

Type-2 Fuzzy Functional Inference Method

Hirosato Seki

Dept. Technology Management
Osaka Institute of Technology
Osaka, Japan
e-mail: h_seki@ieee.org

Masaharu Mizumoto

Dept. Engineering Informatics
Osaka Electro-Communication University
Osaka, Japan
e-mail: mizumoto3003@yahoo.co.jp

Abstract—This paper proposes a *type-2 fuzzy functional inference method* which extends fuzzy sets of the antecedent parts to type-2 fuzzy sets. This paper next explains that the type-2 fuzzy functional inference method can be classified to the three models: Model 1 is most common method, and Model 2 is a special case of Model 1. We show that the inference results of both Models 1 and 2 can be easily obtained from the area of fuzzy sets of the consequent parts and center of gravity. Moreover, we state that Model 3 can obtain different results compared with the simplified fuzzy inference method and T-S inference method.

Index Terms—Type-2 fuzzy inference, fuzzy functional inference method, interval type-2 fuzzy sets, fuzzy function, T-S inference method.

I. INTRODUCTION

Since Mamdani [1] applied the concept of fuzzy inference to steam engine experimental device, relevant research and applications have been executed in various fields. Especially, researches on the T-S inference method [2], [3], which is widely used as fuzzy control method and so on, are reported in many papers.

Although the antecedent parts of the T-S inference method are fuzzy sets, the consequent parts are functions. However, it may be completely natural that the consequent parts are also fuzzy. From this reason, Seki and Mizumoto [4], [5] have proposed a *fuzzy functional inference method* in which the consequent parts of the T-S inference method is generalized to fuzzy functions, and shown the equivalence [6]–[15] between the fuzzy functional inference method and the weighted T-S inference method.

On the other hand, Type-2 fuzzy inference model [16]–[19] also has the structure, in which the antecedent parts and consequent parts are fuzzy sets. Type-2 fuzzy inference model has achieved a great success in the various fields.

In this paper, we propose a *type-2 fuzzy functional inference method* which extends fuzzy sets of the antecedent parts of fuzzy functional inference method to type-2 fuzzy sets. Firstly, we state Model 1 in which the antecedent parts and consequent parts are type-2 fuzzy sets and fuzzy function, respectively. Secondly, we state Model 2 where the antecedent parts and consequent parts are type-1 fuzzy sets and fuzzy function, respectively. And, this paper show that the Model 2 is equivalent to the type-2 TSK fuzzy model, if fuzzy function of consequent parts of Model 2 is limited to a fuzzy function of linear expression. Thirdly, we state Model 3 in which the antecedent parts and/or consequent parts are interval value,

respectively. Moreover, we clarify that Model 3 can obtain different results compared with the simplified fuzzy inference method and the T-S inference method.

II. FUZZY INFERENCE METHODS

In this section we review the min-max-gravity method, product-sum-gravity method, T-S inference method, and fuzzy functional inference method.

A. Min-Max-Gravity method

We firstly explain the *min-max-gravity method* as *Mamdani's fuzzy inference method* [1] for the fuzzy inference form (see Fig. 1). The rules of the min-max-gravity method are given as follows:

$$\text{Rule } R_i = \begin{cases} x_1 = A_i^1, x_2 = A_i^2, \dots, x_n = A_i^n \\ \longrightarrow y = B_i \end{cases} \quad (1)$$

where $x_1, x_2, \dots, x_n$ are variables of the antecedent part, $A_i^1, A_i^2, \dots, A_i^n$ fuzzy sets, B_i fuzzy sets of the consequent part, $i = 1, 2, \dots, M$ and M is the total number of rules.

Each inference result B'_i which is inferred from the fact " $x_1^0, x_2^0, \dots, x_n^0$ " and the fuzzy rule " $A_i^1, A_i^2, \dots, A_i^n \longrightarrow B_i$ " is given in the following.

The degree of fitness, h_i , of the fact " $x_1^0, x_2^0, \dots, x_n^0$ " to the antecedent parts " $A_i^1, A_i^2, \dots, A_i^n$ " is given as

$$h_i = \min\{A_i^1(x_1^0), A_i^2(x_2^0), \dots, A_i^n(x_n^0)\} \quad (2)$$

Thus, the inference result B'_i is given as

$$B'_i(y) = \min\{h_i, B_i(y)\} \quad (3)$$

The final consequence B' of (1) is aggregated from $B'_1, B'_2, \dots, B'_M$ by using the max. Namely,

$$B'(y) = \max\{B'_1(y), B'_2(y), \dots, B'_M(y)\} \quad (4)$$

The representative point y_0 for the resulting fuzzy set B' is obtained as the center of gravity of B' :

$$y_0 = \frac{\int y \cdot B'(y) dy}{\int B'(y) dy} \quad (5)$$

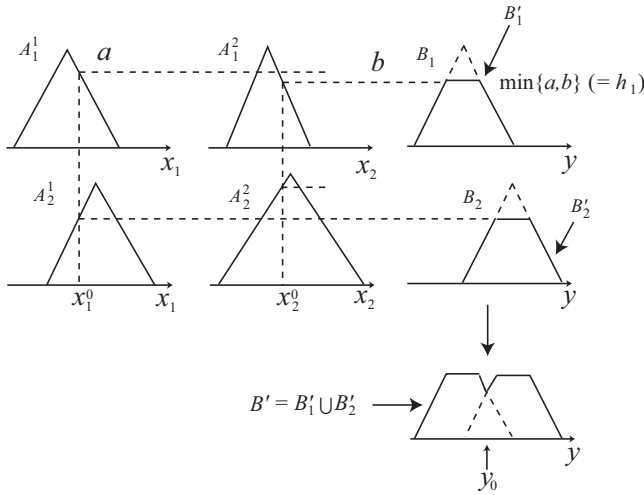


Fig. 1. Min-Max-gravity method.

B. Product-Sum-Gravity method

We secondly explain a fuzzy inference method called *product-sum-gravity method* [20]–[23] for the fuzzy inference form (see Fig. 2). The rules of the product-sum-gravity method are also given as (1).

Each inference result B'_i which is inferred from the fact “ $x_1^0, x_2^0, \dots, x_n^0$ ” and the fuzzy rule “ $A_i^1, A_i^2, \dots, A_i^n \rightarrow B_i$ ” is given in the following.

The degree of fitness, h_i , of the fact “ $x_1^0, x_2^0, \dots, x_n^0$ ” to the antecedent parts “ $A_i^1, A_i^2, \dots, A_i^n$ ” is given as

$$h_i = A_i^1(x_1^0) \cdot A_i^2(x_2^0) \cdots A_i^n(x_n^0) \quad (6)$$

where $\cdot$ stands for the algebraic product. Thus, the inference result B'_i is given as

$$\begin{aligned} B'_i(y) &= A_i^1(x_1^0) \cdots A_i^n(x_n^0) \cdot B_i(y) \\ &= h_i \cdot B_i(y) \end{aligned} \quad (7)$$

The final consequence B' of (1) is aggregated from

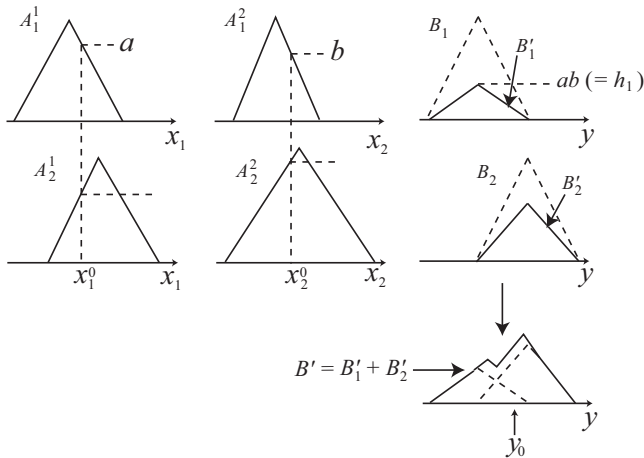


Fig. 2. Product-sum-gravity method.

$B'_1, B'_2, \dots, B'_M$ by using the algebraic sum (+). Namely,

$$B'(y) = B'_1(y) + B'_2(y) + \cdots + B'_M(y) \quad (8)$$

The representative point y_0 for the resulting fuzzy set B' is obtained as the center of gravity of B' :

$$y_0 = \frac{\int y \cdot B'(y) dy}{\int B'(y) dy} \quad (9)$$

C. T-S Inference Method

We thirdly explain the *T-S inference method* [2], [3].

Rules in the T-S inference method are constituted as

$$\text{Rule } R_i = \left\{ \begin{array}{l} x_1 \text{ is } A_i^1, x_2 \text{ is } A_i^2, \dots, x_n \text{ is } A_i^n \\ \rightarrow y = f_i(x_1, x_2, \dots, x_n) \end{array} \right. \quad (10)$$

where $x_1, x_2, \dots, x_n$ are variables of the antecedent part, $A_i^1, A_i^2, \dots, A_i^n$ fuzzy sets, and $f_i(x_1, x_2, \dots, x_n)$ functions of the consequent part, $i = 1, 2, \dots, M$ and M is the total number of fuzzy rules. The inference result y_0 for inputs $x_1^0, x_2^0, \dots, x_n^0$ is given as

$$y_0 = \frac{\sum_{i=1}^M h_i f_i(x_1^0, x_2^0, \dots, x_n^0)}{\sum_{i=1}^M h_i}, \quad h_i = \prod_{j=1}^n A_i^j(x_j^0) \quad (11)$$

where h_i is the degree of the i th rule.

The T-S method becomes the simplified fuzzy inference method when function $f_i(x_1, x_2, \dots, x_n)$ in the consequent part is a constant.

D. Fuzzy Functional Inference Method

Although the antecedent parts of the conventional T-S inference method are fuzzy sets, the consequent parts are function. However, this structure is hard to understand from the different of antecedent parts and consequent parts. Rather, it may be completely natural that the consequent parts are also fuzzy.

In this subsection, we review a *fuzzy functional inference method* [4], [5] in which the consequent parts of the T-S inference method are generalized to fuzzy function. This method can make it easier to understand the structure of rules compared with the ordinary T-S inference method.

The rules in the fuzzy functional inference method are constituted as

$$\text{Rule } R_i = \left\{ \begin{array}{l} x_1 \text{ is } A_i^1, x_2 \text{ is } A_i^2, \dots, x_n \text{ is } A_i^n \\ \rightarrow y = F_i(x_1, x_2, \dots, x_n) \end{array} \right. \quad (12)$$

where $x_1, x_2, \dots, x_n$ are variables of the antecedent part, $A_i^1, A_i^2, \dots, A_i^n$ fuzzy sets, and $F_i(x_1, x_2, \dots, x_n)$ are fuzzy functions.

Since the consequent parts $F_i(x_1, x_2, \dots, x_n)$ constitutes fuzzy sets, the final output y_0 of the fuzzy functional inference method should be obtained from Mamdani type fuzzy inference method. For example, the inference result y_0 is obtained

as follows when the product–sum–gravity method is used (See Fig. 2).

Given the rules of (12), inputs $x_1^0, x_2^0, \dots, x_n^0$ and each antecedent part $A_1^i, A_2^i, \dots, A_n^i$, the degree h_i is obtained as (6), and the inference result $F_i(x_1^0, x_2^0, \dots, x_n^0)'$ of each fuzzy rule is obtained as

$$F_i(x_1^0, x_2^0, \dots, x_n^0)'(y) = h_i \cdot F_i(x_1^0, x_2^0, \dots, x_n^0)(y) \quad (13)$$

where $\cdot$ is the algebraic product.

The final output F' of (12) is obtained by the summation of each inference result $F_i(x_1^0, x_2^0, \dots, x_n^0)'$ as in (8). Namely,

$$\begin{aligned} F'(y) &= F_1(x_1^0, x_2^0, \dots, x_n^0)'(y) + \dots \\ &\quad + F_M(x_1^0, x_2^0, \dots, x_n^0)'(y) \end{aligned} \quad (14)$$

The representative point y_0 of conclusion F' is obtained as follows:

$$y_0 = \frac{\int y \cdot F'(y) dy}{\int F'(y) dy} \quad (15)$$

III. TYPE-2 FUZZY FUNCTIONAL INFERENCE METHOD

In this section, we propose a *type-2 fuzzy functional inference method* which extends fuzzy sets of the antecedent parts of fuzzy functional inference method to type-2 fuzzy sets. Rules of the type-2 fuzzy functional inference method are constructed as follows:

$$\text{Rule } R_i = \left\{ \begin{array}{l} x_1 \text{ is } \tilde{A}_1^1, x_2 \text{ is } \tilde{A}_2^2, \dots, x_n \text{ is } \tilde{A}_n^n \\ \longrightarrow y = F_i(x_1, x_2, \dots, x_n) \end{array} \right. \quad (16)$$

where $x_1, x_2, \dots, x_n$ are variables of the antecedent part, $\tilde{A}_1^1, \tilde{A}_2^2, \dots, \tilde{A}_n^n$ ($i = 1, 2, \dots, M$) type-2 fuzzy sets, and $F_i(x_1, x_2, \dots, x_n)$ are fuzzy functions.

Given input $x_1^0, x_2^0, \dots, x_n^0$, the degree $H(x_1^0, x_2^0, \dots, x_n^0)$ is obtained by the extension principle as

$$H_i(x_1^0, x_2^0, \dots, x_n^0) = \tilde{A}_1^1(x_1^0) * \tilde{A}_2^2(x_2^0) * \dots * \tilde{A}_n^n(x_n^0) \quad (17)$$

where the operation $*$ is t -norm or average operation [5].

The inference result F' is obtained as follows:

$$F' = S_{i=1}^M H_i(x_1^0, x_2^0, \dots, x_n^0) * F_i(x_1^0, x_2^0, \dots, x_n^0) \quad (18)$$

where S is t -conorm, and the operation $*$ is t -norm or average operation, as used in the product–sum–gravity method, min–max–gravity method, and so on. For example, the inference result F' obtained by using the product–sum–gravity method, is given as follows:

$$\begin{aligned} F' &= \sum_{i=1}^M H_i(x_1^0, x_2^0, \dots, x_n^0) \cdot F_i(x_1^0, x_2^0, \dots, x_n^0) \\ &= \sum_{i=1}^M F_i'(x_1^0, x_2^0, \dots, x_n^0) \end{aligned} \quad (19)$$

where $\cdot$ is the algebraic product.

The representative point y_0 of conclusion F' is obtained as follows:

$$y_0 = \frac{\int y \cdot F'(y) dy}{\int F'(y) dy} \quad (20)$$

The representative point y_0 is easily obtained from the area of fuzzy sets of the consequent parts and center of gravity by using the product–sum–gravity method [4], [5].

Note that it is possible to apply the *min–max–gravity method* by replacing algebraic product and algebraic sum with min and max, respectively, in (19).

IV. PROPERTY OF TYPE-2 FUZZY FUNCTIONAL INFERENCE METHOD

The type-2 fuzzy functional inference method can be considered as the following three models as well as the type-2 fuzzy logic systems [17].

- Model 1: Antecedent parts are type-2 fuzzy sets, and consequent parts are fuzzy function (see section III).
- Model 2: Antecedent parts are type-1 fuzzy sets, and consequent parts are fuzzy function (see subsection II-D).
- Model 3: Antecedent parts and/or consequent parts are interval value.

In this section, we discuss the property of the type-2 fuzzy functional inference method.

A. Model 1

Model 1 has the structure in which the antecedent parts and consequent parts are type-2 fuzzy sets and fuzzy function, respectively. The inference results of the conventional type-2 inference systems are obtained by using extension principle and type-reduction. However, the inference results of the type-2 fuzzy functional inference method are obtained from each area of (19) and its center of gravity if the additive fuzzy functional inference method [5] is used.

Therefore, we consider the type-2 fuzzy functional inference method in the case of using the product–sum–gravity method. By using the product–sum–gravity method, the inference result F_i' can be obtained as (19).

At first, let the area and the center of gravity of $F_i'(x_1^0, x_2^0, \dots, x_n^0)$ of (19) be $S_i'(x_1^0, x_2^0, \dots, x_n^0)$ and $f_i'(x_1^0, x_2^0, \dots, x_n^0)$, respectively.

We use the following symbol for short:

$$F_i' = F_i'(x_1^0, x_2^0, \dots, x_n^0) \quad (21)$$

$$f_i' = f_i'(x_1^0, x_2^0, \dots, x_n^0) \quad (22)$$

$$S_i' = S_i'(x_1^0, x_2^0, \dots, x_n^0) \quad (23)$$

From [4] and [5], the inference result y_0 of (20) of the type-2 fuzzy functional inference method can be transformed

to

$$\begin{aligned}
y_0 &= \frac{\int y \cdot F'_i(x)(y)dy}{\int F'_i(x)(y)dy} \\
&= \frac{\int y \cdot \{F'_1(y) + \dots + F'_M(y)\}dy}{\int \{F'_1(y) + \dots + F'_M(y)\}dy} \\
&= \frac{\int y \cdot F'_1(y)dy + \dots + \int y \cdot F'_M(y)dy}{\int F'_1(y)dy + \dots + \int F'_M(y)dy} \\
&= \frac{S'_1 \cdot f'_1 + \dots + S'_M \cdot f'_M}{S'_1 + \dots + S'_M} \quad (24)
\end{aligned}$$

From (24), it turns out that the inference result y_0 of the type-2 fuzzy functional inference method is obtained from the area S'_i and the center of gravity f'_i of each inference result F'_i , in case of an additive fuzzy inference method, which uses the addition as in (19) in the aggregation of each inference result.

Remark 1: In (18), the relation of (24) holds as long as the addition is used in the aggregation, even if instead of using algebraic product, min is used to the operation. Because the area, form and center of gravity of F'_i parts only change. $\square$

B. Model 2

Model 2 has the structure in which the antecedent parts and consequent parts are type-1 fuzzy sets and fuzzy function, respectively. The rules of Model 2 are given as (12). The rules of Model 2 is equivalent to that of the type-2 TSK fuzzy system [17], when fuzzy function is limited to fuzzy function of linear expression, i.e., the rules are constructed as follows:

$$\text{Rule } R_i = \left\{ \begin{array}{l} x_1 \text{ is } A_i^1, x_2 \text{ is } A_i^2, \dots, x_n \text{ is } A_i^n \\ \longrightarrow y = C_i^0 + C_i^1 x_1 + C_i^2 x_2 + \dots + C_i^n x_n \end{array} \right. \quad (25)$$

where $x_1, x_2, \dots, x_n$ are variables of the antecedent part, and $A_i^1, A_i^2, \dots, A_i^n$ and $C_i^1, C_i^2, \dots, C_i^n$ are type-1 fuzzy sets. Therefore, it turns out that Model 2 is extended version of the type-2 TSK fuzzy system.

As well as Model 1, the representative point y_0 of Model 2 is obtained from the area of fuzzy sets of the consequent parts and center of gravity (See [4], [5] for detail).

C. Model 3

Researches on fuzzy inference used interval-valued fuzzy sets are reported in [16]–[19], [24], [25]. Finally, we state a *interval type-2 fuzzy functional inference method* (IT2 fuzzy functional inference method) as Model 3.

Model 3 has the structure in which the antecedent parts and/or consequent parts are interval value.

We shall consider the following simple example (See Fig. 3).

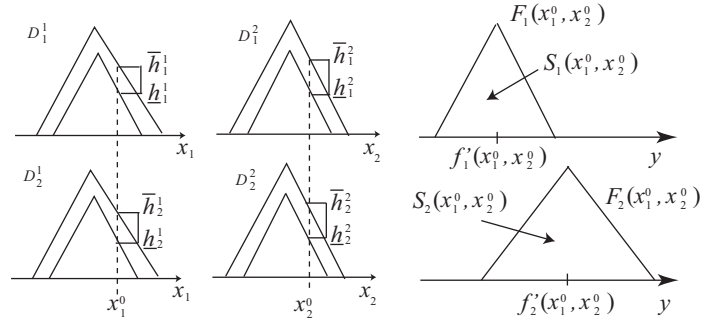


Fig. 3. Type-2 fuzzy functional inference method of Model 3.

$$\text{Rule 1 : } x_1 \text{ is } \tilde{D}_1^1, x_2 = \tilde{D}_1^2 \longrightarrow y = F_1(x_1, x_2) \quad (26)$$

$$\text{Rule 2 : } x_1 \text{ is } \tilde{D}_2^1, x_2 = \tilde{D}_2^2 \longrightarrow y = F_2(x_1, x_2) \quad (27)$$

where $\tilde{D}_j^i$'s are IT2 fuzzy sets, and $F_i(x_1, x_2)$'s are fuzzy functions.

Given input x_1^0 and x_2^0 , the j th degree in the i th input x_i^0 is given as interval value as follows:

$$\tilde{D}_i^j(x_i^0) = [\underline{h}_i^j, \bar{h}_i^j] \quad (28)$$

where $\underline{h}_i^j$ and $\bar{h}_i^j$ represent lower and upper of membership function, respectively.

The inference results of Model 3 are also obtained from the area and center of gravity as (24). That is,

$$y = \frac{[\underline{h}_1^1 \underline{h}_1^2, \bar{h}_1^1 \bar{h}_1^2] \cdot S_1 f'_1 + [\underline{h}_2^1 \underline{h}_2^2, \bar{h}_2^1 \bar{h}_2^2] \cdot S_2 f'_2}{[\underline{h}_1^1 \underline{h}_1^2, \bar{h}_1^1 \bar{h}_1^2] + [\underline{h}_2^1 \underline{h}_2^2, \bar{h}_2^1 \bar{h}_2^2]} \quad (29)$$

where $S_i (= S_i(x_1^0, x_2^0))$ and $f'_i (= f'_i(x_1^0, x_2^0))$ are the area and center of gravity of fuzzy sets of the consequent part, respectively. In the following discussion, we use $y_i = S_i f'_i = S_i(x_1^0, x_2^0) \cdot f'_i(x_1^0, x_2^0)$ ($i = 1, 2$) for short.

From (17) and (18), we obtain the following inference results.

$$\begin{aligned}
y &= \frac{[\underline{h}_1^1 \underline{h}_1^2, \bar{h}_1^1 \bar{h}_1^2] \cdot y_1 + [\underline{h}_2^1 \underline{h}_2^2, \bar{h}_2^1 \bar{h}_2^2] \cdot y_2}{[\underline{h}_1^1 \underline{h}_1^2, \bar{h}_1^1 \bar{h}_1^2] + [\underline{h}_2^1 \underline{h}_2^2, \bar{h}_2^1 \bar{h}_2^2]} \\
&= \frac{[y_1 \underline{h}_1^1 \underline{h}_1^2 + y_2 \underline{h}_2^1 \underline{h}_2^2, y_1 \bar{h}_1^1 \bar{h}_1^2 + y_2 \bar{h}_2^1 \bar{h}_2^2]}{[\underline{h}_1^1 \underline{h}_1^2 + \underline{h}_2^1 \underline{h}_2^2, \bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2]} \\
&= \left[\frac{y_1 \underline{h}_1^1 \underline{h}_1^2 + y_2 \underline{h}_2^1 \underline{h}_2^2}{\underline{h}_1^1 \underline{h}_1^2 + \underline{h}_2^1 \underline{h}_2^2}, \frac{y_1 \bar{h}_1^1 \bar{h}_1^2 + y_2 \bar{h}_2^1 \bar{h}_2^2}{\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2} \right] \quad (30)
\end{aligned}$$

where algebraic product and algebraic sum are used as operations.

Note that the following relation holds for interval $X = [a, b]$, $Y = [c, d]$.

$$X \cdot Y = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}] \quad (31)$$

$$X/Y = [a, b] \cdot [1/d, 1/c] \quad (32)$$

The representative point y_0 of (30) is given as the average of inference result y , i.e., it is obtained as (33), as shown at the top of the next page.

Since the inference result obtained from the above IT2 fuzzy functional inference method, is expressed as a function

$$\begin{aligned}
y_0 &= \frac{1}{2} \left(\frac{y_1 \bar{h}_1^1 \bar{h}_1^2 + y_2 \bar{h}_2^1 \bar{h}_2^2}{\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2} + \frac{y_1 \bar{h}_1^1 \bar{h}_1^2 + y_2 \bar{h}_2^1 \bar{h}_2^2}{\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2} \right) \\
&= \frac{1}{2} \left(\frac{(y_1 \bar{h}_1^1 \bar{h}_1^2 + y_2 \bar{h}_2^1 \bar{h}_2^2)(\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2) + (y_1 \bar{h}_1^1 \bar{h}_1^2 + y_2 \bar{h}_2^1 \bar{h}_2^2)(\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2)}{(\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2)(\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2)} \right) \\
&= \frac{1}{2} \left(\frac{y_1 (\bar{h}_1^1 \bar{h}_1^2)^2 + (y_1 + y_2)(\bar{h}_1^1 \bar{h}_1^2 \bar{h}_2^1 \bar{h}_2^2) + y_2 (\bar{h}_2^1 \bar{h}_2^2)^2 + y_1 (\bar{h}_1^1 \bar{h}_1^2)^2 + (y_1 + y_2)(\bar{h}_1^1 \bar{h}_1^2 \bar{h}_2^1 \bar{h}_2^2) + y_2 (\bar{h}_2^1 \bar{h}_2^2)^2}{(\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2)(\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2)} \right)
\end{aligned} \tag{33}$$

of the square, it can obtain different results compared with the conventional fuzzy inference methods. This is the reason computation of the inference results of the IT2 fuzzy functional inference method, is inverted by division of the numerator and the denominator in interval as (32).

Remark 2: In (30), the inference results can not be obtained by (32), if $\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2$ or $\bar{h}_1^1 \bar{h}_1^2 + \bar{h}_2^1 \bar{h}_2^2$ are 0. $\square$

V. CONCLUSION

In this paper, we have proposed type-2 fuzzy functional inference method which extends fuzzy sets of the antecedent parts to type-2 fuzzy sets. This paper has explained that the type-2 fuzzy functional inference method can be classified to the three Models. Model 1 is most common method, and Model 2 is a special case of Model 1. We have shown that the inference results of both Models 1 and 2 can be easily obtained from the area of fuzzy sets of the consequent parts and center of gravity. Moreover, we have stated that Model 3 can obtain different results compared with the conventional fuzzy inference methods. However, Model 3 requires cautious use when the firing level becomes 0.

Further studies are required to apply the type-2 fuzzy functional inference methods to real systems.

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