. These may be used to explain practical fuzzy models only use a small number of fuzzy rules to achieve successful applications.

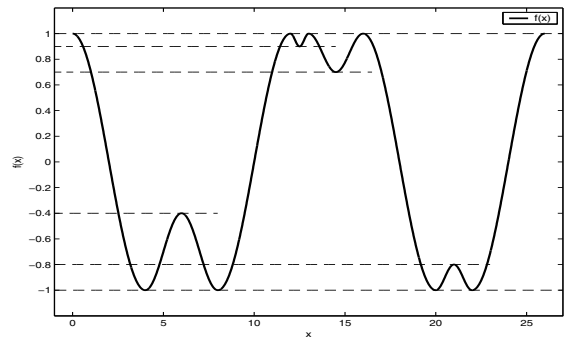


Figure 3. The desired function $f(x)$ in example 2.

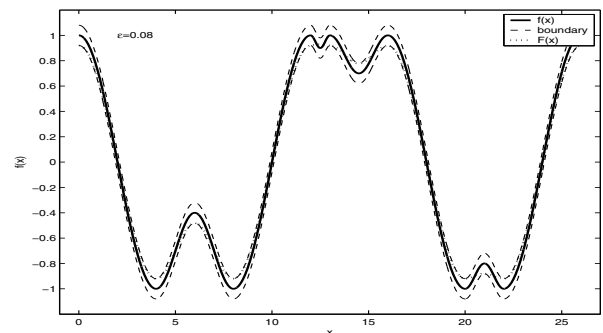


Figure 4. A fuzzy system $F(x)$ is constructed to approximate $f(x)$ in example 2 with accuracy 0.08.

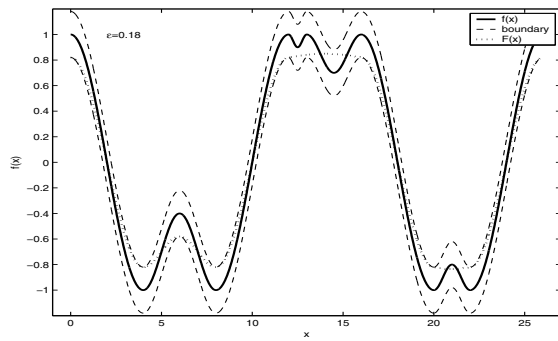


Figure 5. A fuzzy system $F(x)$ is constructed to approximate $f(x)$ in example 2 with accuracy 0.18.

V. CONCLUSIONS

This paper is concerned with the necessary conditions for general SISO Mamdani fuzzy systems as approximators within a given accuracy. The necessary conditions proposed in this paper expand the existing ones so that new conclusions are suitable for more practical cases. It has been proved that the proposed conditions are optimal. Finally, some examples are used to demonstrate the strength and limitation of the general SISO Mamdani fuzzy systems.

REFERENCES

- [1] J. J. Buckley, "Universal fuzzy controllers", *Automatica*, vol. 28, no. 6, pp. 1245-1248, 1992.
- [2] B. Kosko, "Fuzzy systems as universal approximators", *IEEE Trans. Computers*, vol. 43, no. 11, pp. 1329-1333, 1994.
- [3] G. C. Mouzouris and J. M. Mendel, "Nonsingleton fuzzy logic systems: theory and application", *IEEE Trans. Fuzzy Systems*, vol. 5, no. 1, pp. 56-71, 1997.
- [4] H. T. Nguyen, V. Kreionovich and O. Sirisaengtaksin, "Fuzzy control as universal control tool", *Fuzzy Sets and Systems*, vol. 80, no. 1, pp. 71-86, 1997.
- [5] X. J. Zeng and M. G. Singh, "Approximation theory of fuzzy systems-MIMO case", *IEEE Trans. Fuzzy Systems*, vol. 3, no. 2, pp. 219-235, 1995.
- [6] Y. Yam, P. Baranyi and C.T. Yang, "Reduction of fuzzy rule base via singular value decomposition", *IEEE Trans. Fuzzy Systems*, vol. 7, no. 2, pp. 120-132, 1999.
- [7] J. Yen and L. Wang, "Simplifying fuzzy rule-based models using orthogonal transformation methods", *IEEE Trans. Systems Man Cybernetics. B* vol. 29, no. 1, pp. 13-24, 1999.
- [8] T. Sudkamp, A. Knapp and J. Knapp, "Rule base reduction: some comments on the use of orthogonal transforms", *IEEE Trans. Systems Man Cybernetics. C*, vol. 31, no. 3, pp. 199-206, 2001.
- [9] H. Ishibuchi, K. Nozaki, N. Yamamoto and H. Tanaka, "Selecting fuzzy if-then rules for classification problems using genetic algorithms", *IEEE Trans. Fuzzy Systems*, vol. 3, no. 4, pp. 260-270, 1995.
- [10] H. Roubos and M. Setnes, "Compact and transparent fuzzy models and classifiers through iterative complexity reduction", *IEEE Trans. Fuzzy Systems*, vol. 9, no. 4, pp. 516-524, 2001.
- [11] M. Setnes, "Supervised fuzzy clustering for rule extraction", *IEEE Trans. Fuzzy Systems*, vol. 8, no. 4, pp. 416-424, 2000.
- [12] T. Sudkamp, A. Knapp and J. Knapp, "Model generation by domain refinement and rule reduction", *IEEE Trans. Systems Man Cybernetics. B*, vol. 33, no. 1, pp. 45-55, 2003.
- [13] P. Manley-Cooke and M. Razaz, "An efficient approach for reduction of membership functions and rules in fuzzy systems", *IEEE International Conference on Fuzzy Systems*, pp. 1-5, 2007.
- [14] H. Ying, "Sufficient conditions on general fuzzy systems as function approximations", *Automatica*, vol. 30, no.3, pp. 521-525, 1994.
- [15] X. J. Zeng and M. G. Singh, "Approximation accuracy analysis of fuzzy systems as function approximators", *IEEE Trans. Fuzzy Systems*, vol. 4, no. 1, pp. 44-63, 1996.
- [16] K. Zeng, N. Y. Zhang and W. L. Xu, "A comparative study on sufficient conditions for Takagi-Sugeno fuzzy systems as universal approximators", *IEEE Trans. Fuzzy Systems*, vol. 8, no. 6, pp. 773-780, 2000.
- [17] H. Ying, "General SISO Takagi-Sugeno fuzzy systems with linear rule consequent are universal approximators", *IEEE Trans. Fuzzy System*, vol. 6, pp. 582-587, 1998.
- [18] H. Ying, "Sufficient conditions on uniform approximation of multivariate functions by general Takagi-Sugeno fuzzy systems with linear rule consequent", *IEEE Trans. System, Man, Cybernetics*, vol. 28, pp. 515-520, 1998.
- [19] H. Ying and G. R. Chen, "Necessary conditions for some typical fuzzy systems as universal approximators", *Automatica*, vol. 33, no. 7, pp. 1333-1338, 1997.
- [20] H. Ying, Y. S. Ding, S. K. Li and S. H. Shao, "Comparison of necessary conditions for typical Takagi-Sugeno and Mamdani fuzzy systems as universal approximators", *IEEE Trans. Man, System, Cybernetics*, vol. 29, no.5, pp. 508-514, 1999.
- [21] Y. S. Ding, H. Ying and S. H. Shao, "Necessary conditions on minimal system configuration for general MISO Mamdani fuzzy systems as universal approximators", *IEEE Trans. on Systems, Man and Cybernetics*, vol. 30, no. 6, pp.857-864, 2000.
- [22] F. Wan, H. L. Shang, L. X. Wang and Y. X. Sun, "How to determine the minimum number of fuzzy rules to achieve given accuracy: a computational geometric approach to SISO case", *Fuzzy Sets and Systems*, vol. 150, 199-209, 2005.
- [23] F. C. Sun and N. Zhao, "Universal approximation for Takagi-Sugeno fuzzy systems using dynamically constructive method-SISO cases", *IEEE 22nd International Symposium on Intelligent Control*, pp. 150-155, 2007.