

# Graded Equipollence And Fuzzy C-measures Of Finite Fuzzy Sets

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**Abstract**—In the classical set theory, there is the well known correspondence between the equipollence of finite sets and the equality of natural numbers expressing their cardinalities. In the fuzzy set theory, an equivalent correspondence between the equipollence of finite fuzzy sets and the equality of generalized cardinals has been proposed in [1]. This paper is devoted to the situation when we consider the mentioned relations to be graded, that means, the crisp equipollence of fuzzy sets is replaced now by the graded equipollence of fuzzy sets and the equality of fuzzy sets by their similarity.

**Index Terms**—residuated-dually residuated lattice; graded equipollence; fuzzy c-measure

## I. INTRODUCTION

In the classical set theory, two sets have the same number of elements if and only if there exists a one-to-one mapping of one set onto the second one. Thus, there is a nice relation between the functional approach to the cardinality of sets, which does not suppose any knowledge about the numbers of elements in sets, and the approach based on the ordinal numbers expressing the numbers of elements of sets.

The above two mentioned approaches to the cardinality of (finite) fuzzy sets have been also studied in the fuzzy set theory. Nevertheless, for fuzzy sets, the situation is much more complicated by the graduation of membership of elements of fuzzy sets.

The classical-like approaches to the cardinality of fuzzy sets based on the notion of the equipollence of fuzzy sets were proposed by D. Klaua [2] and S. Gottwald [3], [4]. In the second case, a degree in which two fuzzy sets are equipollent is defined using fuzzy uniqueness of fuzzy mappings. Equivalence classes (fuzzy sets) are *fuzzy cardinals*. Note that fuzzy mappings are here defined as special fuzzy relations. Since these approaches are purely theoretical, a further substantial research in this direction has not been realized yet. Another approach was suggested by M. Wygralak [5], where the notion of equipollence of fuzzy sets is introduced using the cardinality of  $\alpha$ -cuts of fuzzy sets. More precisely, two fuzzy sets  $A$  and  $B$  are equipollent, if

$$\bigvee \{ \alpha \in [0, 1] \mid |A_\alpha| \geq i \} = \bigvee \{ \alpha \in [0, 1] \mid |B_\alpha| \geq i \} \quad (1)$$

for any cardinal number  $i$ . Some modifications of (1) can be found in [1], [6], [7], [8]. In [9] (see also [10], [11]), we proposed a new approach to the equipollence of fuzzy sets over a universe of sets (e.g., a universe of all finite or countable sets, or all sets, or a Grothendieck universe).

Analogously to the Gottwald's approach, a graded equipollence is considered, where, however, the degrees of being equipollent are obtained by a simpler way than in the Gottwald's approach. More precisely, this approach is based on a graded one-to-one correspondence between fuzzy sets where the crisp mappings between universes are considered.

The approaches to the cardinality of finite fuzzy sets based on objects expressing the number of elements in fuzzy sets can be divided to two directions, namely, to the scalar and fuzzy approaches. In this contribution, we are interested in the second approach, where the cardinality of fuzzy sets is expressed by special fuzzy sets defined over the set of natural numbers  $\mathbb{N}$  or over universes containing a class of cardinal numbers. In this case, convex fuzzy sets over a class of cardinals have a dominant role. These fuzzy sets are usually called *generalized natural numbers* or *generalized cardinals* (see e.g. [12], [13], [14]). This type of cardinality of fuzzy sets is usually referred to the *fuzzy cardinality*.

Let us mention that the first definition of fuzzy cardinality of finite fuzzy sets, by means of functions from  $\mathbb{N}$  to  $[0, 1]$ , was done by L.A. Zadeh in [15]. A substantial improvement of the original definition was done again by L.A. Zadeh in [16] and gave rise to three types of cardinality, namely, *FGCount*, *FLCount* and *FECCount*. A generalization of FG-Count, FLCount and FECCount using  $t$ -norms and  $t$ -conorms was proposed by M. Wygralak in [14] and further developed in [1]. Both publications are an attempt to develop the cardinal theory for fuzzy sets including a relation between equipollence and equality of cardinal numbers. An axiomatic approach to fuzzy cardinality of finite fuzzy sets covering some of the well-known definition, defined by means of the generalized natural numbers, was proposed by J. Casasnovas and J. Torrens [17].

In [18], we generalized the Casasnovas-Torrens axiomatic approach and defined "fuzzy measures" extending the cardinality of finite sets to a cardinality finite fuzzy sets.<sup>1</sup> Although, the proposed system of fuzzy measures generalizes the cardinality of sets, some properties like the valuation property or, as will be seen later, the fundamental correspondence between the graded equipollence of fuzzy sets and the equality of their generalized cardinals are failed in general. Therefore, we call these measures as the *c-measures*, where the letter "c" is an

<sup>1</sup>Note that, in contrast to the cardinality of finite sets, we can define a large scale of fuzzy measures expressing the number of elements of finite fuzzy sets.

abbreviation of “cardinal”.

The aim of this contribution is to show relations between functional approach based on the graded equipollence introduced in [9] (see also [10], [11]) and the approach based on the fuzzy c-measures proposed in [18], where the cardinality of finite fuzzy sets is expressed by generalized cardinals.

The paper is structured as follows. In Preliminaries, we propose a residuated-dually residuated lattice as an algebraical structure of membership degrees of fuzzy sets and (countable and finite) fuzzy sets defined in the class of all countable sets. The graded equipollence of (countable) fuzzy sets is introduced in Section III. The section IV is devoted to fuzzy c-measures including the important properties as a representation of fuzzy c-measures by suitable homomorphisms between reducts of the rdr-lattices. In Section V, we shows the relation between the graded equipollence of finite fuzzy sets and the similarity of generalized cardinals obtained by fuzzy c-measures. The last section is the conclusion.

## II. PRELIMINARIES

### A. Algebraic Structures of Membership Degrees of Fuzzy Sets

In this paper, the membership degrees of fuzzy sets are interpreted in a complete residuated lattice which is, moreover, extended by an adjoint pair of dual operations. We shall say that an algebra  $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \oplus, \ominus, \perp, \top \rangle$  with six binary operations and two constants is a *residuated-dually residuated lattice* (shortly, an *rdr-lattice*), if

- (i)  $\langle L, \wedge, \vee, \perp, \top \rangle$  is a bounded lattice, where  $\perp$  is the least element and  $\top$  is the greatest element of  $L$ , respectively,
- (ii)  $\langle L, \otimes, \top \rangle$  and  $\langle L, \oplus, \perp \rangle$  are commutative monoids,
- (iii) the pairs  $\langle \otimes, \rightarrow \rangle$  and  $\langle \oplus, \ominus \rangle$  form adjoint pairs, i.e.

$$a \leq b \rightarrow c \quad \text{if and only if} \quad a \otimes b \leq c, \quad (2)$$

$$a \leq b \oplus c \quad \text{if and only if} \quad a \ominus b \leq c \quad (3)$$

hold for each  $a, b, c \in L$  ( $\leq$  denotes the corresponding lattice ordering).

The operations  $\otimes$ ,  $\rightarrow$ ,  $\oplus$  and  $\ominus$  are called *multiplication*, *residuum*, *addition* and *difference*, respectively.<sup>2</sup> We shall say that an rdr-lattice is *complete (linearly ordered)*, if  $\langle L, \wedge, \vee, \perp, \top \rangle$  is a complete (linearly ordered) lattice, respectively. Further, an rdr-lattice is *divisible*, if  $a \otimes (a \rightarrow b) = a \wedge b$  holds for all  $a, b, c \in L$ , and *dually divisible*, if  $(a \ominus b) \oplus b = a \vee b$  holds for all  $a, b, c \in L$ .

**EXAMPLE 1** Let  $T$  and  $S$  be a left continuous  $t$ -norm and a right continuous  $t$ -conorm, respectively, and define  $\rightarrow_T$  and  $\ominus_S$  by

$$a \rightarrow_T b = \bigvee \{c \in [0, 1] \mid T(a, c) \leq b\},$$

$$a \ominus_S b = \bigwedge \{c \in [0, 1] \mid S(b, c) \geq a\}.$$

The algebra  $\mathbf{L} = \langle [0, 1], \min, \max, T, \rightarrow_T, S, \ominus_S, 0, 1 \rangle$  is a complete rdr-lattice. If  $T$  and  $S$  is the product (or the

<sup>2</sup>Notice that if the adjoint pair  $\langle \oplus, \ominus \rangle$  (or  $\langle \otimes, \rightarrow \rangle$ ) is forgotten in the rdr-lattice  $\mathbf{L}$  we obtain the *residuated lattice* (or *dually residuated lattice*).

Łukasiewicz conjunction) and the probabilistic sum (or the Łukasiewicz disjunction), respectively, then we shall use  $\mathbf{L}_P$  ( $\mathbf{L}_L$ ) to denote the Goguen (Łukasiewicz) rdr-lattice.

Let us define the following additional operations for any  $a, b \in L$  and set  $\{a_i \mid i \in I\}$  of elements from  $L$  over a countable (possibly empty) index set  $I$ :

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \quad (\text{biresiduum})$$

$$\neg a = a \rightarrow \perp, \quad (\text{negation})$$

$$\bigotimes_{i \in I} a_i = \begin{cases} \top, & I = \emptyset, \\ \bigwedge_{K \in \mathfrak{Fin}(I)} \bigotimes_{i \in K} a_i, & \text{otherwise,} \end{cases} \quad (\text{countable multiplication})$$

$$\bigoplus_{i \in I} a_i = \begin{cases} \perp, & I = \emptyset, \\ \bigvee_{K \in \mathfrak{Fin}(I)} \bigoplus_{i \in K} a_i, & \text{otherwise,} \end{cases} \quad (\text{countable addition})$$

where  $\mathfrak{Fin}(I)$  denotes the set of all finite subsets of  $I$ . If  $|I| = n$  is a finite index set and  $a_i = a$  for any  $i \in I$ , then we will write  $a^n = \bigotimes_{i \in I} a_i$  or  $na = \bigoplus_{i \in I} a_i$ .

To integrate some of alternative constructions based on the operations of  $\wedge$  and  $\otimes$ , we shall use the common symbol  $\odot$  (or  $\odot$ ) for them. Analogously, we shall use  $\odot$  (or  $\odot$ ) to denote one of the operations  $\vee$  or  $\oplus$ .

### B. Fuzzy Sets

Since we are interested in finite fuzzy sets and consequently also denumerable fuzzy sets, let us consider the class  $\mathfrak{Count}$  of all countable sets as a framework for our theory.<sup>3</sup> Further, we shall use  $\mathfrak{Fin}$  to denote the class of all finite sets. A fuzzy set in  $\mathfrak{Count}$  is defined as follows.

**DEFINITION 1** A mapping  $A : x \rightarrow L$  is called a fuzzy set in  $\mathfrak{Count}$ , if  $x$  is a set in  $\mathfrak{Count}$ .

In what follows, we shall use  $\mathfrak{Fcount}$  to denote the class of all fuzzy sets in  $\mathfrak{Count}$ . Although, the fuzzy sets in  $\mathfrak{Fcount}$  may be referred to “countable” fuzzy sets, we shall not use this notation and say only fuzzy sets. Notice that, for example, each fuzzy set or pair of fuzzy sets belong to  $\mathfrak{Count}$ . Hence, we obtain that  $\mathfrak{Fcount} \subseteq \mathfrak{Count}$  and  $\mathfrak{Fcount} \times \mathfrak{Fcount} \subseteq \mathfrak{Count}$ .

Let us summarize several well known notions of the fuzzy set theory and suppose that  $A : z \rightarrow L$  is a fuzzy set from  $\mathfrak{Fcount}$ . The set  $z = \text{Dom}(A)$  is called a *universe of discourse of A*,  $\text{Supp}(A) = \{x \in z \mid A(x) > \perp\}$  is called a *support of fuzzy set A* and  $A_a = \{x \in z \mid A(x) \geq a\}$  is called an *a-cut of fuzzy set A*. The fuzzy set  $A$  is *crisp*, if  $A(x) \in \{\perp, \top\}$  for any  $x \in z$ , and is called a *singleton*, if  $A(x) > \perp$  for some  $x \in z$  and  $A(y) = \perp$  for any  $y \in z$  with  $y \neq x$ . For simplicity, we shall use the same symbol to denote a crisp set as a fuzzy set (e.g.,  $\emptyset$  denotes the empty set as well as the empty fuzzy

<sup>3</sup>Note that we need to deal with the set  $\mathbb{N}$  of all natural numbers and fuzzy sets defined on it, and, therefore, the countable sets are considered. In general, we can consider an arbitrary universe of sets containing  $\mathbb{N}$  satisfying the axioms U1-U5 of the definition of a universe of sets over  $\mathbf{L}$  (see e.g. [9], [18], [10]). Note that the entire set theory can be introduced inside such universes of sets.

set) and  $\{a/x\}$  to denote a singleton  $A$  with  $A(x) = a > \perp$ . Finally, if  $f : z \rightarrow z'$  and  $A : z \rightarrow L$  be a fuzzy set, then the *image of  $A$  under  $f$*  is denoted by  $f^{\rightarrow}(A)$  and defined by  $f^{\rightarrow}(A)(y) = \bigvee_{x \in z: f(x)=y} A(x)$  for any  $y \in z'$ .

An essential predicate in our theory is a binary relation saying that two fuzzy sets are the same or equivalent fuzzy sets. In the classical set theory, we say that two sets are the same sets, if they have the same elements. In our theory, we define the analogous notion as follows.

**DEFINITION 2** We shall say that fuzzy sets  $A$  and  $B$  are *equivalent* (symbolically,  $A = B$ ), if  $\text{Supp}(A) = \text{Supp}(B)$  and  $A(x) = B(x)$  for any  $x \in \text{Supp}(A)$ .

For example,  $A = \{0.9/a, 0/b\}$  and  $B = \{0.9, /a\}$  are the equivalent fuzzy sets. We shall use  $\text{cls}(A)$  to denote the class of all fuzzy sets from  $\mathfrak{Fcount}$  which are equivalent with  $A$ . A graded generalization of this relation may be given as follows.

**DEFINITION 3** We shall say that fuzzy sets  $A$  and  $B$  are *the equivalent fuzzy sets* in the degree  $a$  (symbolically,  $[A \approx B] = a$ ), if

$$a = \bigwedge_{x \in \text{Dom}(A) \cup \text{Dom}(B)} (A'(x) \leftrightarrow B'(x)), \quad (4)$$

holds for  $A' \in \text{cls}(A)$ ,  $B' \in \text{cls}(B)$  with  $\text{Dom}(A') = \text{Dom}(B') = \text{Dom}(A) \cup \text{Dom}(B)$ .

One can simply prove that  $[A \approx B] = [A' \approx B']$  holds for any  $A' \in \text{cls}(A)$  and  $B' \in \text{cls}(B)$  and, hence, the previous definition is correct. Moreover,  $[A \approx B] = \top$  if and only if  $A = B$ . For our investigation, we need only the operations of union and intersection defined as follows.<sup>4</sup> Let  $A, B \in \mathfrak{Fcount}$ ,  $x = \text{Dom}(A) \cup \text{Dom}(B)$  and  $A' = A$ ,  $B' = B$  such that  $\text{Dom}(A') = \text{Dom}(B') = x$ . Then

- the *union* of  $A$  and  $B$  is a mapping  $A \cup B : x \rightarrow L$  defined by

$$(A \cup B)(a) = A'(a) \vee B'(a) \quad (5)$$

for any  $a \in x$ ,

- the *intersection* of  $A$  and  $B$  is a mapping  $A \cap B : x \rightarrow L$  defined by

$$(A \cap B)(a) = A'(a) \wedge B'(a) \quad (6)$$

for any  $a \in x$ ,

One can check easily that our definition is correct, i.e., there is no dependence on the choice of fuzzy sets  $A'$  and  $B'$  in the previous definitions.

As one could notice, the major object of our theory is a finite fuzzy set. We propose the following definition based on the finiteness of universes of discourse (cf. [1], [14]).

**DEFINITION 4** We shall say that a fuzzy set  $A$  from  $\mathfrak{Fcount}$  is *finite*, if there exists  $A' \in \text{cls}(A)$  such that  $\text{Dom}(A')$  is a finite set.

<sup>4</sup>Further definitions of operations between fuzzy sets and also a definition of a fuzzy power set can be found in [10].

We shall use  $\mathfrak{Ffin}$  to denote the class of all finite fuzzy sets in  $\mathfrak{Count}$ . Obviously,  $A$  is a finite fuzzy set if and only if  $\text{Supp}(A)$  is a finite set,  $\mathfrak{Ffin}$  is a subclass of  $\mathfrak{Fcount}$ , the union or intersection of finite fuzzy sets is again a finite fuzzy set.

To define measures of finite fuzzy sets which are similar to the cardinality for finite sets, we need to introduce a concept of generalized cardinal over  $\mathbb{N}$ . A reasonable condition (see e.g. [1], [14]) for this concept is a “convexity” of fuzzy sets. The convexity of fuzzy sets may be defined over linearly ordered sets as follows. Recall that  $\odot \in \{\wedge, \otimes\}$ .

**DEFINITION 5** Let  $x$  be a set in  $\mathfrak{Count}$  equipped with a linear ordering  $\leq$ . We shall say that  $A : x \rightarrow L$  is an  $\odot$ -convex fuzzy set, if  $A(a) \odot A(c) \leq A(b)$  holds for any  $a, b, c \in x$  with  $a \leq b \leq c$ .

**DEFINITION 6** A *generalized cardinal*  $A$  (over  $\mathbb{N}$ ) is an  $\odot$ -convex fuzzy set  $A : \mathbb{N} \rightarrow L$ .

One can see that generalized cardinals need not be finite fuzzy sets. We shall use  $\mathfrak{N}$  to denote the set of all generalized cardinals. To be able to manipulate with generalized cardinals, we establish, by the Zadeh’ extension principle, the operation of addition on  $\mathfrak{N}$  as

$$(A + B)(i) = \bigvee_{\substack{k, l \in \mathbb{N} \\ k+l=i}} (A(k) \odot B(l)). \quad (7)$$

Let us define  $\mathbf{0} : \mathbb{N} \rightarrow L$  by  $\mathbf{0}(k) = \top$ , if  $k = 0$ , and  $\mathbf{0}(k) = \perp$ , otherwise. One can easily prove the following statement saying that our definition of addition is correct.

**THEOREM 1** The triplet  $(\mathfrak{N}, +, \mathbf{0})$  is a commutative monoid, where the divisibility of the rdr-lattice  $\mathbf{L}$  has to be supposed for  $\odot = \wedge$ .

### III. GRADED EQUIPOLLENCE OF FUZZY SETS

In order to introduce a graded equipollence for fuzzy sets, we shall follow the classical idea and use “one-to-one mappings between fuzzy sets”. Nevertheless, contrary to the classical approach, it seems to be reasonable to distinguish what one-to-one mapping is better than another one. This motivates us to introduce a degree in which a one-to-one mapping between suitable sets is a one-to-one mapping between fuzzy sets as follows.

**DEFINITION 7** Let  $A, B \in \mathfrak{Fcount}$ ,  $x, y \in \mathfrak{Count}$  and  $f : x \rightarrow y$  be a one-to-one mapping of  $x$  onto  $y$  in  $\mathfrak{Count}$ . We shall say that  $f$  is a *one-to-one mapping of  $A$  onto  $B$  in the degree  $a$*  with respect to  $\odot$ , if  $\text{Supp}(A) \subseteq x \subseteq \text{Dom}(A)$  and  $\text{Supp}(B) \subseteq y \subseteq \text{Dom}(B)$  and

$$a = \bigodot_{z \in x} (A(z) \leftrightarrow B(f(z))). \quad (8)$$

We shall write  $[A \sim_f^\odot B] = a$ , if  $f$  is a one-to-one mapping of  $A$  onto  $B$  in the degree  $a$  with respect to  $\odot$ . Before we introduce the definition of being equipollent in

the degree  $\alpha$ , let us denote  $\text{Bij}(A, B)$  the set of all one-to-one correspondences between  $A$  and  $B$ . Obviously, the set  $\text{Bij}(A, B)$  can be characterized as follows.

**THEOREM 2** *Let  $A, B \in \mathfrak{Fcount}$ . A mapping  $f : x \rightarrow y$  belongs to the set  $\text{Bij}(A, B)$ , if*

- (i)  $f$  is a one-to-one mapping of  $x$  onto  $y$ ,
- (ii)  $\text{Supp}(A) \subseteq x \subseteq \text{Dom}(A)$ , and
- (iii)  $\text{Supp}(B) \subseteq y \subseteq \text{Dom}(B)$ .

In what follows, we shall use some results on the graded equipollence for fuzzy sets to simplify the original definition (see [9], [11]). Moreover, we shall consider a more general case, when the operation  $\wedge$  is replaced by  $\odot$ .<sup>5</sup>

Let  $z \in \mathfrak{Count}$ . We shall use  $\text{Perm}(z)$  to denote the set of all permutations on  $z$ , that is, the one-to-one mappings of  $z$  onto  $z$ .

**DEFINITION 8** Let  $A, B \in \mathfrak{Fcount}$ . We shall say that  $A$  is equipollent with  $B$  in a degree  $a$  with respect to  $\odot$  (symbolically,  $[A \sim^\odot B] = a$ ), if

$$a = \bigvee_{f \in \text{Perm}(z)} [C \sim_f^\odot D] \quad (9)$$

where  $C \in \text{cls}(A)$ ,  $D \in \text{cls}(B)$  are arbitrary fuzzy sets with  $z = \text{Dom}(C) = \text{Dom}(D)$  such that  $|\text{Supp}(A)| \leq |z \setminus \text{Supp}(B)|$  and  $|\text{Supp}(B)| \leq |z \setminus \text{Supp}(A)|$ .

One could be surprised that the degree in which fuzzy sets are equipollent is not dependent on the choice of fuzzy sets  $C$  and  $D$ . As we have mentioned above, the previous definition uses the results presented in [9], [11]. More precisely, we use the fact (described for the case  $\odot = \wedge$ ) that the degree of equipollence of  $A$  and  $B$  can be obtained by fuzzy sets  $C$  and  $D$  being equivalent with  $A$  and  $B$ , respectively, for which it is sufficient to suppose that their domains have the same cardinality, or even, their domains are identical (after a suitable extension). Analogously, one could prove the sufficiency of this presumption on  $\text{Dom}(C) = \text{Dom}(D)$  for  $\odot = \otimes$ .

The following useful lemma shows that the degree in which finite fuzzy sets are equipollent can be derived from finite sets.

**THEOREM 3** *Let  $A, B \in \mathfrak{Ffin}$  and  $C \in \text{cls}(A)$ ,  $D \in \text{cls}(B)$  be such that  $z = \text{Dom}(C) = \text{Dom}(D)$  and  $|z| = n$ . Then*

$$[A \sim^\odot B] = \bigvee_{f \in \text{Perm}(z)} [C \sim_f^\odot D]. \quad (10)$$

As we have mentioned above, the class  $\mathfrak{Fcount} \times \mathfrak{Fcount}$  is a subclass of  $\mathfrak{Count}$ . Hence, we can define the following “fuzzy class” (or also “fuzzy class relation”) in  $\mathfrak{Count}$ , which is analogous to the equivalence relation “to be equipollent” on the class of all sets (for details about fuzzy classes in a universe of sets, we refer to [9], [10]).

**THEOREM 4** *The fuzzy class relation  $\sim^\odot : \mathfrak{Fcount} \times \mathfrak{Fcount} \rightarrow L$  is a similarity relation ( $\otimes$ -equivalence) on the class  $\mathfrak{Fcount}$ , i.e.,*

$$[A \sim^\odot A] = \top, \quad (11)$$

$$[A \sim^\odot B] = [B \sim^\odot A], \quad (12)$$

$$[A \sim^\odot B] \otimes [B \sim^\odot C] \leq [A \sim^\odot C] \quad (13)$$

hold for arbitrary finite fuzzy sets  $A, B, C \in \mathfrak{Fcount}$ .

The fuzzy class relation  $\sim^\odot$  is called the *graded equipollence* of finite fuzzy sets.

The rest of this section is devoted to the mappings (fuzzy classes) which can well characterize under some condition the finite fuzzy sets from the graded equipollence perspective.

Let  $A \in \mathfrak{Ffin}$  be a finite fuzzy set and define a fuzzy class  $p_A^\odot : \mathbb{N} \times \mathfrak{Ffin} \rightarrow L$  by

$$p_A^\odot(i, y) = \bigvee_{\substack{z \subseteq y, \\ |z|=i}} \bigodot_{x \in z} A(x) \quad (14)$$

for any  $i \in \mathbb{N}$ . For simplicity, we shall omit the symbol  $\odot$  in  $p_A^\odot(-, y)$  in such cases, when we consider both interpretations of  $\odot$ , and thus no confusion can arise. Several properties of the fuzzy sets  $p_A(-, y)$  are summarized in the following lemma.

**LEMMA 5** *Let  $A \in \mathfrak{Ffin}$ . Then, for  $i, j \in \mathbb{N}$ , we have*

$$p_A(i, y) \leq p_A(j, y), \text{ if } i \geq j, \quad (15)$$

$$p_A(i, y) = \perp, \text{ if } i > \min(|y|, |\text{Supp}(A)|), \quad (16)$$

$$p_A(0, y) = \top, \quad (17)$$

$$p_A(1, y) = \bigvee_{x \in y \cap \text{Supp}(A)} A(x). \quad (18)$$

According to the previous lemma, one can see that  $p_A(-, y)$  is a generalized cardinal which is a finite fuzzy set. The following statement shows an interesting relation between the equipollence of finite fuzzy sets and the equivalence of generalized cardinals derived from them.

**THEOREM 6** *Let  $A, B \in \mathfrak{Ffin}$  and  $y \in \mathfrak{Ffin}$  such that  $\text{Supp}(A) \cup \text{Supp}(B) \subseteq y$ . Then*

$$[A \sim^\odot B] \leq [p_A^\odot(-, y) \approx p_B^\odot(-, y)]. \quad (19)$$

Simple consequences of the inequality (19) are

$$p_A^\odot(-, y) = p_B^\odot(-, y), \quad (20)$$

$$[p_A^\odot(-, y) \sim^\odot p_B^\odot(-, y)] = \top, \quad (21)$$

whenever  $A \in \text{cls}(B)$  and  $\text{Supp}(A) \subseteq y$ .

Let us suppose in the rest of this section that  $\mathbf{L}$  be a complete linearly ordered rdr-lattice. Then one may simply prove the following equality for  $\odot = \wedge$ .

**THEOREM 7** *Let  $\mathbf{L}$  be a complete linearly ordered rdr-lattice,  $A, B \in \mathfrak{Ffin}$  and  $y \in \mathfrak{Ffin}$  such that  $\text{Supp}(A) \cup \text{Supp}(B) \subseteq y$ . Then*

$$[A \sim^\wedge B] = [p_A^\wedge(-, y) \approx p_B^\wedge(-, y)]. \quad (22)$$

<sup>5</sup>For interesting readers, we refer to [9], [10], [11].

A straightforward consequence of the previous theorem is the following lemma shows that under some presumption on domains of considered finite fuzzy sets there exists a permutation determining the degree in which these fuzzy sets are equipollent.

**COROLLARY 8** *Let  $\mathbf{L}$  be a linearly ordered residuated lattice and  $A, B \in \mathfrak{F}\text{fin}$  be such that  $z = \text{Dom}(A) = \text{Dom}(B)$ . Then there exists a permutation  $f$  on  $z$  such that*

$$[A \sim^\wedge B] = [f^\rightarrow(A) \approx B]. \quad (23)$$

**EXAMPLE 2** Let  $\mathbf{L}$  be the Łukasiewicz algebra and  $z = \{c_1, \dots, c_5\}$  be a set of criterion. Let us suppose that the following fuzzy sets

$$\begin{aligned} A &= \{0.3/c_1, 0.8/c_2, 0.4/c_3, 0/c_4, 0/c_5\} \\ B &= \{0/c_1, 0.6/c_2, 0.9/c_3, 0/c_4, 0.7/c_5\} \end{aligned}$$

express the satisfaction of all criterion by two applicants and our goal is to determine, whether the numbers of satisfied criterion are more or less the same for both applicants. According to (14), we can establish

$$\begin{aligned} p_A^\wedge(-, z) &= \{1/0, 0.8/1, 0.4/2, 0.3/3, 0/4, 0/5, \dots\}, \\ p_B^\wedge(-, z) &= \{1/0, 0.9/1, 0.7/2, 0.6/3, 0/4, 0/5, \dots\} \end{aligned}$$

and, using Theorem 7, we obtain

$$[A \sim^\wedge B] = \bigwedge_{i \in N} (p_A^\wedge(i, z) \leftrightarrow p_B^\wedge(i, z)) = 0.7.$$

Hence, we can conclude that both applicants have more or less the same number of satisfied criterion.

In the end of this section, we shall introduce an analogous notion to  $p_A^\odot(i, y)$  for  $\odot$ . Let  $A \in \mathfrak{F}\text{fin}$  be a finite fuzzy set and define a fuzzy class  $p_A^\odot : \mathbb{N} \times \mathfrak{F}\text{fin} \rightarrow L$  by

$$p_A^\odot(i, y) = \bigwedge_{\substack{z \subseteq y, \\ |z|=i}} \bigodot_{x \in z} A(x) \quad (24)$$

for any  $i \in \mathbb{N}$ . Again, for simplicity, we shall omit the symbol  $\odot$  in  $p_A^\odot$  in such cases, when we consider both interpretations of  $\odot$ , and thus no confusion can arise.

**LEMMA 9** *Let  $A \in \mathfrak{F}\text{fin}$ . Then, for  $i, j \in \mathbb{N}$ , we have*

$$p_A(i, y) \geq p_A(j, y), \text{ if } i \geq j, \quad (25)$$

$$p_A(i, y) = \top, \text{ if } i > \min(|y|, |\text{Supp}(A)|), \quad (26)$$

$$p_A(0, y) = \perp, \quad (27)$$

$$p_A(1, y) = \bigwedge_{x \in y \cap \text{Supp}(A)} A(x). \quad (28)$$

In contrast to  $p_A^\odot(-, y)$ , the mappings  $p_A^\odot(-, y)$  are denumerable fuzzy sets, i.e.,  $|\text{Supp}(p_A^\odot(-, y))| = \aleph_0$ . Nevertheless, one can simply check that

$$[p_A^\odot(-, y) \approx p_B^\odot(-, y)] = \top, \quad (29)$$

whenever  $A \in \text{cls}(B)$  and  $\text{Supp}(A) \subseteq y$ .

## IV. FUZZY C-MEASURES

In the following text, we shall suppose that  $\mathbf{L}$  is a complete rdr-lattice which is, moreover, divisible, whenever  $\wedge$  is considered as  $\odot$ .

**DEFINITION 9** A class mapping  $\mathfrak{C} : \mathfrak{F}\text{fin} \rightarrow \mathfrak{N}$  is a *fuzzy c-measure of finite fuzzy sets* with respect to  $\odot$ , if, for arbitrary  $A, B \in \mathfrak{F}\text{fin}$ , it satisfies the following axioms

- C1: if  $A \cap B = \emptyset$ , then  $\mathfrak{C}(A \cup B) = \mathfrak{C}(A) + \mathfrak{C}(B)$ ,
- C2: if  $i, j \in N$  and  $i > |\text{Supp}(A)|$ ,  $j > |\text{Supp}(B)|$ , then  $\mathfrak{C}(A)(i) = \mathfrak{C}(B)(j)$ ,
- C3: if  $A$  is a crisp set, then  $\mathfrak{C}(A)$  is a crisp set and  $\mathfrak{C}(A)(|A|) = \top$ ,
- C4: if  $a \in L$ ,  $x, y \in \mathcal{U}$  and  $i \in N$ , then  $\mathfrak{C}(\{a/x\})(i) = \mathfrak{C}(\{a/y\})(i)$ ,
- C5: if  $a, b \in L$  and  $x \in \mathcal{U}$ , then

$$\begin{aligned} \mathfrak{C}(\{a \odot b/x\})(0) &= \mathfrak{C}(\{a/x\})(0) \odot \mathfrak{C}(\{b/x\})(0), \\ \mathfrak{C}(\{a \odot b/x\})(1) &= \mathfrak{C}(\{a/x\})(1) \odot \mathfrak{C}(\{b/x\})(1). \end{aligned}$$

The axioms C1-C5 are called the *additivity*, *variability*, *consistency*, *singleton independency*, *preservation of non-existence* and *existence*, respectively. In the following text, for simplicity, we shall often speak only about “c-measure” instead of “fuzzy c-measure of finite fuzzy sets” and we shall omit “with respect to  $\odot$ ”, when no confusion can arise.

The first three axioms are adopted from [17] and their motivation is as follows. The additivity of c-measure is the standard property of cardinality of sets. The variability states that the c-measure of fuzzy sets is not influenced by the elements that do not belong to supports. A consequence of this axiom is correctness of our axiomatic system, it means, the equivalent finite fuzzy sets<sup>6</sup> have the same c-measure. The axiom of consistency ensures the fact that the c-measures are extensions of the cardinality measure (for finite sets). The singleton independency ensures that a form of elements does not influence the value of a c-measure. Finally, a value  $\mathfrak{C}(\{A(x)/x\})(0)$  expresses a degree to which  $\{A(x)/x\}$  may be considered as the empty set  $\{0/x\}$ , i.e.  $x$  does not belong to  $A$ . Analogously, a value  $\mathfrak{C}(\{A(x)/x\})(1)$  determines a degree to which  $\{A(x)/x\}$  may be considered as a singleton  $\{1/x\}$ , i.e.  $x$  belongs to  $A$ . Since  $a \odot b \geq a \wedge b$ , then one could deduce that the singletons  $\{a/x\}$  and  $\{b/x\}$  are closer to the empty set  $\{0/x\}$  than the singleton  $\{a \odot b/x\}$  and, hence,

$$\mathfrak{C}(\{a \odot b/x\})(0) \leq \mathfrak{C}(\{a/x\})(0) \wedge \mathfrak{C}(\{b/x\})(0).$$

Since  $a \odot b \leq a \wedge b$ , we may obtain

$$\mathfrak{C}(\{a/x\})(0) \odot \mathfrak{C}(\{b/x\})(0) \leq \mathfrak{C}(\{a \odot b/x\})(0),$$

but this is in a contradiction with the relation mentioned above. A compromise is the first “strong” relation of the axiom C5.

<sup>6</sup>See Definition 2.

Analogously, one could deduce the second “strong” relation of the axiom C5.<sup>7</sup>

The following proposition shows the FGCount, FLCount and FECount by means of c-measures of finite fuzzy sets for  $\odot = \wedge$  (cf. [17], [8], [1]). Define a complement of  $A$  by  $\bar{A}(x) = A(x) \rightarrow \perp$  for any  $x \in \text{Supp}(A)$  and  $\bar{A}(x) = \perp$  for any  $x \in \text{Dom}(A) \setminus \text{Supp}(A)$ .<sup>8</sup>

**PROPOSITION 10** *The mappings  $\mathfrak{C}_k : \mathfrak{Ffin} \rightarrow \mathfrak{N}$ ,  $k = 1, 2, 3$ , defined by*

$$\mathfrak{C}_1(A)(i) = \bigvee \{a \mid a \in L \text{ and } |A_a| \geq i\}, \quad (30)$$

$$\mathfrak{C}_2(A)(i) = \mathfrak{C}_1(\bar{A})(\bar{i}), \quad (31)$$

$$\mathfrak{C}_3(A)(i) = \mathfrak{C}_1(A)(i) \wedge \mathfrak{C}_2(A)(i) \quad (32)$$

are c-measures of finite fuzzy sets with respect to  $\wedge$ , where  $\bar{i} = \max(0, |\text{Dom}(A)| - i)$ .

**EXAMPLE 3** Let  $\mathbf{L}$  be the Łukasiewicz algebra, i.e.  $a \otimes b = \max(0, a + b - 1)$  and  $A = \{0.5/a, 0.8/b, 0.1/c, 0.4/d, 0/e\}$ . Then

$$\mathfrak{C}_1(A) = \{1/0, 0.8/1, 0.5/2, 0.4/3, 0.1/4, 0/5, 0/6 \dots\},$$

$$\mathfrak{C}_2(A) = \{0/0, 0.2/1, 0.5/2, 0.6/3, 0.9/4, 1/5, 1/6, \dots\},$$

$$\mathfrak{C}_3(A) = \{0/0, 0.2/1, 0.5/2, 0.4/3, 0.1/4, 0/0, 0/6 \dots\}.$$

Note that  $\mathfrak{C}_1$  does not define a c-measure with respect to  $\otimes$  in general. Two examples of c-measures with respect to  $\otimes$  are given below.

**PROPOSITION 11** *Let  $\mathfrak{C}_1$  be the c-measure with respect to  $\wedge$  defined above. Then mappings  $\mathfrak{C}_k : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  defined by*

$$\mathfrak{C}_4(A)(i) = \mathfrak{C}_1(A_{\top})(i), \quad (33)$$

$$\mathfrak{C}_5(A)(i) = \begin{cases} \top, & i = 0, \\ \mathfrak{C}_5(A)(i-1) \otimes \mathfrak{C}_1(A)(i), & i > 0 \end{cases} \quad (34)$$

are c-measures with respect to  $\otimes$ .

**EXAMPLE 4** Let  $A$  be the fuzzy set from Example 3. Then

$$\mathfrak{C}_4(A) = \{1/0, 0/1, 0/2, \dots\},$$

$$\mathfrak{C}_5(A) = \{1/0, 0.8/1, 0.3/2, 0/3, 0/4, 6 \dots\},$$

where, for example,  $0.3 = 0.8 \otimes 0.5 = \max(0, 0.8 + 0.5 - 1)$ .

The following proposition shows that the form of the c-measure can be various. Recall that  $\mathbf{0}$  is the neutral element in the monoid  $(\mathfrak{N}, +, \mathbf{0})$  and  $\mathbb{N}$  can also denote the fuzzy set of all natural numbers, i.e.,  $\mathbb{N}(n) = \top$  for any  $n \in \mathbb{N}$ .

<sup>7</sup>Note that one could imagine a weaker relation, where the equalities in C5 are replaced by the inequality  $\leq$ . Nevertheless, this inequality does not lead to nicer results.

<sup>8</sup>One can verify that  $A = B$  implies  $\bar{A} = \bar{B}$  and thus  $=$  is a congruence with respect to the complement in  $\mathcal{U}$ . Note that the same result cannot be obtained for a definition of  $\bar{A}$  where  $\bar{A}(x) = \top$  for any  $x \in \text{Dom}(A) \setminus \text{Supp}(A)$ .

**PROPOSITION 12** *Let  $\mathfrak{C} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a c-measure. Then  $\mathfrak{C}(\emptyset) = \mathbf{0}$  or  $\mathfrak{C}(\emptyset) = \mathbb{N}$ . If  $\mathfrak{C}(\emptyset) = \mathbb{N}$ , then  $\mathfrak{C}(A)$  is a non-decreasing mapping.*

A simple consequence of this proposition is the fact that  $\mathfrak{C}(A)(i) \in \{\perp, \top\}$  for any  $i \in \mathbb{N}$  such that  $i > |\text{Supp}(A)|$ .

A consequence of the additivity of c-measure is the following theorem.

**THEOREM 13** *Let  $\mathfrak{C} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a c-measure and  $A \in \mathfrak{Ffin}$  such that  $\text{Supp}(A) = \{x_1, \dots, x_m\}$ . Then*

$$\mathfrak{C}(A)(i) = \bigvee_{\substack{i_1, \dots, i_m \in \{0,1\} \\ i_1 + \dots + i_m = i}} \bigodot_{k=1}^m \mathfrak{C}(\{A(x_k)/x_k\})(i_k) \quad (35)$$

for each  $i \in \mathbb{N}$ . Moreover,  $\mathfrak{C}(A)(i) = \perp$  or  $\mathfrak{C}(A)(i) = \top$  hold for any  $m < i$ .

In [17], a representation of cardinalities of finite fuzzy sets using two monotonic mappings  $f, g : [0, 1] \rightarrow [0, 1]$  was shown. In order to introduce an analogical representation for the c-measures, we need to establish a generalization of monotonic mappings.

Let  $\mathbf{L}_1, \mathbf{L}_2$  be complete rdr-lattices. We shall say that  $h : L_1 \rightarrow L_2$  is an  $\odot$ -homomorphism of  $\mathbf{L}_1$  to  $\mathbf{L}_2$ , if  $h$  is a homomorphism of the reduct  $(L_1, \odot_1, \top_1)$  of  $\mathbf{L}_1$  to the reduct  $(L_2, \odot_2, \top_2)$  of  $\mathbf{L}_2$ , i.e.,  $h(a \odot_1 b) = h(a) \odot_2 h(b)$  and  $h(\top_1) = \top_2$ .<sup>9</sup> Further, we shall say that  $h : L_1 \rightarrow L_2$  is an  $\odot_d$ -homomorphism, if  $h$  is a homomorphism from the reduct  $(L_1, \odot_1, \perp_1)$  of  $\mathbf{L}_1$  to the reduct  $(L_2, \odot_2, \top_2)$  of  $\mathbf{L}_2$ , i.e.  $h(a \odot_1 b) = h(a) \odot h(b)$  and  $h(\perp_1) = \top_2$ .

The following lemma gives a characterization of c-measures using  $\odot$ - and  $\odot_d$ -homomorphisms.

**LEMMA 14** *Let  $f, g : L \rightarrow L$  be  $\odot$ - and  $\odot_d$ -homomorphisms from  $\mathbf{L}$  to  $\mathbf{L}$  such that  $f(\perp) \in \{\perp, \top\}$  and  $g(\top) \in \{\perp, \top\}$ , respectively. Let  $\mathfrak{C}_{f,g} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a mapping defined by the induction:*

$$\begin{aligned} \mathfrak{C}_{f,g}(\{a/x\})(0) &= g(a), \quad \mathfrak{C}_{f,g}(\{a/x\})(1) = f(a), \\ \mathfrak{C}_{f,g}(\{a/x\})(k) &= f(\perp), \quad k > 1 \end{aligned}$$

hold for each singleton  $\{a/x\} \in \mathfrak{Ffin}$  and

$$\mathfrak{C}_{f,g}(A) = \mathfrak{C}_{f,g}(\{A(x_1)/x_1\}) + \dots + \mathfrak{C}_{f,g}(\{A(x_m)/x_m\})$$

holds for each  $A \in \mathfrak{Ffin}$  with  $\text{Supp}(A) = \{x_1, \dots, x_m\}$ . Then  $\mathfrak{C}_{f,g}$  is a c-measure of finite fuzzy sets with respect to  $\odot$ .

**THEOREM 15 (REPRESENTATION OF C-MEASURES)** *Let  $\mathfrak{C} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a mapping satisfying the additivity axiom. Then the following statements are equivalent:*

- (i)  $\mathfrak{C}$  is a c-measure of finite fuzzy sets with respect to  $\odot$ ,

<sup>9</sup>Note that each homomorphism between rdr-lattices (or residuated lattices which are the reducts of original rdr-lattices) is also an  $\odot$ -homomorphism.

(ii) there exist an  $\odot$ -homomorphism  $f : L \rightarrow L$  and an  $\overline{\odot}_d$ -homomorphism  $g : L \rightarrow L$ , such that  $f(\perp) \in \{\perp, \top\}$ ,  $g(\top) \in \{\perp, \top\}$  and

$$\begin{aligned}\mathfrak{C}(\{a/x\})(0) &= g(a), \quad \mathfrak{C}(\{a/x\})(1) = f(a), \\ \mathfrak{C}(\{a/x\})(k) &= f(\perp)\end{aligned}$$

hold for arbitrary  $a \in L$ ,  $x \in \mathfrak{C}_{\text{ount}}$  and  $k \in \mathbb{N}$ ,  $k > 1$ .

We shall use  $\mathfrak{C}_{f,g}$  to denote a c-measure which is determined by an  $\odot$ -homomorphism  $f : L \rightarrow L$  and an  $\overline{\odot}_d$ -homomorphism  $g : L \rightarrow L$  satisfying (ii). Moreover, we shall use  $\mathfrak{C}_f$  or  $\mathfrak{C}_g$ , if  $g$  or  $f$  are the trivial  $\overline{\odot}_d$ -homomorphism or  $\odot$ -homomorphism (i.e.,  $g(a) = \top$  or  $f(a) = \top$  for any  $a \in L$ ), respectively. Note that the trivial homomorphisms have no influence on the form of a c-measure.

## V. GRADED EQUIPOLLENCE AND C-MEASURES

Let  $\mathbf{L}$  be a complete rdr-lattice,  $f : L \rightarrow L$  be a mapping and  $A : x \rightarrow L$  be a fuzzy set. We shall use  $f(A)$  to denote a fuzzy set of  $x$  to  $L$  defined by  $f(A)(z) = f(A(z))$  for any  $z \in x$ . Note that  $f(A) = f \circ A$  and thus  $\text{Dom}(A) = \text{Dom}(f(A))$ .

The first theorem shows a relation between the degrees in which there exist one-to-one correspondences between fuzzy sets (under  $f$  and  $g$ ) and the similarity of generalized cardinals obtained by the corresponding c-measure.

**THEOREM 16** Let  $\mathfrak{C}_{f,g} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a c-measure determined by  $\odot$ - and  $\overline{\odot}_d$ -homomorphisms  $f$  and  $g$ , respectively,  $A, B \in \mathfrak{Ffin}$  and  $h \in \text{Bij}(A, B)$ . Then

$$a_{gh} \odot a_{fh} \leq [\mathfrak{C}_{f,g}(A) \approx \mathfrak{C}_{f,g}(B)], \quad (36)$$

where  $a_{gh}$  and  $a_{fh}$  are the degrees in which  $h$  is a one-to-one mapping of  $g(A)$  and  $f(A)$  onto  $g(B)$  and  $f(B)$ , respectively.

A straightforward consequence of the previous theorem is the following corollary.

**COROLLARY 17** Let  $f : L \rightarrow L$  be a  $\odot$ -homomorphism and  $g : L \rightarrow L$  be a  $\overline{\odot}_d$ -homomorphism. Then

- (i)  $[f(A) \sim^\odot f(B)] \leq [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)]$ ,
- (ii)  $[g(A) \sim^\odot g(B)] \leq [\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)]$

hold for any  $A, B \in \mathfrak{Ffin}$ .

Let  $\mathbf{L}$  be a complete rdr-lattice and  $h : L \rightarrow L$  be a mapping. We shall say that  $h$  is  $(k, \odot)$ -compatible with  $\leftrightarrow$ , if

$$\underbrace{(a \leftrightarrow b) \odot \cdots \odot (a \leftrightarrow b)}_{k\text{-times}} \leq h(a) \leftrightarrow h(b). \quad (37)$$

Obviously,  $h$  is  $(k, \wedge)$ -compatible with  $\leftrightarrow$ , if  $(a \leftrightarrow b) \leq h(a) \leftrightarrow h(b)$ . For example,  $h(a) = (\neg a)^k$  is  $(k, \otimes)$ -compatible with  $\leftrightarrow$ .

**COROLLARY 18** Let  $\mathfrak{C}_{f,g} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a c-measure determined by  $\odot$ - and  $\overline{\odot}_d$ -homomorphisms  $f$  and  $g$  which are

$(k, \odot)$ - and  $(l, \odot)$ -compatible with  $\leftrightarrow$ , respectively,  $A, B \in \mathfrak{Ffin}$  and  $h \in \text{Bij}(A, B)$ . Then

$$a_h^{k+l} \leq [\mathfrak{C}_{f,g}(A) \approx \mathfrak{C}_{f,g}(B)], \quad (38)$$

where  $a_h$  is the degree in which  $h$  is a one-to-one mapping of  $A$  onto  $B$  and  $a^n = a$  for  $\odot = \wedge$ . Especially, we have

$$(A \sim^\odot B)^k \leq [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)] \quad (39)$$

$$(A \sim^\odot B)^l \leq [\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)], \quad (40)$$

where  $a^n = a$  for  $\odot = \wedge$ .

In the previous section, we have introduced the fuzzy classes  $p_A^\odot$  and  $p_A^{\overline{\odot}}$ . The following theorems show the relation between the c-measures and some expressions obtained by the fuzzy sets  $p_A^\odot(-, y)$  and  $p_A^{\overline{\odot}}(-, y)$ .

**THEOREM 19** Let  $\mathfrak{C}_{f,g} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a c-measure determined by  $\odot$ - and  $\overline{\odot}_d$ -homomorphisms  $f$  and  $g$ , respectively,  $A \in \mathfrak{Ffin}$  and  $y \in \mathfrak{Ffin}$  such that  $\text{Supp}(A) \subseteq y$  and  $|y| = m$ . Then

$$\mathfrak{C}_{f,g}(A)(i) \leq g(p_A^{\overline{\odot}}(m - i, y)) \odot f(p_A^\odot(i, y)) \quad (41)$$

holds for any  $i = 0, \dots, m$ .

**THEOREM 20** Let  $\mathbf{L}$  be linearly ordered,  $\mathfrak{C}_{f,g} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a c-measure determined by  $\odot$ - and  $\overline{\odot}_d$ -homomorphisms  $f$  and  $g$ , respectively,  $A \in \mathfrak{Ffin}$  and  $y \in \mathfrak{Ffin}$  such that  $\text{Supp}(A) \subseteq y$  and  $|y| = m$ . Then

$$\mathfrak{C}_{f,g}(A)(i) = g(p_A^{\overline{\odot}}(m - i, y)) \odot f(p_A^\odot(i, y)), \quad (42)$$

$$\mathfrak{C}_{f,g}(A)(i) = p_{g(A) \cap y}^{\overline{\odot}}(m - i, y) \odot p_{f(A)}^\odot(i, y) \quad (43)$$

hold for any  $i = 0, \dots, m$  and  $\mathfrak{C}_{f,g}(A)(i) = f(\perp)$  for any  $i > m$ .

**COROLLARY 21** Let  $\mathbf{L}$  be linearly ordered,  $\mathfrak{C}_{f,g} : \mathfrak{Ffin} \rightarrow \mathfrak{N}$  be a c-measure determined by  $\odot$ - and  $\overline{\odot}_d$ -homomorphisms  $f$  and  $g$ , respectively, and  $A \in \mathfrak{Ffin}$  such that  $|\text{Supp}(A)| = m$ . Then

$$\mathfrak{C}_{f,g}(A)(i) = g(p_A^\wedge(i + 1)) \wedge f(p_A^\wedge(i)) \quad (44)$$

holds for any  $i = 0, \dots, m$  and  $\mathfrak{C}_{f,g}(A)(i) = f(\perp)$  for any  $i > m$ , where  $p_A^\wedge(i) = p_A^\wedge(i, \text{Supp}(A))$  for any  $i \in \mathbb{N}$ .

In the following example, we shall introduce three c-measures based on Theorem 20.

**EXAMPLE 5** Let us suppose that the membership degrees of fuzzy sets are interpreted in the Goguen rdr-lattice  $\mathbf{L}_P$  or Łukasiewicz rdr-lattice  $\mathbf{L}_L$  (see Ex. 1) and

$$A = \{0.5/x_1, 0.2/x_2, 0.9/x_3, 1/x_4, 0.8/x_5, 0.9/x_6, 1/x_7, 0.6/x_8\}$$

is a finite fuzzy set. Further, let us define  $f, g : [0, 1] \rightarrow [0, 1]$  by  $f(a) = a$  (the identity mapping) and  $g(a) = 1 - a$  for any  $a \in [0, 1]$ . Obviously, both mappings are homomorphisms

between the corresponding structures. Let us put  $\odot = T_M$ ,  $\odot = T_P$  and  $\odot = T_L$  (the minimum, product and Łukasiewicz conjunction). Then we can construct three c-measures of finite fuzzy sets denoted by  $\mathfrak{C}_{T_M}$ ,  $\mathfrak{C}_{T_P}$  and  $\mathfrak{C}_{T_L}$ . In particular, if  $T$  is one of the mentioned  $t$ -norms, then c-measure determined by the corresponding homomorphisms has the following form (due to Theorem 20)

$$\mathfrak{C}_T(A)(i) = T(1 - p_A^S(8 - i, y), p_A^T(i, y)), \quad (45)$$

where  $y = \{x_1, \dots, x_8\}$ ,  $S$  is the corresponding  $t$ -conorm to  $T$  and  $i = 1, \dots, 8$ . One can simply verify that  $p_A^T(-, y)$  and  $p_A^S(-, y)$  can be expressed by

$$\begin{aligned} p_A^T(i, y) &= T(p_A^{T_M}(0), \dots, p_A^{T_M}(i)), \\ p_A^S(i, y) &= S(p_A^{S_M}(0), \dots, p_A^{S_M}(i)), \end{aligned}$$

where  $S_M$  denotes the maximum, for any  $i = 0, \dots, 8$ , and  $p_A^T(i, y) = \perp$  and  $p_A^T(i, y) = \top$  for any  $i \in \mathbb{N}$  such that  $i > 8$  (see Lemmas 5 and 9). Hence, we can write the sequences

$$\begin{aligned} p_A^{T_M} &= \{1, 1, 1, 0.9, 0.9, 0.8, 0.6, 0.5, 0.2, 0, 0, \dots\}, \\ p_A^{S_M} &= \{0, 0.2, 0.5, 0.6, 0.8, 0.9, 0.9, 1, 1, 1, \dots\} \end{aligned}$$

and, after a simple computation, we obtain

$$\begin{aligned} \mathfrak{C}_{T_M}(A) &= \{0/0, 0/1, 0.1/2, 0.1/3, 0.2/4, 0.4/5, \\ &\quad 0.5/6, 0.5/7, 0.2/8, 0/9, \dots\}, \\ \mathfrak{C}_{T_P}(A) &= \{0/0, 0/1, 0.00032/2, 0.00288/3, 0.02592/4, \\ &\quad 0.10368/5, 0.15552/6, 0.15552/7, 0.03888/8, 0/9, \dots\}, \\ \mathfrak{C}_{T_L}(A) &= \{0/0, 0/1, 0/2, 0/3, 0/4, 0/5, 0/6, 0/7, 0/8, \dots\}. \end{aligned}$$

Up till now all relationships between the graded equipollence and the similarity of c-measures of finite fuzzy sets have been in the form of the inequality. The following corollary shows that the equality can arise only in very special cases.

**COROLLARY 22** *Let  $\mathbf{L}$  be linearly ordered and  $\mathfrak{C}_{f,g} : \mathfrak{F}\text{fin} \rightarrow \mathfrak{C}\text{ount}$  be a c-measure determined by  $\odot$ - and  $\odot_d$ -homomorphisms  $f$  and  $g$ , respectively. Then*

$$(g(A) \sim^\odot g(B)) \odot (f(A) \sim^\odot f(B)) \leq [\mathfrak{C}_{f,g}(A) \approx \mathfrak{C}_{f,g}(B)] \quad (46)$$

for any  $A, B \in \mathfrak{F}\text{fin}$ . Especially, we have

$$g(A) \sim^\wedge g(B) = [\mathfrak{C}_g(A) \approx \mathfrak{C}_g(B)], \quad (47)$$

$$f(A) \sim^\wedge f(B) = [\mathfrak{C}_f(A) \approx \mathfrak{C}_f(B)] \quad (48)$$

for any  $A, B \in \mathfrak{F}\text{fin}$ .

## VI. CONCLUSION

In this paper, we presented two approaches to the cardinality of finite fuzzy sets. In the first approach, the cardinality of fuzzy sets was expressed using the fuzzy (class) relation called the graded equipollence. In the second one, the number of elements in fuzzy sets was obtained by a fuzzy c-measure with respect to an operation  $\odot$  as a generalized cardinal. Although, both approaches seem to be different, we showed that there

exist several natural relations between them. Nevertheless, a desired relation which interconnects both approaches analogously to the classical relations between the equipollence of finite sets and finite ordinals can be proved only in the special cases, when  $\odot = \wedge$  and some other presumptions on the form of fuzzy c-measures have to be satisfied (see Corollary 22).

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